**École Doctorale de Sciences Mathématiques de Paris Centre (ED 386)**
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# From derived categories to representations of quantum affine algebras

Thèse de doctorat

Discipline : Mathématiques

présentée par

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# Abstract

This thesis explores the interplay between the representation theory of quantum affine algebras and that of quivers with relations and their differential graded enhancements. It is organized into three main chapters.

The first chapter provides a categorical interpretation of Oh and Suh's combinatorial Auslander–Reiten quivers. Starting from a reduced word for the longest element in a simply-laced finite Weyl group, we construct analogues of the module and derived categories of a Dynkin quiver within the derived category of a 2-dimensional Ginzburg dg algebra. This framework allows us to realize Oh and Suh's quivers as Gabriel quivers, generalizing a result of Bédard. Furthermore, we lift additional combinatorial features to the categorical level, notably in the more restrictive setting of Fujita–Oh's Q-data.

In the second chapter, we employ methods from Gorenstein homological algebra to establish a link between Nakajima's graded tensor product varieties and filtered derived categories, generalizing work by Keller–Scherotzke on graded Nakajima quiver varieties.

The third chapter expands the dictionary between the additive and monoidal categorifications of cluster algebras. We introduce a counterpart to Kashiwara–Kim–Oh–Park's $\Lambda$-invariant for representations of quantum affine algebras in the additive setting. To this end, we associate an ice quiver with potential to their monoidal categorifications and prove that, in certain cases, the corresponding relative Ginzburg dg algebra is proper. This yields a canonical quantum structure on the associated Higgs category, which we use to recover the value of the $\Lambda$-invariant on reachable simple modules.

## Keywords

Derived categories, quantum affine algebras, categorification, cluster algebras, Auslander–Reiten quivers, Coxeter combinatorics, Ginzburg dg algebras, Gorenstein projective modules, Nakajima quiver varieties, ice quivers with potential.

# Des catégories dérivées aux représentations des algèbres affines quantiques

## Résumé

Cette thèse explore l'interaction entre la théorie des représentations des algèbres affines quantiques et celle des carquois avec relations et leurs enrichissements différentiels gradués. Elle est organisée en trois chapitres principaux.

Le premier chapitre propose une interprétation catégorique des carquois d'Auslander–Reiten combinatoires de Oh et Suh. En partant d'un mot réduit pour l'élément de plus grande longueur d'un groupe de Weyl fini simplement lacé, nous construisons des analogues des catégories de modules et des catégories dérivées d'un carquois de Dynkin au sein de la catégorie dérivée d'une dg-algèbre de Ginzburg de dimension 2. Ce cadre nous permet de réaliser les carquois de Oh et Suh comme des carquois de Gabriel, généralisant ainsi un résultat de Bédard. De plus, nous relevons d'autres caractéristiques combinatoires au niveau catégorique, notamment dans le cadre plus restrictif des Q-données de Fujita–Oh.

Dans le deuxième chapitre, nous employons des méthodes issues de l'algèbre homologique de Gorenstein pour établir un lien entre les variétés de produits tensoriels graduées de Nakajima et les catégories dérivées filtrées, généralisant un travail de Keller–Scherotzke sur les variétés de carquois de Nakajima graduées.

Le troisième chapitre élargit le dictionnaire entre les catégorifications additives et monoïdales des algèbres amassées. Nous introduisons une contrepartie à l'invariant $\Lambda$ de Kashiwara–Kim–Oh–Park pour les représentations d'algèbres affines quantiques dans le cadre additif. À cette fin, nous associons un carquois glacé à potentiel à leurs catégorifications monoïdales et prouvons que, dans certains cas, la dg-algèbre de Ginzburg relative associée est propre. Cela produit une structure quantique canonique sur la catégorie de Higgs associée, que nous utilisons pour retrouver la valeur de l'invariant $\Lambda$ sur les modules simples atteignables.

## Mots-clés

Catégories dérivées, algèbres affines quantiques, catégorification, algèbres amassées, carquois d'Auslander–Reiten, combinatoire de Coxeter, dg-algèbres de Ginzburg, modules projectifs de Gorenstein, variétés de carquois de Nakajima, carquois glacés à potentiel.

# Contents

# Introduction

## 0.1 Introduction (version française)

La théorie des représentations des carquois est intimement liée à la théorie de Lie depuis ses origines. L'un de ses résultats fondateurs est le théorème de Gabriel [Gab72], qui classifie les carquois de type de représentation fini en termes de diagrammes de Dynkin de type ADE, déjà familiers dans la classification des algèbres de Lie simples de dimension finie. Au-delà de cette classification, Gabriel a mis en évidence un lien plus profond, qui a ensuite été développé par Bernstein–Gelfand–Ponomarev [BGP73] : les vecteurs dimensions des représentations indécomposables sont en bijection avec les racines positives du système de racines correspondant, et leurs composantes donnent les multiplicités des racines simples dans les racines positives correspondantes. Ce lien a rapidement été étendu dans plusieurs directions. Par exemple, Dlab et Ringel [DR75] ont réalisé les diagrammes de Dynkin non simplement lacés au moyen de la théorie des représentations des espèces, tandis que Kac [Kac80, Kac82] a montré que les systèmes de racines des algèbres de Lie de Kac–Moody gouvernent les vecteurs dimension des représentations indécomposables de carquois sans boucles arbitraires.

Une avancée décisive est venue avec la réalisation par Ringel dans [Rin90] de la partie positive de l'algèbre enveloppante quantique d'une algèbre de Lie simple de dimension finie comme algèbre de Hall des représentations d'une espèce de Dynkin sur un corps fini. Dans cette construction, le paramètre de déformation quantique apparaît naturellement, reliant les représentations de carquois à la théorie des groupes quantiques introduite par Drinfeld et Jimbo (voir [Dri87]). Lusztig a ensuite donné une interprétation géométrique de ce cadre au moyen des variétés de représentations et des faisceaux pervers, conduisant à sa théorie des bases canoniques [Lus90]. Nakajima a prolongé cette perspective par l'introduction des variétés de carquois [Nak94, Nak98], qui fournissent des réalisations géométriques des représentations intégrables de plus haut poids des algèbres de Lie de Kac–Moody. De plus, il a utilisé les variétés de carquois graduées dans [Nak01a] pour obtenir des constructions géométriques de représentations de dimension finie des algèbres affines quantiques.

Un développement parallèle, tout aussi central pour la présente thèse, concerne le rôle des catégories dérivées. À la suite de l'interprétation par Happel des catégories dérivées comme cadre naturel pour la théorie du basculement dans la théorie des représentations des algèbres de dimension finie [Hap87, Hap88], les méthodes dérivées et triangulées sont devenues des outils fondamentaux du domaine. En particulier, les constructions d'algèbres de Hall admettent des analogues dérivés. Ceux-ci ont été introduits dans le contexte des algèbres affines quantiques par Hernandez et Leclerc [HL15], qui ont obtenu un isomorphisme entre certaines déformations d'anneaux de Grothendieck d'algèbres affines quantiques et des algèbres de Hall dérivées de carquois de Dynkin.

Cette interaction entre catégories dérivées et algèbres affines quantiques constitue l'un des fils directeurs de cette thèse. Le premier chapitre est motivé par le cadre de Hernandez–Leclerc. L'un de leurs outils principaux est la combinatoire d'Auslander–Reiten de la catégorie dérivée d'un carquois de Dynkin, généralisée par Oh et Suh [OS19a, OS19b] puis développée davantage par Fujita et Oh [FO21] afin de traiter les types non simplement lacés. Nous donnons une interprétation catégorique des carquois combinatoires d'Auslander–Reiten de Oh–Suh, comme première étape vers une extension de l'isomorphisme de Hernandez–Leclerc au-delà du cas simplement lacé.

Un deuxième fil directeur est la réalisation géométrique des représentations via les variétés de carquois de Nakajima et leur interprétation homologique. Hernandez–Leclerc ont montré dans [HL15] que certaines variétés de carquois de Nakajima graduées peuvent être décrites à l'aide des représentations de carquois de Dynkin, et Keller–Scherotzke [KS16] ont ensuite raffiné cette perspective à l'aide des catégories dérivées et de l'algèbre homologique de Gorenstein. Leur travail constitue le point de départ du deuxième chapitre, où nous étendons cette approche des variétés de carquois graduées aux variétés de produits tensoriels graduées de Nakajima [Nak01b] (qui fournissent des constructions géométriques de produits tensoriels de modules standard), établissant un nouveau lien avec les catégories dérivées filtrées.

Dans une autre direction, la théorie des bases canoniques de Lusztig et ses travaux sur la positivité totale [Lus94] ont constitué une source d'inspiration majeure pour l'introduction des algèbres amassées par Fomin–Zelevinsky [FZ02]. Depuis lors, de nombreux liens entre les algèbres amassées et d'autres branches des mathématiques ont été mis en évidence, en particulier avec la théorie des représentations des carquois et celle des algèbres affines quantiques. Dans ces deux domaines, le lien passe par la notion de catégorification, qui consiste à interpréter les algèbres amassées au moyen de constructions catégoriques où la structure supplémentaire fournit un éclairage additionnel sur la combinatoire d'amas.

D'une part, la catégorification *additive* des algèbres amassées est construite à partir des représentations de carquois ou, plus généralement, des représentations d'algèbres jacobiennes et de leurs enrichissements différentiels gradués. Ce programme a été initié avec les catégories amassées de Caldero–Chapoton–Schiffler [CCS06] et de Buan–Marsh–Reineke–

Reiten–Todorov [BMRRT06], puis développé davantage notamment dans les travaux de Geiß–Leclerc–Schröer [GLS06], Buan–Iyama–Reiten–Scott [BIRS09], Derksen–Weyman–Zelevinsky [DWZ08, DWZ10], Amiot [Ami09], et Plamondon [Pla11b, Pla11a], culminant dans les catégories amassées relatives et les catégories de Higgs de Yilin Wu [Wu23b] et Keller–Wu [KW23b]. Ces catégorifications ont trouvé d'importantes applications dans la théorie des algèbres amassées et ont joué un rôle déterminant dans la démonstration de résultats tels que la cohérence de signe des $c$-vecteurs et l'indépendance linéaire des monômes d'amas.

D'autre part, Hernandez–Leclerc ont montré dans [HL10] que certaines catégories de modules de dimension finie sur des algèbres affines quantiques ont des anneaux de Grothendieck isomorphes à des algèbres amassées, de telle sorte que tous les monômes d'amas correspondent à des classes de modules simples, et ils ont conjecturé que cela restait vrai dans un cadre beaucoup plus général (voir aussi [HL16]). Ce fut le début du programme de catégorification *monoïdale* des algèbres amassées. Après des développements initiaux dus à Nakajima [Nak11], Kang–Kashiwara–Kim–Oh [KKKO18] et Fan Qin [Qin17], les conjectures de Hernandez et Leclerc ont été résolues par Kashiwara–Kim–Oh–Park [KKOP20, KKOP24a, KKOP25], qui les ont également généralisées dans d'autres directions, notamment aux algèbres affines quantiques de type tordu. L'un des ingrédients principaux de leur travail est l'introduction de l'invariant $\Lambda$ et de l'invariant $\mathfrak{d}$, issus de la théorie des $R$-matrices.

Notre troisième fil directeur dans cette thèse est le lien entre catégorification additive et catégorification monoïdale. À ce jour, aucune connexion directe entre les deux n'a été établie, mais de nombreux parallèles entre ces deux mondes ont récemment été mis en évidence dans des travaux tels que Duan–Schiffler [DS26b, DS26a], Contu [Con26], Fujita [Fuj24], et Baur–Fu–Li [BFL24]. En nous appuyant sur l'invariant tropical et l'invariant $F$ introduits par Peigen Cao [Cao25], nous développons davantage ce dictionnaire dans le troisième chapitre en présentant un analogue de l'invariant $\Lambda$ de Kashiwara–Kim–Oh–Park dans le cadre des catégories de Higgs associées aux carquois glacés à potentiel Jacobi-finis.

Décrivons maintenant plus en détail nos principaux résultats ainsi que le contenu de chacun des chapitres.

### 0.1.1 Une catégorification des carquois d'Auslander–Reiten combinatoires

Soit $\mathfrak{g}$ une algèbre de Lie simple complexe de dimension finie, et soit $U_q(L\mathfrak{g})$ l'algèbre de lacets quantique associée, où $q$ est une indéterminée. Lorsque $\mathfrak{g}$ est de type simplement lacé (c'est-à-dire de type ADE), des liens intéressants ont émergé entre la théorie des représentations de $U_q(L\mathfrak{g})$ et celle d'un carquois de Dynkin $Q$ de même type que $\mathfrak{g}$. Un travail fondateur dans cette direction est [HL15], où Hernandez et Leclerc construisent une sous-catégorie monoïdale rigide $\mathscr{C}_{\mathbb{Z}}$ de la catégorie des représentations de dimension finie de $U_q(L\mathfrak{g})$ et étudient son anneau de Grothendieck quantique $\mathscr{K}_t(\mathscr{C}_{\mathbb{Z}})$, qui est une déformation de l'anneau de Grothendieck classique. Leur résultat principal est un isomorphisme entre l'algèbre de Hall dérivée [Toë06] de la catégorie dérivée de $Q$ sur un corps fini $F$ et la spécialisation de $\mathscr{K}_t(\mathscr{C}_{\mathbb{Z}})$ en $t = \sqrt{|F|}$.

Un élément clé dans la démonstration de Hernandez et Leclerc est le calcul de l'inverse $\widetilde{C}(q)$ de la matrice de Cartan quantique de $\mathfrak{g}$, en utilisant le carquois d'Auslander–Reiten (AR) de la catégorie dérivée bornée $\mathcal{D}^b(\operatorname{mod} KQ)$ de $Q$ sur un corps $K$. Leurs formules ont été généralisées à tous les types de Dynkin par Fujita et Oh dans [FO21] à travers leur notion de *Q-donnée*, une généralisation d'un carquois de Dynkin muni d'une fonction de hauteur. Pour chaque Q-donnée $\mathcal{Q}$, ils utilisent la combinatoire du groupe de Weyl pour définir un *carquois d'AR tordu* $\Gamma_{\mathcal{Q}}$ et un *élément de Coxeter tordu généralisé* $\tau_{\mathcal{Q}}$. Dans le cas simplement lacé, ces définitions se spécialisent au carquois d'AR de l'algèbre des chemins $KQ$ et à sa translation d'AR. Nous renvoyons le lecteur à la Section 1.2.6 pour des exemples de carquois d'AR tordus. Ce cadre ne se contente pas de décrire $\widetilde{C}(q)$ pour les types non simplement lacés, mais trouve également des applications significatives dans la théorie des représentations de $U_q(L\mathfrak{g})$ (voir par exemple [FO21, FHOO22, FHOO23, KKOP24a]).

Il est important de noter que la combinatoire de Fujita et Oh s'appuie sur une théorie développée antérieurement par Oh et Suh dans [OS19a] et [OS19b]. Rappelons brièvement le contexte de leurs travaux. Soit $\Delta$ un diagramme de Dynkin de type $\mathsf{ADE}$, et soit $\mathsf{W}$ le groupe de Weyl correspondant. L'élément de plus grande longueur de $\mathsf{W}$ est noté $w_0$, et l'on peut considérer des mots réduits $\boldsymbol{i} = (i_1, \ldots, i_N) \in \Delta_0^N$ pour $w_0$. On dit que deux mots réduits $\boldsymbol{i}$ et $\boldsymbol{j}$ sont *équivalents par commutation* si l'un peut être transformé en l'autre par une suite de relations de tresses courtes impliquant des réflexions simples de $\mathsf{W}$ qui commutent. Cela définit une relation de commutation dont les classes d'équivalence sont appelées *classes de commutation*. Si $Q$ est une orientation de $\Delta$, alors l'ensemble des mots réduits de $w_0$ qui sont des suites de sources pour $Q$ forme une classe de commutation, notée $[Q]$. Celle-ci induit un ordre partiel $\preceq_{[Q]}$ sur l'ensemble des racines positives $\mathsf{R}^+$ du système de racines associé à $\Delta$ (voir la Section 1.2.2). Bédard a montré dans [Béd99] qu'il existe un isomorphisme entre le carquois d'AR de l'algèbre des chemins $KQ$ et le carquois de Hasse de l'ordre partiel $\preceq_{[Q]}$, envoyant une représentation indécomposable sur son vecteur dimension. D'autre part, on peut définir de manière analogue un ordre partiel $\preceq_{[\boldsymbol{i}]}$ sur $\mathsf{R}^+$ pour toute classe de commutation $[\boldsymbol{i}]$. Cela a conduit Oh et Suh dans [OS19a] à étudier le carquois de Hasse de cet ordre partiel et à explorer ses applications en théorie des représentations. Ils l'ont appelé le *carquois d'AR combinatoire* $\Upsilon_{[\boldsymbol{i}]}$ associé à $[\boldsymbol{i}]$. Nous soulignons que les carquois d'AR tordus définis par Fujita et Oh sont des exemples particuliers de carquois d'AR combinatoires associés à certaines classes de commutation introduites dans [OS19b].

Si $Q$ est un carquois de Dynkin, alors $\operatorname{mod} KQ$ peut être étudiée à travers la combinatoire de $\Upsilon_{[Q]}$. Cela soulève une question naturelle : pour toute classe de commutation $[\boldsymbol{i}]$, existe-t-il une « catégorie de représentations » décrite par $\Upsilon_{[\boldsymbol{i}]}$ ? De plus, on peut également chercher un analogue de la catégorie dérivée de $Q$. Dans le premier chapitre, nous proposons des candidates pour de telles catégories et étudions leurs propriétés. Dans le cas des Q-données, nous espérons que nos constructions pourront apporter un éclairage nouveau sur les liens avec la théorie des représentations des algèbres de lacets quantiques mentionnés plus haut.

Pour expliquer plus en détail nos constructions, soit $\Pi_Q$ la dg-algèbre de Ginzburg (complète) de dimension 2 associée à une orientation $Q$ de $\Delta$ (voir la Section 1.3.1). Nous travaillons dans sa catégorie dérivée à valeurs parfaites $\operatorname{pvd}(\Pi_Q)$, également appelée catégorie dérivée de dimension finie. Pour chaque sommet $i \in \Delta_0$, le $\Pi_Q$-module simple correspondant $S_i$ est 2-sphérique en tant qu'objet de $\operatorname{pvd}(\Pi_Q)$, et induit donc le foncteur de twist sphérique $T_i : \operatorname{pvd}(\Pi_Q) \longrightarrow \operatorname{pvd}(\Pi_Q)$ (voir [ST01]). Il est bien connu que $\operatorname{pvd}(\Pi_Q)$ et ses foncteurs de twist sphériques catégorifient le système de racines de $\Delta$ ainsi que l'action de son groupe de Weyl (voir la Section 1.3.4).

Étant donné un mot réduit $\boldsymbol{i} = (i_1, \dots, i_N) \in \Delta_0^N$ pour $w_0$, nous définissons la *catégorie de représentations* $\mathcal{C}([\boldsymbol{i}])$ associée à la classe de commutation $[\boldsymbol{i}]$ comme la sous-catégorie additive de $\operatorname{pvd}(\Pi_Q)$ engendrée par les objets

$$T_{i_1} \cdots T_{i_{k-1}}(S_{i_k})$$

pour $1 \leq k \leq N$. Lorsque $\boldsymbol{i}$ est une suite de sources pour une orientation $Q'$ de $\Delta$, cette construction redonne la catégorie $\operatorname{mod} KQ'$. Plus généralement, on remarque que les objets indécomposables ci-dessus peuvent être vus comme les quotients successifs d'une filtration particulière de l'algèbre préprojective de type $\Delta$, introduits par Amiot–Iyama–Reiten–Todorov [AIRT12].

Notre premier résultat principal montre comment retrouver le carquois d'AR combinatoire à partir de la catégorie de représentations $\mathcal{C}([\boldsymbol{i}])$. On peut le voir comme une généralisation du résultat de Bédard.

**Théorème 0.1.1** (Théorème 1.5.5)**.** *Soit $[\boldsymbol{i}]$ une classe de commutation de mots réduits pour $w_0$. Le carquois d'Auslander–Reiten combinatoire $\Upsilon_{[\boldsymbol{i}]}$ est isomorphe au carquois obtenu à partir du carquois de Gabriel de $\mathcal{C}([\boldsymbol{i}])$ en supprimant toutes les flèches parallèles à des chemins de longueur au moins deux.*

Pour des classes de commutation générales, $\mathcal{C}([\boldsymbol{i}])$ n'est plus une catégorie abélienne. Néanmoins, nous construisons une catégorie $\mathcal{D}([\boldsymbol{i}])$ munie d'équivalences spécifiques qui rappellent $\mathcal{D}^b(\mathrm{mod}\, KQ)$ et les foncteurs de réflexion classiques de Bernstein–Gelfand–Ponomarev [BGP73]. Cette catégorie est obtenue comme quotient de la sous-catégorie pleine de $\mathrm{pvd}(\Pi_Q)$ engendrée par les objets de $\mathcal{C}([\boldsymbol{i}])$ et leurs décalages. Voir la Section 1.4 pour une définition précise. Nous l'appelons la *catégorie dérivée combinatoire de* $[\boldsymbol{i}]$.

Bien que $\mathcal{D}([\boldsymbol{i}])$ ne possède pas de structure naturelle de catégorie triangulée, nous montrons l'existence de certains triangles distingués particuliers dans $\mathrm{pvd}(\Pi_Q)$ dont les sommets appartiennent à $\mathcal{D}([\boldsymbol{i}])$ (Théorème 1.6.5). Ils doivent être vus comme des analogues des triangles d'AR dans $\mathcal{D}^b(\mathrm{mod}\, KQ)$ [Hap88]. En conséquence (Corollaire 1.6.8), nous donnons une preuve alternative de la propriété $\mathfrak{g}$-additive de [FO21, Theorem 3.41], en l'étendant à des classes de commutation arbitraires.

La catégorie dérivée combinatoire est munie d'un foncteur de suspension $\Sigma$, ce qui permet de définir des groupes d'extensions entre objets $M, N \in \mathcal{D}([\boldsymbol{i}])$ par

$$\mathrm{Ext}^n_{[\boldsymbol{i}]}(M, N) = \mathrm{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M, \Sigma^n N).$$

Si $M, N \in \mathcal{C}([\boldsymbol{i}])$, ces foncteurs s'annulent sauf pour $n = 0$ ou $n = 1$, ce qui implique que $\mathcal{C}([\boldsymbol{i}])$ est « héréditaire », de manière analogue au fait que $KQ$ est une algèbre héréditaire. À l'aide de ces groupes, nous construisons une forme d'Euler sur $\mathcal{D}([\boldsymbol{i}])$, et nous montrons que sa symétrisation induit la forme bilinéaire symétrique sur le réseau des racines de $\Delta$ (Théorème 1.7.6).

Considérons maintenant le cas d'une classe de commutation $[\mathcal{Q}]$ donnée par une Q-donnée $\mathcal{Q} = (\Delta, \sigma, \xi)$ au sens de [FO21]. Nous catégorifions l'élément de Coxeter tordu généralisé $\tau_{\mathcal{Q}}$ de [FO21, Section 3.6] et l'interprétons comme une auto-équivalence de $\mathcal{D}([\mathcal{Q}])$. Si $r$ est l'ordre de l'automorphisme $\sigma$ de $\Delta$, soit $S_{\mathcal{Q}} : \mathcal{D}([\mathcal{Q}]) \longrightarrow \mathcal{D}([\mathcal{Q}])$ la composée de $\tau_{\mathcal{Q}}^r$ avec le foncteur de suspension $\Sigma$. Elle se comporte presque comme un foncteur de Serre [BK89] et met en évidence une symétrie remarquable de $\mathcal{D}([\mathcal{Q}])$.

**Théorème 0.1.2** (Théorème 1.8.10)**.** *Soit $\mathcal{Q} = (\Delta, \sigma, \xi)$ une Q-donnée et soit $M \in \mathcal{D}([\mathcal{Q}])$ un objet indécomposable de résidu $\imath \in \Delta_0$. Si l'orbite de $\imath$ sous l'action de $\sigma$ comporte $r$ éléments, alors il existe un isomorphisme*

$$\mathrm{Hom}_{\mathcal{D}([\mathcal{Q}])}(M, N) \xrightarrow{\sim} D\mathrm{Hom}_{\mathcal{D}([\mathcal{Q}])}(N, S_{\mathcal{Q}} M)$$

*naturel en la variable $N \in \mathcal{D}([\mathcal{Q}])$, où $D$ désigne le foncteur de dualité des espaces vectoriels sur $K$.*

Comme application, nous réinterprétons une formule de [FO21] pour l'inverse $\widetilde{C}(q)$ de la matrice de Cartan quantique de $\mathfrak{g}$ en termes de la forme d'Euler $\langle -, - \rangle_{\mathcal{Q}}$ sur la catégorie dérivée combinatoire associée à une Q-donnée $\mathcal{Q}$ de même type. Pour énoncer ce résultat, introduisons quelques notations. Pour $u \in \mathbb{Z}$, notons $\widetilde{c}_{ij}(u)$ le coefficient de

$q^u$ dans le développement en série de Laurent formelle de l'entrée $(i,j)$ de $\widetilde{C}(q)$. Soit $I$ le diagramme de Dynkin de $\mathfrak{g}$, et $D = \mathrm{diag}(d_i \mid i \in I)$ le symétriseur à gauche minimal de sa matrice de Cartan. Soit $\widehat{I} \subset I \times \mathbb{Z}$ le *carquois de répétition replié* associé ([FO21, Section 3.8]). Étant donné une Q-donnée $\mathcal{Q}$ pour $\mathfrak{g}$, il existe une bijection $H_{\mathcal{Q}}$ de $\widehat{I}$ vers l'ensemble des classes d'isomorphisme d'objets indécomposables de la catégorie dérivée combinatoire $\mathcal{D}([\mathcal{Q}])$. Voir la Section 1.8.3 pour plus de détails.

**Théorème 0.1.3** (Proposition 1.8.15)**.** *Soient $(i,p),(j,s) \in \widehat{I}$ tels que $p - s + d_i \geq 0$. Si l'on a $\max\{d_i, d_j\} = r$, alors*

$$\widetilde{c}_{ij}(p - s + d_i) = \left\langle H_{\mathcal{Q}}(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i,p)) \right\rangle_{\mathcal{Q}}$$

*pour toute Q-donnée $\mathcal{Q} = (\Delta, \sigma, \xi)$ pour $\mathfrak{g}$.*

Lorsque $\mathfrak{g}$ est simplement lacée, ce résultat retrouve [Fuj22, Corollary 3.6], qui découle d'une formule de Hernandez–Leclerc dans [HL15].

### 0.1.2 Catégories de filtrations scindées et variétés de carquois graduées

La théorie des variétés de carquois (graduées) et ses applications à la théorie des représentations ont été développées par Nakajima dans une série d'articles [Nak94, Nak98, Nak01a, Nak04], motivée notamment par des travaux antérieurs de Ringel [Rin90] et Lusztig [Lus90]. En particulier, Nakajima a utilisé ces variétés pour donner une réalisation géométrique de certaines représentations de dimension finie de l'algèbre affine quantique $U_q(\widehat{\mathfrak{g}})$ associée à une algèbre de Lie simple complexe de dimension finie $\mathfrak{g}$, obtenant ainsi des informations importantes sur leur structure et sur la théorie de leurs $q$-caractères. La version graduée des variétés de carquois joue également un rôle important dans l'étude par Nakajima des algèbres amassées et de leurs catégorifications monoïdales [Nak11], qui a ensuite été généralisée par Kimura–Qin [KQ14] aux variétés de carquois graduées associées à des carquois acycliques arbitraires.

Une approche fructueuse de l'étude des variétés de carquois graduées a été initiée par Hernandez–Leclerc [HL15] et développée davantage par Leclerc–Plamondon [LP13] et Keller–Scherotzke [KS16]. Dans ces deux derniers travaux, il a été montré que les variétés de carquois graduées affines sont isomorphes à des variétés de représentations d'une catégorie $\mathcal{S}$, appelée la *catégorie de Nakajima singulière*. De plus, pour tout carquois de Dynkin $Q$ de même type que $\mathfrak{g}$, Keller et Scherotzke ont construit un foncteur

$$\Phi : \mathrm{mod}\,\mathcal{S} \longrightarrow \mathcal{D}^b(\mathrm{mod}\,kQ) \tag{1}$$

de la catégorie des $\mathcal{S}$-modules de dimension finie vers la catégorie dérivée bornée des représentations de $Q$ sur le corps $k = \mathbb{C}$ des nombres complexes. Ils ont montré que ce foncteur retrouve la stratification régulière de Nakajima sur la variété de carquois graduée affine au sens suivant : deux $\mathcal{S}$-modules $M$ et $N$ appartiennent à la même strate si et seulement si $\Phi(M)$ et $\Phi(N)$ sont isomorphes dans $\mathcal{D}^b(\operatorname{mod} kQ)$. Ces résultats ont trouvé plusieurs applications, en particulier dans la théorie des groupes quantiques et de leurs représentations (voir, par exemple, [KS14, Qin16, SS16, LW21, Fuj22]).

L'objectif du deuxième chapitre est d'étendre ce cadre à une autre classe de variétés définies indépendamment par Nakajima [Nak01b] (voir aussi [Nak11] pour la version graduée), Malkin [Mal03], et Varagnolo–Vasserot [VV03]. Nous appelons ces variétés les *variétés de produits tensoriels graduées affines d'ordre* $n$, où $n \geq 1$ est un entier positif. Par les constructions de Nakajima, elles fournissent une réalisation géométrique du produit tensoriel d'une suite de $n$ modules standard sur l'algèbre affine quantique. Ces variétés ont des applications, par exemple, comme outils précieux dans la réalisation géométrique de la dualité de Schur–Weyl affine quantique [Fuj20] et dans l'étude des $R$-matrices et de leurs dénominateurs [Fuj22, FH26].

Notre première étape consiste à identifier la variété de produits tensoriels graduée affine d'ordre $n$ à une variété de représentations appropriée. Ceci est réalisé en considérant une catégorie de filtrations de $\mathcal{S}$-modules de longueur $n$ munies d'un scindage, que nous appelons *filtrations scindées* pour abréger. Décrivons cette construction plus précisément, et dans une plus grande généralité, car elle peut présenter un intérêt indépendant. Fixons un corps de base $k$. Soit $\mathcal{A}$ une $k$-catégorie petite et soit $\mathcal{B}$ une sous-catégorie (pas nécessairement pleine) de $\mathcal{A}$ contenant tous les objets. La *catégorie des filtrations scindées de $\mathcal{A}$-modules sur $\mathcal{B}$* est la catégorie $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ dont les objets sont des $\mathcal{A}$-modules $M$ munis d'une filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

par des sous-modules de $\mathcal{A}$, ainsi que d'applications $\mathcal{B}$-linéaires $r_i^M : M_i \to M_{i-1}$ qui sont des rétractions des inclusions $M_{i-1} \to M_i$. En particulier, en restreignant la structure de module à $\mathcal{B}$, chaque inclusion $M_{i-1} \to M_i$ devient un monomorphisme scindé. Nous montrons que $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ est elle-même une catégorie de modules.

**Théorème 0.1.4** (Proposition 2.3.3)**.** *La catégorie* $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ *est équivalente à* $\operatorname{Mod} \mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ *pour une certaine $k$-catégorie* $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$*.*

La catégorie $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ est un exemple de *catégorie de matrices triangulaires*. Cette notion généralise les anneaux de matrices triangulaires, dans l'esprit de la philosophie de Mitchell selon laquelle « les catégories additives sont des anneaux à plusieurs objets » [Mit72]. Les catégories de matrices triangulaires apparaissent sous diverses formes dans la littérature (voir, par exemple, [Tab08, KL15, LOS23]). Par exemple, lorsque $n = 2$, on a

$$\mathcal{T}^2_{\mathcal{B}}(\mathcal{A}) = \begin{pmatrix} \mathcal{A} & 0 \\ \ker(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A}) & \mathcal{A} \end{pmatrix},$$

où le $\mathcal{A}$-$\mathcal{A}$-bimodule dans le coin inférieur gauche est le noyau du morphisme induit par la composition de $\mathcal{A}$. Une description similaire vaut pour $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ pour tout $n$ (voir le Lemme 2.3.7).

Nous nous intéressons principalement au cas où $\mathcal{A} = \mathcal{S}$ est la catégorie de Nakajima singulière et $\mathcal{B} = k\mathcal{S}_0$ est la plus petite sous-catégorie $k$-linéaire de $\mathcal{A}$ contenant tous les objets. Nous définissons la *catégorie de Nakajima singulière de filtrations scindées de longueur* $n$ comme la catégorie $\mathcal{S}^{n\text{-filt}} = \mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$. Comme conséquence immédiate de [LP13, Theorem 2.4] et de la preuve de [Nak01b, Lemma 3.6], les variétés de produits tensoriels graduées affines d'ordre $n$ sont réalisées comme des variétés de représentations de $\mathcal{S}^{n\text{-filt}}$. Bien que ces variétés ne dépendent que de la filtration par des sous-modules de $\mathcal{S}$, nous soulignons que le scindage sur $k\mathcal{S}_0$ est encodé dans l'action de groupe considérée dans [Nak01b] et est nécessaire pour que le Théorème 0.1.4 soit vrai.

Nous fournissons également une description alternative de $\mathcal{S}^{n\text{-filt}}$ en termes de catégories de mailles, en nous appuyant sur la construction de la catégorie $\mathcal{S}$ par Keller–Scherotzke (voir la Proposition 2.4.3). En conséquence, nous montrons dans la Section 2.4 que de nombreux résultats et arguments de [KS16] s'étendent à notre cadre. La réalisation de $\mathcal{S}^{n\text{-filt}}$ comme catégorie de matrices triangulaires joue un rôle crucial dans ces généralisations, en particulier dans la preuve du Lemme 2.4.15. Notre résultat principal est une généralisation de [KS16, Theorem 5.18], décrivant la catégorie stable $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ des $\mathcal{S}^{n\text{-filt}}$-modules projectifs de Gorenstein (pseudo-cohérents), définis via des résolutions complètes totalement acycliques dont les termes sont projectifs de type fini.

**Théorème 0.1.5** (Théorème 2.4.22)**.** *Il existe une équivalence de catégories triangulées*

$$\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} k\overleftarrow{\mathsf{A}}_n \otimes kQ),$$

*où $k\overleftarrow{\mathsf{A}}_n$ désigne l'algèbre des chemins d'un carquois de Dynkin de type $\mathsf{A}_n$ linéairement orienté.*

Nous définissons le *foncteur de stratification* $\Phi^n$ de la catégorie des $\mathcal{S}^{n\text{-filt}}$-modules de dimension finie vers $\mathcal{D}^b(k\overleftarrow{\mathsf{A}}_n \otimes kQ)$ comme la composée

$$\operatorname{mod} \mathcal{S}^{n\text{-filt}} \xrightarrow{\Omega} \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}) \xrightarrow{\sim} \mathcal{D}^b(k\overleftarrow{\mathsf{A}}_n \otimes kQ),$$

où $\Omega$ est le foncteur de syzygie, envoyant un $\mathcal{S}^{n\text{-filt}}$-module $M$ sur le noyau d'une couverture projective $P \to M$. Lorsque $n = 1$, on retrouve ainsi le foncteur $\Phi$ de Keller–Scherotzke dans (1). Nous montrons que $\Phi^n$ est étroitement lié à la stratification de la variété de carquois graduée affine.

**Théorème 0.1.6** (Corollaire 2.4.32)**.** *Soient $M$ et $N$ deux points dans une variété de produits tensoriels graduée affine d'ordre $n$, vus comme des $\mathcal{S}^{n\text{-filt}}$-modules. Si $\Phi^n(M) \cong \Phi^n(N)$, alors pour tous $0 \leq i < j \leq n$, les sous-quotients $M_j/M_i$ et $N_j/N_i$ des filtrations de $M$ et $N$ appartiennent à la même strate d'une variété de carquois graduée affine.*

Comme le montre l'Exemple 2.4.33, la réciproque n'est pas vraie en général. Néanmoins, il serait intéressant de déterminer dans quels cas elle l'est, et plus généralement, de donner une description géométrique de la partition de la variété de produits tensoriels graduée affine d'ordre $n$ induite par $\Phi^n$.

### 0.1.3 Catégorification additive de l'invariant $\Lambda$ monoïdal

#### Contexte et aperçu

Les algèbres amassées sont une classe d'algèbres commutatives introduite par Fomin–Zelevinsky [FZ02] au début des années 2000. Par définition, elles possèdent un ensemble distingué de générateurs construits par une procédure itérative appelée mutation. Une approche fructueuse pour comprendre leur combinatoire complexe consiste à les interpréter au moyen de certaines catégories, dont la riche structure fournit des outils plus conceptuels pour leur étude. Il existe deux types de catégorification des algèbres amassées : additive et monoïdale. Bien qu'elles reposent sur des techniques très différentes, des travaux récents de nombreux auteurs ont mis en évidence des correspondances entre elles qui vont bien au-delà de ce que l'on pouvait initialement attendre. Parmi celles-ci figurent notamment les travaux de Fujita [Fuj22, Fuj24], Fujita–Oh [FO21], Fujita–Murakami [FM23], Duan–Schiffler [DS26b, DS26a], Contu [Con26], Baur–Fu–Li [BFL24], Cao [Cao25] et Casbi [Cas26]. Le troisième chapitre contribue à cette interaction en proposant une réinterprétation additive de l'invariant $\Lambda$ de Kashiwara–Kim–Oh–Park [KKOP20], un outil important dans leur étude de la catégorification monoïdale [KKOP24a, KKOP25].

Le programme de catégorification *additive* a été initié peu après l'introduction des algèbres amassées, à partir de l'observation de similitudes entre la combinatoire des mutations et celle de la théorie du basculement pour les algèbres héréditaires de dimension finie (cf. [MRZ03]). Après l'invention des catégories amassées par Caldero–Chapoton–Schiffler [CCS06] et Buan–Marsh–Reineke–Reiten–Todorov [BMRRT06], il s'est développé et étendu dans de nombreux articles, tels que [CC06, GLS06, CK08, CK06, BIRS09, FK10, DWZ08, DWZ10, Ami09, Pal08, Pla11b, Pla11a, Pre17], culminant avec la catégorification des algèbres amassées antisymétriques à coefficients par Yilin Wu [Wu23b] et Keller–Wu [KW23b]. Dans ces deux articles, à un carquois glacé à potentiel $(Q, F, W)$ on associe une algèbre différentielle graduée $\widehat{\mathbf{\Gamma}}(Q, F, W)$ appelée *dg-algèbre de Ginzburg relative complétée*. À partir de sa catégorie dérivée, les auteurs de [Wu23b, KW23b] construisent la *catégorie de Higgs* $\mathcal{H}(Q, F, W)$ ainsi qu'un *caractère d'amas* qui, sous des hypothèses modérées, catégorifient l'algèbre amassée $\mathcal{A}_{Q,F}$ (à coefficients non inversibles) associée au carquois glacé $(Q, F)$. Par exemple, le caractère d'amas établit une bijection entre les classes d'isomorphisme des objets indécomposables rigides atteignables de $\mathcal{H}(Q, F, W)$ et les variables d'amas de $\mathcal{A}_{Q,F}$.

Dans une autre direction, Hernandez–Leclerc [HL10] ont introduit les catégorifications monoïdales des algèbres amassées, motivées par des identités de caractères en théorie des représentations des algèbres affines quantiques. Une catégorie abélienne monoïdale de longueur fini $\mathcal{C}$ est une *catégorification monoïdale* d'une algèbre amassée $\mathcal{A}$ si elle est munie d'un isomorphisme de $\mathbb{Z}$-algèbres entre l'anneau de Grothendieck $K_0(\mathcal{C})$ et $\mathcal{A}$ qui identifie l'ensemble des monômes d'amas de $\mathcal{A}$ à un sous-ensemble de l'ensemble des classes d'isomorphisme des objets simples de $\mathcal{C}$, dont les éléments sont appelés les objets simples *atteignables*. Pour une algèbre de Lie simple complexe de dimension finie $\mathfrak{g}_0$, Hernandez–Leclerc [HL10, HL16] ont défini des sous-catégories monoïdales $\mathscr{C}_\ell$ $(\ell \geq 1)$ et $\mathscr{C}^-$ de la catégorie des modules de dimension finie sur l'algèbre affine quantique non tordue $U_q'(\widehat{\mathfrak{g}}_0)$, et ont montré que leurs anneaux de Grothendieck admettent naturellement des structures d'algèbres amassées. Ils ont conjecturé que ces catégories fournissent des catégorifications monoïdales de ces algèbres amassées. Après des travaux pionniers de Hernandez–Leclerc et Nakajima [Nak11] (voir aussi Kimura–Qin [KQ14]), une avancée majeure est venue des travaux de Kang–Kashiwara–Kim–Oh [KKKO18] et Qin [Qin17]. S'appuyant sur les outils développés dans [KKKO18], les conjectures de Hernandez–Leclerc ont été entièrement démontrées par Kashiwara–Kim–Oh–Park [KKOP20, KKOP24a, KKOP25], qui les ont généralisées à une famille beaucoup plus vaste de sous-catégories monoïdales $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ de la catégorie des représentations d'une algèbre affine quantique arbitraire $U_q'(\mathfrak{g})$.

Pour une paire $(V, W)$ de modules simples intégrables de dimension finie sur $U_q'(\mathfrak{g})$, Kashiwara–Kim–Oh–Park [KKOP20] ont introduit les invariants $\Lambda$ et $\mathfrak{d}$, notés respectivement $\Lambda(V, W) \in \mathbb{Z}$ et $\mathfrak{d}(V, W) \in \mathbb{Z}_{\geq 0}$, en s'inspirant d'invariants analogues construits dans [KKKO18] pour les représentations des algèbres de Hecke carquois. Ces invariants sont extraits du morphisme de commutation $V \otimes W \to W \otimes V$ issu de la $R$-matrice universelle, et ils encodent des informations importantes sur la structure de ces produits tensoriels. Ils jouent un rôle fondamental dans les démonstrations de [KKOP20, KKOP24a, KKOP25] et permettent à Kashiwara–Kim–Oh–Park d'établir un résultat plus fin : la catégorie $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ est une *catégorification $\Lambda$-monoïdale*. Cela implique notamment que l'anneau commutatif $K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0})$ possède une structure de Poisson compatible [GSV03] et est une *algèbre $\Lambda$-amassée*. Une algèbre $\Lambda$-amassée est une algèbre amassée dans laquelle chaque graine, munie de sa matrice d'échange étendue $\widetilde{B}$, est également dotée d'une matrice d'entiers antisymétrique $\Lambda$, appelée *matrice des coefficients de Poisson*, telle que la paire $(\widetilde{B}, \Lambda)$ est compatible au sens de Berenstein–Zelevinsky [BZ05]. De plus, les paires compatibles associées à deux graines voisines sont reliées par une mutation au sens de loc. cit. Dans le cas de la catégorie $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$, si $\{V_i\}_{i\in K}$ est la famille de modules simples correspondant aux variables d'amas d'une graine, alors la matrice des coefficients de Poisson associée $\Lambda = (\lambda_{ij})_{i,j\in K}$ est donnée par $\lambda_{ij} = \Lambda(V_i, V_j)$ pour $i, j \in K$.

Motivé en partie par ces résultats, Peigen Cao [Cao25] (voir aussi [CL20]) a introduit l'*invariant tropical* et l'*invariant $F$* pour des paires de monômes d'amas (ou, plus généralement, de *bons éléments*) dans une algèbre $\boldsymbol{\Lambda}$-amassée (supérieure) arbitraire. Ces invariants peuvent être calculés à l'aide des $F$-polynômes et des $g$-vecteurs étendus, et il a montré qu'ils retrouvent l'invariant $\Lambda$ et l'invariant $\mathfrak{d}$ pour des paires d'objets simples atteignables dans les algèbres $\boldsymbol{\Lambda}$-amassées de Kashiwara–Kim–Oh–Park. De plus, il a démontré que l'invariant $F$ coïncide avec l'invariant $E$ des représentations décorées de carquois à potentiel dans le cadre de la catégorification additive de Derksen–Weyman–Zelevinsky [DWZ08, DWZ10] (voir aussi [DF15]), confirmant ainsi une conjecture de Fujita [Fuj24].

Notre objectif dans le troisième chapitre est de compléter ce tableau en fournissant une interprétation additive de l'invariant tropical (et donc de l'invariant $\Lambda$) dans le cadre des catégories de Higgs. Pour un carquois glacé à potentiel non dégénéré $(Q, F, W)$ tel que $\widehat{\boldsymbol{\Gamma}}(Q, F, W)$ soit une dg-algèbre *propre*, nous démontrons que l'algèbre amassée associée $\mathcal{A}_{Q,F}$ admet une structure d'algèbre $\boldsymbol{\Lambda}$-amassée définie en termes d'*extensions négatives* dans la catégorie de Higgs $\mathcal{H}(Q, F, W)$, et nous donnons une formule homologique permettant de calculer les invariants tropical et $F$ correspondants. Inspirés par les travaux de Contu [Con26], nous enrichissons la matrice d'échange initiale de Kashiwara–Kim–Oh–Park pour $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ en un carquois glacé à potentiel lorsque l'intervalle $[a, b]$ est fini, et considérons la catégorie de Higgs associée. Lorsque $U_q'(\mathfrak{g})$ est non tordue de type simplement lacé et que la paire PBW $(\mathcal{D}, \underline{w}_0)$ est *adaptée*, nous montrons que la dg-algèbre de Ginzburg relative correspondante est propre et que notre structure d'algèbre $\boldsymbol{\Lambda}$-amassée coïncide avec celle définie par Kashiwara–Kim–Oh–Park, fournissant ainsi une catégorification additive de leurs invariants $\Lambda$ et $\mathfrak{d}$.

Nous expliquons maintenant nos principaux résultats de manière plus détaillée.

### Une structure d'algèbre $\boldsymbol{\Lambda}$-amassée via la catégorification additive

Soit $(Q, F)$ un carquois glacé fini. Nous autorisons les flèches entre sommets gelés, qui peuvent être soit non gelées, soit gelées. Supposons que les ensembles de sommets soient $Q_0 = \{1, 2, \ldots, m\}$ et $F_0 = \{n + 1, \ldots, m\}$ pour $1 \leq n \leq m$. Nous considérons la matrice

$m \times m$ $\widehat{B} = (b_{ij})$ définie par

$$b_{ii} = \begin{cases} 0 & \text{si } 1 \leq i \leq n, \\ 1 & \text{sinon,} \end{cases}$$

et, pour $i \neq j$,

$$b_{ij} = \#\{\text{flèches non gelées } i \to j \text{ dans } Q\} - \#\{\text{flèches } j \to i \text{ dans } Q\}.$$

Les $n$ premières colonnes forment la matrice d'échange étendue $\widetilde{B}$ du carquois glacé $(Q, F)$. Si $\widehat{B}$ est inversible sur $\mathbb{Q}$, nous définissons

$$\Lambda = |\det \widehat{B}| \cdot (\widehat{B}^{-T} - \widehat{B}^{-1}).$$

Dans ce cas, nous montrons que la paire $(\widetilde{B}, \Lambda)$ est compatible (voir la Proposition 3.4.1) et nous l'appelons la *paire compatible associée à* $(Q, F)$. Cette construction est compatible avec la mutation des carquois glacés (en tenant compte des flèches entre sommets gelés !) telle que définie par Pressland [Pre20] (voir le Lemme 3.4.3).

Si l'on munit $(Q, F)$ d'un potentiel $W$, alors la matrice $\widehat{B}$ est la matrice de la forme d'Euler de la catégorie dérivée à valeurs parfaites $\mathrm{pvd}(\widehat{\mathbf{\Gamma}})$ dans la base des dg-modules simples, où $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}(Q, F, W)$ est la dg-algèbre de Ginzburg relative complétée. Lorsque $\widehat{\mathbf{\Gamma}}(Q, F, W)$ est propre, il s'ensuit que $\widehat{B}$ est inversible (voir le Lemme 3.4.4), et donc que la paire compatible associée à $(Q, F)$ est définie. Nous allons interpréter sa matrice des coefficients de Poisson à l'aide de la catégorie de Higgs $\mathcal{H} = \mathcal{H}(Q, F, W)$. Dans ce but, nous définissons

$$[M, N]_{\mathcal{H}} = \sum_{p \geq 0} (-1)^p (\dim \mathrm{Ext}_{\mathcal{H}}^{-p}(M, N) - \dim \mathrm{Ext}_{\mathcal{H}}^{-p}(N, M)), \tag{2}$$

pour $M, N \in \mathcal{H}$, où les extensions négatives sont calculées dans la catégorie amassée relative (voir la Section 3.3.3). Sous l'hypothèse que $\widehat{\mathbf{\Gamma}}(Q, F, W)$ est propre, cette somme est finie (voir le Lemme 3.4.5).

À tout sommet $t$ de l'arbre $n$-régulier étiqueté $\mathbb{T}_n$, nous associons un carquois glacé à potentiel $(Q_t, F_t, W_t)$ obtenu à partir de $(Q, F, W)$ par mutations itérées le long du chemin de $\mathbb{T}_n$ reliant la racine à $t$. Notre premier résultat principal est le suivant.

**Théorème 0.1.7** (Théorème 3.4.6)**.** *Soit $(Q, F, W)$ un carquois glacé à potentiel non dégénéré tel que $\widehat{\mathbf{\Gamma}}(Q, F, W)$ est propre. Soit $\mathcal{H} = \mathcal{H}(Q, F, W)$ la catégorie de Higgs associée. Alors la paire compatible associée à $(Q_t, F_t)$ est définie pour tout $t \in \mathbb{T}_n$, et sa matrice des* *coefficients de Poisson* $\Lambda_t = (\lambda_{ij}^t)$ *est donnée par*

$$\lambda_{ij}^t = [M_i^t, M_j^t]_{\mathcal{H}},$$

*où* $M_i^t \in \mathcal{H}$ *est l'objet indécomposable rigide atteignable correspondant à la* $i$*-ème variable d'amas dans la graine de* $\mathcal{A}_{Q,F}$ *associée à* $t$*. En particulier, cela munit* $\mathcal{A}_{Q,F}$ *d'une structure d'algèbre* $\Lambda$*-amassée.*

Si, de plus, $\widehat{\mathbf{\Gamma}}(Q, F, W)$ est concentrée en degré zéro, alors la catégorie de Higgs est une catégorie exacte de Frobenius d'après un résultat de Yilin Wu [Wu23b, Theorem 6.2]. Comme observé dans la Remarque 3.4.7, il s'ensuit que

$$[M, N]_{\mathcal{H}} = \dim \operatorname{Hom}_{\mathcal{H}}(M, N) - \dim \operatorname{Hom}_{\mathcal{H}}(N, M)$$

pour $M, N \in \mathcal{H}$. Par conséquent, la structure d'algèbre $\Lambda$-amassée sur $\mathcal{A}_{Q,F}$ issue du Théorème 0.1.7 coïncide avec celle de Geiß–Leclerc–Schröer [GLS13] et de Grabowski–Pressland [GP25, Theorem 6.22].

Le résultat suivant fournit une interprétation homologique des invariants tropical et $F$ de Cao pour des paires de monômes d'amas dans $\mathcal{A}_{Q,F}$, calculés à l'aide de la structure d'algèbre $\Lambda$-amassée du Théorème 0.1.7.

**Théorème 0.1.8** (Proposition 3.4.8)**.** *Soit* $(Q, F, W)$ *un carquois glacé à potentiel non dégénéré tel que* $\widehat{\mathbf{\Gamma}}(Q, F, W)$ *est propre. Soit* $\mathcal{H} = \mathcal{H}(Q, F, W)$ *la catégorie de Higgs associée. Pour des objets rigides atteignables* $M, N \in \mathcal{H}$*, on a*

$$\langle CC(M), CC(N)\rangle_{\mathrm{trop}} = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}}$$

*et*

$$(CC(M) \mid\mid CC(N))_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N),$$

*où* $CC$ *désigne le caractère d'amas canonique sur* $\mathcal{H}$*.*

Comme expliqué dans la Remarque 3.4.9, un résultat de Plamondon [Pla11b] nous permet d'exprimer $\dim \operatorname{Ext}^1_{\mathcal{H}}(M, N)$ dans le théorème ci-dessus comme un invariant $E$ entre les représentations décorées correspondant à $M$ et $N$. Nous retrouvons ainsi, dans notre cadre, le lien entre l'invariant $E$ et l'invariant $F$ établi dans [Cao25, Theorem 6.11].

### Catégorification additive de la matrice $\Lambda$ monoïdale

Rappelons les principaux ingrédients intervenant dans la définition des catégories $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ et de leurs graines monoïdales initiales, introduites par Kashiwara–Kim–Oh–Park. Soit $\mathfrak{g}$ une algèbre de Lie de Kac–Moody affine et considérons l'algèbre affine quantique associée $U_q'(\mathfrak{g})$. D'après [KKOP22, Theorem 3.6], on peut associer à $U_q'(\mathfrak{g})$ un diagramme de Dynkin simplement lacé $\Delta$ de type fini dont le système de racines contrôle la décomposition en blocs de la catégorie des $U_q'(\mathfrak{g})$-modules intégrables de dimension finie. Par exemple, si $\mathfrak{g}$ est de type $\mathsf{A}^{(1)}$, $\mathsf{D}^{(1)}$ ou $\mathsf{E}^{(1)}$, alors $\Delta$ est le diagramme de Dynkin de type fini correspondant. Dans [KKOP24b], la notion de *paire PBW complète* est introduite. Elle consiste en une paire $(\mathcal{D}, \underline{w}_0)$, où $\mathcal{D}$ est une *donnée de dualité complète* et $\underline{w}_0$ est une expression réduite de l'élément de plus grande longueur du groupe de Weyl de $\Delta$. La catégorie $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ dépend du choix d'un intervalle d'entiers $[a,b]$, où $a,b \in \mathbb{Z} \cup \{\pm\infty\}$, ainsi que d'une paire PBW complète $(\mathcal{D}, \underline{w}_0)$.

Lorsque $b \in \mathbb{Z}$, Kashiwara–Kim–Oh–Park décrivent le carquois glacé d'une graine initiale pour la structure d'algèbre amassée sur $K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0})$ en adaptant des constructions antérieures de [BFZ05, BIRS09, GLS11]. Nous le notons $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$, puisqu'il ne dépend pas de $\mathcal{D}$. Lorsque l'intervalle $[a,b]$ est fini, nous munissons ce carquois glacé d'un potentiel $W^{[a,b]}(\underline{w}_0)$, en nous appuyant sur [BIRS11, HL16, Con26]. La définition du carquois glacé à potentiel ainsi obtenu est donnée dans la Section 3.5.3. Cela nous permet d'étudier la catégorie de Higgs [Wu23b] qui catégorifie l'algèbre amassée correspondante. Notons que Contu [Con26] a déjà considéré un analogue additif de la catégorification monoïdale de Kashiwara–Kim–Oh–Park. Cependant, pour ses objectifs, il suffisait de catégorifier l'algèbre amassée *sans* coefficients, tandis que la présence de coefficients est essentielle pour nos résultats.

Soit $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ la dg-algèbre de Ginzburg relative complétée du carquois glacé à potentiel $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$. Nous conjecturons qu'il s'agit toujours d'une dg-algèbre propre.

**Conjecture 0.1.9** (Conjecture 3.8.1)**.** La dg-algèbre de Ginzburg relative $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ est propre.

Le résultat suivant confirme cette conjecture dans deux cas particuliers.

**Théorème 0.1.10** (Corollaires 3.8.8 et 3.8.20)**.** *Soit $[a,b]$ un intervalle d'entiers fini et $\underline{w}_0$ une expression réduite de l'élément $w_0$ de plus grande longueur du groupe de Weyl de $\Delta$. La dg-algèbre de Ginzburg relative complétée $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ est une dg-algèbre propre si l'une des conditions suivantes est satisfaite :*

*(1) $\underline{w}_0$ est une suite de sources pour une orientation de $\Delta$ ;*

*(2) $b-a+1$ est un multiple de la longueur de $w_0$.*

Pour démontrer (1), nous montrons d'abord que $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ peut être remplacée par la dg-algèbre de Ginzburg relative *non complétée* $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ (voir le Lemme 3.3.3). D'après une version relative d'un résultat de Keller [Kel11a] (voir le Théorème 3.7.6), $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ est un complétion de Calabi–Yau relative au sens de Yeung [Yeu22]. Cette completion est propre si un certain bimodule dualisant inverse relatif est nilpotent. Nous montrons que c'est effectivement le cas en le décomposant et en ramenant la démonstration au cas $\Delta = \mathsf{A}_1$. Au passage, nous établissons un résultat de « recollement » pour des bimodules dualisants inverses relatifs qui peut être d'intérêt indépendant (voir la Proposition 3.7.8). Pour démontrer (2), nous étudions l'effet des mouvements de commutation et de tresse sur $\underline{w}_0$, en suivant des idées de Contu [Con26, Section 6.1]. Voir également la Proposition 3.8.19 pour un léger renforcement du théorème.

Nous pouvons maintenant énoncer le résultat principal du troisième chapitre. Nous supposons que $\mathfrak{g}$ est non tordue de type simplement lacé, c'est-à-dire de type $\mathsf{A}^{(1)}$, $\mathsf{D}^{(1)}$ ou $\mathsf{E}^{(1)}$. Nous supposons en outre que la paire PBW $(\mathcal{D}, \underline{w}_0)$ est *adaptée à une orientation de* $\Delta$, ce qui signifie que $\mathcal{D}$ est la donnée de dualité complète associée à une fonction de hauteur sur $\Delta$ (voir la Section 3.5.3) et que $\underline{w}_0$ est une suite de sources pour l'orientation correspondante. Soit $[a, b]$ un intervalle d'entiers fini. D'après le Théorème 0.1.10, $\widehat{\boldsymbol{\Gamma}}^{[a,b]}(\underline{w}_0)$ est propre, de sorte que le crochet (2) est défini pour les objets de la catégorie de Higgs associée $\mathcal{H}^{[a,b]}(\underline{w}_0)$. Soit $CC$ le caractère d'amas canonique de $\mathcal{H}^{[a,b]}(\underline{w}_0)$ vers l'algèbre amassée correspondante, notée $\mathcal{A}^{[a,b]}(\underline{w}_0)$, et soit $\varphi : K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}) \to \mathcal{A}^{[a,b]}(\underline{w}_0)$ l'isomorphisme définissant la catégorification $\Lambda$-monoïdale de Kashiwara–Kim–Oh–Park.

**Théorème 0.1.11** (Théorème 3.9.2)**.** *Supposons que* $\mathfrak{g}$ *est non tordue de type simplement lacé. Soit* $(\mathcal{D}, \underline{w}_0)$ *une paire PBW complète adaptée à une orientation de* $\Delta$*, et soient* $a, b \in \mathbb{Z}$ *tels que* $a \leq b$*. Pour des objets simples atteignables* $V, W \in \mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$*, on a*

$$\Lambda(V, W) = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}},$$

*où* $\mathcal{H} = \mathcal{H}^{[a,b]}(\underline{w}_0)$ *et* $M, N \in \mathcal{H}$ *sont les objets rigides atteignables correspondants tels que* $\varphi(V) = CC(M)$ *et* $\varphi(W) = CC(N)$ *dans* $\mathcal{A}^{[a,b]}(\underline{w}_0)$*.*

Nous démontrons ce théorème en montrant que les structures d'algèbre $\Lambda$-amassée sur $\mathcal{A}^{[a,b]}(\underline{w}_0)$ provenant du Théorème 0.1.7 et de la catégorification $\Lambda$-monoïdale coïncident. Dans ce but, nous utilisons une formule de Kashiwara–Kim–Oh–Park exprimant l'invariant $\Lambda$ en fonction de l'invariant $\mathfrak{d}$ et du foncteur de dualité à gauche $\mathscr{D}^{-1}$ (voir la Proposition 3.5.1), à savoir

$$\Lambda(V, W) = \mathfrak{d}(V, W) + \sum_{n \geq 1} (-1)^{n-1} \left[ \mathfrak{d}(V, \mathscr{D}^{-n}(W)) - \mathfrak{d}(W, \mathscr{D}^{-n}(V)) \right].$$

Cette expression ressemble fortement à la définition du crochet $[M, N]_{\mathcal{H}}$ donnée dans (2). Une comparaison des deux formules montre qu'il suffit de démontrer que

$$\mathfrak{d}(V, \mathscr{D}^{-n}(W)) = \dim \operatorname{Ext}^{1-n}_{\mathcal{H}}(M, N)$$

pour tout $n \geq 0$. Cela est établi dans la Proposition 3.9.3 lorsque $V$ et $W$ appartiennent à la graine monoïdale initiale, qui avait été décrite auparavant par Hernandez–Leclerc dans [HL16]. En exploitant une suite infinie de mutations considérée dans loc. cit., nous montrons dans la Proposition 3.6.9 que $\mathscr{D}^{-1}(W)$ peut être calculé à l'aide d'une suite verte maximale. À quelques détails techniques près, l'égalité recherchée découle alors de résultats de Keller [Kel11b, Kel12], combinés aux catégorifications additive et monoïdale de l'invariant $F$. Signalons que le lien entre le foncteur de dualité du côté monoïdal et le foncteur de suspension du côté additif avait déjà été observé auparavant ; voir par exemple [KK19, Section 3.6] et [Qin23, Table 3].

En gardant à l'esprit que nous conjecturons que $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ est propre pour tout intervalle fini $[a,b]$ et toute expression réduite $\underline{w}_0$, nous nous attendons à ce que les hypothèses sur $\mathfrak{g}$ et $(\mathcal{D}, \underline{w}_0)$ dans le Théorème 3.1.5 ne soient pas nécessaires et puissent être supprimées.

## 0.2 Introduction (English version)

The representation theory of quivers has been intertwined with Lie theory since its inception. One of its foundational results is Gabriel's theorem [Gab72], which classifies quivers of finite representation type in terms of Dynkin diagrams of type $\mathsf{ADE}$, already familiar from the classification of finite-dimensional simple Lie algebras. Beyond this classification, Gabriel observed a deeper connection, which was further developed by Bernstein–Gelfand–Ponomarev [BGP73]: the dimension vectors of indecomposable representations are in bijection with the positive roots of the corresponding root system, and their entries record the multiplicities of the simple roots in the corresponding positive roots. This connection was soon extended in several directions. For instance, Dlab and Ringel [DR75] realized non-simply laced Dynkin diagrams through the representation theory of species, while Kac [Kac80, Kac82] showed that the root systems of Kac–Moody Lie algebras govern the dimension vectors of indecomposable representations of arbitrary quivers without loops.

A major breakthrough came with Ringel's realization in [Rin90] of the positive part of the quantized enveloping algebra of a finite-dimensional simple Lie algebra as the Hall algebra of representations of a Dynkin species over a finite field. In this construction, the quantum deformation parameter appears naturally, linking quiver representations to the theory of quantum groups introduced by Drinfeld and Jimbo (see [Dri87]). Lusztig subsequently gave a geometric interpretation of this picture through representation varieties and perverse sheaves, leading to his theory of canonical bases [Lus90]. Nakajima pushed this perspective further through his introduction of quiver varieties [Nak94, Nak98], which provide geometric realizations of integrable highest-weight representations of Kac–Moody

Lie algebras. Moreover, he used graded quiver varieties in [Nak01a] to obtain geometric constructions of finite-dimensional representations of quantum affine algebras.

A parallel development, equally central to the present thesis, concerns the role of derived categories. Following Happel's realization that derived categories provide a natural framework for tilting theory in the representation theory of finite-dimensional algebras [Hap87, Hap88], derived and triangulated methods became fundamental tools in the subject. In particular, Hall algebra constructions admit derived analogues, which were brought into the context of quantum affine algebras by Hernandez and Leclerc [HL15], who obtained an isomorphism between certain deformations of Grothendieck rings of quantum affine algebras and derived Hall algebras of Dynkin quivers.

This interaction between derived categories and quantum affine algebras forms one of the guiding themes of this thesis. The first chapter is motivated by the framework of Hernandez–Leclerc. One of their main tools is the Auslander–Reiten combinatorics of the derived category of a Dynkin quiver, generalized by Oh and Suh [OS19a, OS19b] and further developed by Fujita and Oh [FO21] to treat non-simply laced types. We give a categorical interpretation of Oh–Suh's combinatorial Auslander–Reiten quivers, as a first step toward extending Hernandez–Leclerc's isomorphism beyond the simply-laced case.

A second guiding theme is the geometric realization of representations through Nakajima quiver varieties and their homological interpretation. Hernandez–Leclerc showed in [HL15] that certain graded Nakajima quiver varieties can be described using representations of Dynkin quivers, and Keller–Scherotzke [KS16] later refined this perspective using derived categories and Gorenstein homological algebra. Their work provides the starting point for the second chapter, where we extend this approach from graded quiver varieties to Nakajima's graded tensor product varieties [Nak01b] (which provide geometric constructions of tensor products of standard modules), establishing a new connection with filtered derived categories.

In another direction, Lusztig's theory of canonical bases and his work on total positivity [Lus94] were a major source of inspiration for Fomin–Zelevinsky's introduction of cluster algebras [FZ02]. Since then, cluster algebras have found connections with many other branches of mathematics, in particular with both the representation theory of quivers and that of quantum affine algebras. In these two subjects, the link came through the notion of categorification, which consists of interpreting cluster algebras via category-theoretic constructions where the extra structure provides additional insight into cluster combinatorics.

On the one hand, the *additive* categorification of cluster algebras is built from quiver representations or, more generally, representations of Jacobian algebras and their differential graded enhancements. This program was initiated with the cluster categories of Caldero–Chapoton–Schiffler [CCS06] and Buan–Marsh–Reineke–Reiten–Todorov [BM-

RRT06], and was further developed in particular in the works of Geiß–Leclerc–Schröer [GLS06], Buan–Iyama–Reiten–Scott [BIRS09], Derksen–Weyman–Zelevinsky [DWZ08, DWZ10], Amiot [Ami09], and Plamondon [Pla11b, Pla11a], culminating in the relative cluster categories and the Higgs categories of Yilin Wu [Wu23b] and Keller–Wu [KW23b]. These categorifications found important applications in the theory of cluster algebras, and were instrumental in proving results such as the sign-coherence of $c$-vectors and the linear independence of cluster monomials.

On the other hand, Hernandez–Leclerc proved in [HL10] that certain categories of finite-dimensional modules over quantum affine algebras have Grothendieck rings isomorphic to cluster algebras in such a way that all cluster monomials correspond to classes of simple modules, and conjectured that this was the case much more generally (see also [HL16]). This was the beginning of the *monoidal* categorification program of cluster algebras. After initial developments by Nakajima [Nak11], Kang–Kashiwara–Kim–Oh [KKKO18], and Fan Qin [Qin17], Hernandez and Leclerc's conjectures were resolved by Kashiwara–Kim–Oh–Park [KKOP20, KKOP24a, KKOP25], who also generalized them in other directions, including to quantum affine algebras of twisted types. One of the main ingredients in their work is their introduction of the $\Lambda$-invariant and the $\mathfrak{d}$-invariant arising from the theory of $R$-matrices.

Our third guiding theme in this thesis is the relationship between additive and monoidal categorification. At present, no direct connection between the two has been established, but recently many parallels have been shown between these two worlds in works such as Duan–Schiffler [DS26b, DS26a], Contu [Con26], Fujita [Fuj24], and Baur–Fu–Li [BFL24]. Building on the tropical and $F$-invariants introduced by Peigen Cao [Cao25], we further develop this dictionary in the third chapter by presenting a counterpart to Kashiwara–Kim–Oh–Park's $\Lambda$-invariant in the setting of Higgs categories of Jacobi-finite ice quivers with potential.

For further details, we refer the reader to the introductions at the beginning of each chapter; see Sections 1.1, 2.1, and 3.1.

# Chapter 1

# A categorification of combinatorial Auslander–Reiten quivers

This chapter consists of the article [Can26a], published on the *Journal of the London Mathematical Society*.

## 1.1 Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra, and let $U_q(L\mathfrak{g})$ denote the associated quantum loop algebra, with $q$ an indeterminate. When $\mathfrak{g}$ is of simply laced type (i.e., type $\mathsf{ADE}$), interesting connections have emerged between the representation theory of $U_q(L\mathfrak{g})$ and that of a Dynkin quiver $Q$ of the same type as $\mathfrak{g}$. A seminal work in this direction is [HL15], where Hernandez and Leclerc construct a rigid monoidal subcategory $\mathscr{C}_{\mathbb{Z}}$ of the category of finite-dimensional representations of $U_q(L\mathfrak{g})$ and study its quantum Grothendieck ring $\mathscr{K}_t(\mathscr{C}_{\mathbb{Z}})$, a deformation of the classical Grothendieck ring. Their main result is an isomorphism between the derived Hall algebra [Toë06] of the derived category of $Q$ over a finite field $F$ and the specialization of $\mathscr{K}_t(\mathscr{C}_{\mathbb{Z}})$ at $t = \sqrt{|F|}$.

A key element in Hernandez and Leclerc's proof is their computation of the inverse $\widetilde{C}(q)$ of the quantum Cartan matrix of $\mathfrak{g}$, using the Auslander–Reiten (AR) quiver of the bounded derived category $\mathcal{D}^b(\operatorname{mod} KQ)$ of $Q$ over a field $K$. Their formulas were generalized to arbitrary Dynkin types by Fujita and Oh in [FO21] through their concept of a *Q-datum*, a generalization of a Dynkin quiver with a height function. For each Q-datum $\mathcal{Q}$, they use Weyl group combinatorics to define a *twisted AR quiver* $\Gamma_{\mathcal{Q}}$ and a *generalized twisted Coxeter element* $\tau_{\mathcal{Q}}$. In the simply laced case, these definitions specialize to the AR quiver of the path algebra $KQ$ and its AR translation. We refer the reader to Section 1.2.6 for examples of twisted AR quivers. This framework not only describes $\widetilde{C}(q)$ for nonsimply laced types but also finds significant applications in the representation theory of $U_q(L\mathfrak{g})$ (see, e.g., [FO21, FHOO22, FHOO23, KKOP24a]).

It is important to note that Fujita and Oh's combinatorics build upon a theory developed earlier by Oh and Suh in [OS19a] and [OS19b]. Let us briefly recall the context of their work. Given a Dynkin diagram $\Delta$ of type $\mathsf{ADE}$, let $\mathsf{W}$ denote its corresponding Weyl group. The longest element of $\mathsf{W}$ is denoted by $w_0$, and one can consider reduced words $\boldsymbol{i} = (i_1, \dots, i_N) \in \Delta_0^N$ for $w_0$. We say that two reduced words $\boldsymbol{i}$ and $\boldsymbol{j}$ are *commutation-equivalent* if one can be transformed into the other via a sequence of short braid relations involving commuting simple reflections of $\mathsf{W}$. This gives rise to a commutation relation whose equivalence classes are called *commutation classes*. If $Q$ is an orientation of $\Delta$, then the set of all reduced words for $w_0$ that are source sequences for $Q$ forms a commutation class, denoted by $[Q]$. It induces a partial order $\preceq_{[Q]}$ on the set of positive roots $\mathsf{R}^+$ of the root system associated with $\Delta$ (see Section 1.2.2). Bédard showed in [Béd99] that there is an isomorphism between the AR quiver of the path algebra $KQ$ and the Hasse quiver of the partial order $\preceq_{[Q]}$, sending an indecomposable representation to its dimension vector. On the other hand, one can similarly define a partial order $\preceq_{[\boldsymbol{i}]}$ on $\mathsf{R}^+$ for any commutation class $[\boldsymbol{i}]$. This led Oh and Suh in [OS19a] to study the Hasse quiver of this partial order and to explore its applications to representation theory. They called it the *combinatorial AR quiver* $\Upsilon_{[\boldsymbol{i}]}$ associated with $[\boldsymbol{i}]$. We highlight that the twisted AR quivers defined by Fujita and Oh are specific examples of combinatorial AR quivers associated with certain commutation classes introduced in [OS19b].

If $Q$ is a Dynkin quiver, then $\operatorname{mod} KQ$ can be studied through the combinatorics of $\Upsilon_{[Q]}$. This raises a natural question: for any commutation class $[\boldsymbol{i}]$, is there a "category of representations" that can be described by $\Upsilon_{[\boldsymbol{i}]}$? Additionally, one might also ask for an analog of the derived category of $Q$. In this chapter, we provide candidates for such categories and investigate their properties. In the case of Q-data, we hope that our constructions could shed new light on the connections with the representation theory of quantum loop algebras mentioned earlier.

To explain our constructions in more detail, let $\Pi_Q$ be the (complete) 2-dimensional Ginzburg dg algebra associated with an orientation $Q$ of $\Delta$ (see Section 1.3.1). We work within its perfectly valued derived category $\operatorname{pvd}(\Pi_Q)$, also known as the finite-dimensional derived category. For each vertex $i \in \Delta_0$, the corresponding simple $\Pi_Q$-module $S_i$ is 2-spherical as an object in $\operatorname{pvd}(\Pi_Q)$; hence, it yields the spherical twist functor $T_i : \operatorname{pvd}(\Pi_Q) \longrightarrow \operatorname{pvd}(\Pi_Q)$ (see [ST01]). It is well known that $\operatorname{pvd}(\Pi_Q)$ and its spherical twist functors categorify the root system of $\Delta$ and the action of its Weyl group (see Section 1.3.4).

Given a reduced word $\boldsymbol{i} = (i_1, \dots, i_N) \in \Delta_0^N$ for $w_0$, we define the *category of representations* $\mathcal{C}([\boldsymbol{i}])$ associated with the commutation class $[\boldsymbol{i}]$ to be the additive subcategory of $\mathrm{pvd}(\Pi_Q)$ generated by the objects

$$T_{i_1}\cdots T_{i_{k-1}}(S_{i_k})$$

for $1 \leq k \leq N$. When $\boldsymbol{i}$ is a source sequence for an orientation $Q'$ of $\Delta$, this construction recovers the category $\operatorname{mod} KQ'$. More generally, we remark that the indecomposable objects above can be seen as the consecutive quotients of a particular filtration of the preprojective algebra of type $\Delta$, introduced by Amiot–Iyama–Reiten–Todorov [AIRT12].

Our first main result demonstrates how to recover the combinatorial AR quiver from the category of representations $\mathcal{C}([\boldsymbol{i}])$. It can be seen as a generalization of Bédard's result.

**Theorem 1.1.1** (Theorem 1.5.5)**.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. The combinatorial Auslander–Reiten quiver $\Upsilon_{[\boldsymbol{i}]}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}([\boldsymbol{i}])$ by removing all arrows parallel to paths of length at least two.*

For general commutation classes, $\mathcal{C}([\boldsymbol{i}])$ is no longer an abelian category. Nevertheless, we construct a category $\mathcal{D}([\boldsymbol{i}])$ equipped with specific equivalences that resemble $\mathcal{D}^b(\operatorname{mod} KQ)$ and the classical reflection functors of Bernstein–Gelfand–Ponomarev [BGP73]. This category is obtained as a quotient of the full subcategory of $\mathrm{pvd}(\Pi_Q)$ generated by the objects of $\mathcal{C}([\boldsymbol{i}])$ and their shifts. See Section 1.4 for a precise definition. We call it the *c-derived category of* $[\boldsymbol{i}]$.

Although $\mathcal{D}([\boldsymbol{i}])$ has no natural structure of triangulated category, we show the existence of some particular distinguished triangles in $\mathrm{pvd}(\Pi_Q)$ with corners in $\mathcal{D}([\boldsymbol{i}])$ (Theorem 1.6.5). They should be thought of as analogs to the AR triangles in $\mathcal{D}^b(\operatorname{mod} KQ)$ [Hap88]. As a result (Corollary 1.6.8), we provide an alternative proof of the $\mathfrak{g}$-additive property from [FO21, Theorem 3.41], extending it to arbitrary commutation classes.

The c-derived category is endowed with a suspension functor $\Sigma$, allowing us to define extension groups between objects $M, N \in \mathcal{D}([\boldsymbol{i}])$ as

$$\mathrm{Ext}^n_{[\boldsymbol{i}]}(M,N) = \mathrm{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M, \Sigma^n N).$$

If $M, N \in \mathcal{C}([\boldsymbol{i}])$, these functors vanish unless $n = 0$ or $n = 1$, implying that $\mathcal{C}([\boldsymbol{i}])$ is "hereditary" in a similar manner to the fact that $KQ$ is a hereditary algebra. Using these groups, we construct an Euler form on $\mathcal{D}([\boldsymbol{i}])$, and we prove that its symmetrization induces the symmetric bilinear form on the root lattice of $\Delta$ (Theorem 1.7.6).

Let us now consider the case of a commutation class $[\mathcal{Q}]$ given by a Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$, in the sense of [FO21]. We categorify the generalized twisted Coxeter element $\tau_{\mathcal{Q}}$ of [FO21, Section 3.6] and view it as an autoequivalence of $\mathcal{D}([\mathcal{Q}])$. If $r$ is the order of the automorphism $\sigma$ of $\Delta$, let $S_{\mathcal{Q}} : \mathcal{D}([\mathcal{Q}]) \longrightarrow \mathcal{D}([\mathcal{Q}])$ be the composition of $\tau_{\mathcal{Q}}^r$ with the suspension functor $\Sigma$. It behaves almost like a Serre functor [BK89] and reveals a remarkable symmetry of $\mathcal{D}([\mathcal{Q}])$.

**Theorem 1.1.2** (Theorem 1.8.10)**.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take an indecomposable object $M \in \mathcal{D}([\mathcal{Q}])$ of residue $\imath \in \Delta_0$. If the orbit of $\imath$ under the action of $\sigma$ has $r$ elements, then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}([\mathcal{Q}])}(M, N) \xrightarrow{\sim} D\mathrm{Hom}_{\mathcal{D}([\mathcal{Q}])}(N, S_{\mathcal{Q}}M)$$

*natural in the variable $N \in \mathcal{D}([\mathcal{Q}])$, where $D$ denotes the duality functor for $K$-vector spaces.*

As an application, we reinterpret a formula from [FO21] for the inverse $\widetilde{C}(q)$ of the quantum Cartan matrix of $\mathfrak{g}$ in terms of the Euler form $\langle -, - \rangle_{\mathcal{Q}}$ on the c-derived category associated with a Q-datum $\mathcal{Q}$ of the same type.

To state this result, let us introduce some notation. For $u \in \mathbb{Z}$, denote by $\widetilde{c}_{ij}(u)$ the coefficient of $q^u$ in the expansion of the $(i, j)$-entry of $\widetilde{C}(q)$ as a formal Laurent series in the indeterminate $q$. Let $I$ be the Dynkin diagram of $\mathfrak{g}$, and $D = \mathrm{diag}(d_i \mid i \in I)$ the minimal left symmetrizer of its Cartan matrix. Let $\widehat{I} \subset I \times \mathbb{Z}$ be the associated *folded repetition quiver* ([FO21, Section 3.8]). Given a Q-datum $\mathcal{Q}$ for $\mathfrak{g}$, there is a bijection $H_{\mathcal{Q}}$ from $\widehat{I}$ to the set of isomorphism classes of indecomposable objects of the c-derived category $\mathcal{D}([\mathcal{Q}])$. See Section 1.8.3 for more details.

**Theorem 1.1.3** (Proposition 1.8.15)**.** *Let $(i, p), (j, s) \in \widehat{I}$ be such that $p - s + d_i \geq 0$. If we have $\max\{d_i, d_j\} = r$, then*

$$\widetilde{c}_{ij}(p - s + d_i) = \left\langle H_{\mathcal{Q}}(j, s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i, p)) \right\rangle_{\mathcal{Q}}$$

*for any Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$ for $\mathfrak{g}$.*

When $\mathfrak{g}$ is simply laced, this result recovers [Fuj22, Corollary 3.6], which comes from a formula of Hernandez–Leclerc in [HL15].

## Organization

This chapter is organized as follows. In Section 1.2, we recall the main definitions and results of [OS19a], [OS19b] and [FO21]. In Section 1.3, the 2-dimensional Ginzburg dg algebra $\Pi_Q$, the category $\mathrm{pvd}(\Pi_Q)$, and its spherical twists are introduced, and we explain how they categorify the root system of the same type. The category of representations $\mathcal{C}([\boldsymbol{i}])$ and the c-derived category $\mathcal{D}([\boldsymbol{i}])$, together with an auxiliary repetition category $\mathcal{R}([\boldsymbol{i}])$, are constructed in Section 1.4. In Section 1.5, we describe the Gabriel quiver of these categories in terms of combinatorial AR quivers and combinatorial repetition quivers. In Section 1.6, we prove the existence of certain distinguished triangles in $\mathrm{pvd}(\Pi_Q)$ with corners in $\mathcal{R}([\boldsymbol{i}])$ and deduce the generalized $\mathfrak{g}$-additive property. In Section 1.7, we introduce projective and injective objects, extension groups, and the Euler form. The case of commutation classes coming from Q-data is treated in Section 1.8, where we categorify the generalized twisted Coxeter element $\tau_{\mathcal{Q}}$, prove the partial Serre duality property, and reinterpret some results of [FO21] on inverse quantum Cartan matrices. Finally, in Section 1.9, we provide a description of the indecomposable objects of $\mathcal{C}([\mathcal{Q}])$ for a Q-datum $\mathcal{Q}$ in all nonsimply laced Dynkin types.

## 1.2 Combinatorial AR quivers and Q-data combinatorics

The main goal of this section is to recall the definition of combinatorial Auslander–Reiten (AR) quivers, Q-data, and their combinatorics, following [OS19a], [OS19b] and [FO21]. We also prove some lemmas needed for our main results. Along the way, we introduce most of the language and notation used in the subsequent sections.

### 1.2.1 Simply laced root systems

From now on, $\Delta$ denotes a simply laced Dynkin diagram (i.e., of type $\mathsf{ADE}$) with vertex set $\Delta_0$. For $i, j \in \Delta_0$, we write $i \sim j$ if the vertices $i$ and $j$ are adjacent in $\Delta$.

Let $\mathsf{g}$ be the complex finite-dimensional simple Lie algebra associated with $\Delta$, and denote by $\mathsf{h}$ a Cartan subalgebra. Denote by $\mathsf{R} \subset \mathsf{h}^*$ the associated root system, where $\mathsf{h}^*$ is the dual of $\mathsf{h}$. Let $\{\alpha_i\}_{i \in \Delta_0}$ and $\{\varpi_i\}_{i \in \Delta_0}$ denote a choice of simple roots and the corresponding fundamental weights, respectively, which live inside $\mathsf{h}^*$. With this choice, denote by $\mathsf{R}^+$ the set of positive roots and define $N = |\mathsf{R}^+|$. The simple roots form a basis for the root lattice $\mathsf{Q}$, while the fundamental weights form a basis for the weight lattice $\mathsf{P}$, both contained in $\mathsf{h}^*$. Notice that $\mathsf{Q} \subset \mathsf{P}$. We have a symmetric bilinear form $(-,-) : \mathsf{P} \times \mathsf{P} \longrightarrow \mathbb{Q}$ defined by $(\varpi_i, \alpha_j) = \delta_{ij}$.

Denote by $\mathsf{W} \subset \mathrm{Aut}(\mathsf{P})$ the Weyl group associated with the root system $\mathsf{R}$. It is generated by the simple reflections $s_i : \mathsf{P} \longrightarrow \mathsf{P}$ ($i \in \Delta_0$) defined by $s_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$ for $\lambda \in \mathsf{P}$. Such reflections are involutions subject to the *commutation relation* $s_i s_j = s_j s_i$, if $i \not\sim j$, and the *braid relation* $s_i s_j s_i = s_j s_i s_j$, if $i \sim j$. In this way, the pair $(\mathsf{W}, \{s_i\}_{i \in \Delta_0})$ forms a finite Coxeter system. Let $w_0$ be the longest element of $\mathsf{W}$, which has length $N$. It induces an involution $i \mapsto i^*$ of $\Delta_0$ determined by $w_0(\alpha_i) = -\alpha_{i^*}$.

### 1.2.2 Commutation classes and combinatorial AR quivers

A sequence $\boldsymbol{i} = (i_1, \dots, i_t)$ of vertices in $\Delta$ is a *reduced word for* $w \in \mathsf{W}$ if $w$ has length $t$ and we have $w = s_{i_1} \cdots s_{i_t}$. If $\boldsymbol{i}$ can be obtained from another reduced word $\boldsymbol{i'}$ by the direct application of a commutation relation, we say that $\boldsymbol{i}$ is obtained from $\boldsymbol{i'}$ by a *commutation move*. Two reduced words for $w \in \mathsf{W}$ are *commutation equivalent* if one can be obtained from the other by a sequence of commutation moves. This defines an equivalence relation on the set of reduced words for $w$. The equivalence class containing a reduced word $\boldsymbol{i}$ is called a *commutation class* and is denoted by $[\boldsymbol{i}]$.

Let $\boldsymbol{i} = (i_1, \dots, i_t)$ be a reduced word for $w \in \mathsf{W}$. Define $\beta_k^{\boldsymbol{i}} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$ for $1 \leq k \leq t$. They are distinct positive roots and form a subset $\mathsf{R}^+(w) \subseteq \mathsf{R}^+$ which depends only on $w$ ([Bou68, p. 158, Corollaire 2]). In particular, if $w = w_0$ is the longest element, we have $\mathsf{R}^+(w_0) = \mathsf{R}^+$. There is a total order $<_{\boldsymbol{i}}$ on $\mathsf{R}^+(w)$ defined by $\beta_k^{\boldsymbol{i}} <_{\boldsymbol{i}} \beta_l^{\boldsymbol{i}}$ if $k < l$. For a commutation class $[\boldsymbol{i}]$, we define a partial order $\preceq_{[\boldsymbol{i}]}$ on $\mathsf{R}^+(w)$ given by $\alpha \preceq_{[\boldsymbol{i}]} \beta$ if $\alpha = \beta$ or $\alpha <_{\boldsymbol{j}} \beta$ for all $\boldsymbol{j} \in [\boldsymbol{i}]$.

For a commutation class $[\boldsymbol{i}]$ and a positive root $\alpha \in \mathsf{R}^+(w)$, we define the *residue of* $\alpha$ *with respect to* $[\boldsymbol{i}]$, denoted by $\mathrm{res}^{[\boldsymbol{i}]}(\alpha)$, to be $i_k \in \Delta_0$ if we have $\boldsymbol{i} = (i_1, \dots, i_t)$ and $\alpha = \beta_k^{\boldsymbol{i}}$. As the notation suggests, this is well defined and independent of the choice of representative for the commutation class.

When $w = w_0$, we define the notion of injective and projective positive roots with respect to $[\boldsymbol{i}]$. Write $\boldsymbol{i} = (i_1, \dots, i_N)$. The *injective positive root associated with* $i \in \Delta_0$ is $\beta_k^{\boldsymbol{i}}$ where $1 \leq k \leq N$ is the first index such that $i_k = i$. Dually, the *projective positive root associated with* $i \in \Delta_0$ is $\beta_l^{\boldsymbol{i}}$ where $1 \leq l \leq N$ is the last index such that $i_l = i^*$. Both definitions do not depend on the choice of representative for $[\boldsymbol{i}]$.

Following [OS19a] (which builds on [Béd99]), we define the *combinatorial AR quiver* $\Upsilon_{[\boldsymbol{i}]}$ associated with a commutation class $[\boldsymbol{i}]$ of reduced words for $w \in \mathsf{W}$ as follows. Its vertex set $(\Upsilon_{[\boldsymbol{i}]})_0$ is the set $\mathsf{R}^+(w)$ of positive roots associated with $w$. After having chosen a representative $\boldsymbol{i} = (i_1, \dots, i_t)$ for $[\boldsymbol{i}]$, we define an arrow in $\Upsilon_{[\boldsymbol{i}]}$ from $\beta_k^{\boldsymbol{i}}$ to $\beta_l^{\boldsymbol{i}}$ if $1 \leq l < k \leq t$, $i_k \sim i_l$ and there is no index $l < j < k$ such that $i_j = i_k$ or $i_j = i_l$. This definition is also independent of the choice of the representative for the commutation class.

**Theorem 1.2.1** ([OS19a, Theorem 2.22])**.** *Let* $[\boldsymbol{i}]$ *be a commutation class of reduced words for* $w \in \mathsf{W}$*. The combinatorial AR quiver* $\Upsilon_{[\boldsymbol{i}]}$ *is isomorphic to the Hasse quiver of the partial order* $\preceq_{[\boldsymbol{i}]}$*. More precisely, for* $\alpha, \beta \in \mathsf{R}^+(w)$*, we have* $\alpha \preceq_{[\boldsymbol{i}]} \beta$ *if and only if there is a path from* $\beta$ *to* $\alpha$ *in* $\Upsilon_{[\boldsymbol{i}]}$*.*

Let $X \subseteq (\Upsilon_{[\boldsymbol{i}]})_0$. A total ordering $(x_1, x_2, \dots, x_l)$ of $X$ is a *compatible reading of* $X$ if we have $k \leq k'$ whenever there is an oriented path from $x_{k'}$ to $x_k$ in $\Upsilon_{[\boldsymbol{i}]}$. Given such a compatible reading, we define the element $w[X] \in \mathsf{W}$ to be

$$w[X] = s_{i_1} s_{i_2} \cdots s_{i_l},$$

where $i_k = \mathrm{res}^{[\boldsymbol{i}]}(x_k)$ for $1 \leq k \leq l$. By adapting the proof of [FO21, Lemma 3.18], one can show that $w[X]$ is independent of the choice of compatible reading.

**Theorem 1.2.2** ([OS19a, Theorem 2.22])**.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w \in \mathsf{W}$. A sequence $(i_1, \dots, i_t)$ of vertices in $\Delta_0$ is in $[\boldsymbol{i}]$ if and only if there is a compatible reading $(x_1, \dots, x_t)$ of $(\Upsilon_{[\boldsymbol{i}]})_0$ such that $i_k = \mathrm{res}^{[\boldsymbol{i}]}(x_k)$ for all $1 \leq k \leq t$.*

### 1.2.3 Combinatorial reflection functors and repetition quivers

In this subsection, we only work with reduced words for the longest element $w_0 \in \mathsf{W}$. We say that $i \in \Delta_0$ is a *source* (resp. *sink*) of a commutation class $[\boldsymbol{i}]$ of reduced words for $w_0$ if there is a reduced word $\boldsymbol{j} \in [\boldsymbol{i}]$ starting with $i$ (resp. ending with $i^*$). By [OS19a, Proposition 4.4], $i$ is a source (resp. sink) of $[\boldsymbol{i}]$ if and only if $\alpha_i$ is a sink (resp. source) of $\Upsilon_{[\boldsymbol{i}]}$.

If $\boldsymbol{j} = (j_1, \dots, j_N)$ is a reduced word for $w_0$, then so is $r_{j_1}\boldsymbol{j} = (j_2, \dots, j_N, j_1^*)$. If $i \in \Delta_0$ is a source of a commutation class $[\boldsymbol{i}]$, we define $r_i[\boldsymbol{i}] = [r_i\boldsymbol{j}]$ where $\boldsymbol{j} \in [\boldsymbol{i}]$ is a reduced word starting with $i$. This operation on commutation classes is well defined and is called a *(combinatorial) reflection functor*. It is injective and its image is the set of commutation classes for which $i$ is a sink. We denote its inverse by $r_i^{-1}$. A sequence of vertices $(i_1, \dots, i_k)$ of $\Delta$ is a *source sequence for* $[\boldsymbol{i}]$ if $i_l$ is a source of $r_{i_{l-1}} r_{i_{l-2}} \cdots r_{i_1}[\boldsymbol{i}]$ for all $1 \leq l \leq k$. Similarly, a sequence of vertices $(i_1, \dots, i_k)$ of $\Delta$ is a *sink sequence for* $[\boldsymbol{i}]$ if $i_l$ is a sink of $r_{i_{l-1}}^{-1} r_{i_{l-2}}^{-1} \cdots r_{i_1}^{-1}[\boldsymbol{i}]$ for all $1 \leq l \leq k$. If two commutation classes can be obtained from one another by applying a sequence of reflection functors (or their inverses), we say that they are *in the same $r$-cluster point*.

For a reduced word $\boldsymbol{i} = (i_1, \dots, i_N)$ for $w_0$, denote by $\widehat{\boldsymbol{i}} = (i_k)_{k \in \mathbb{Z}}$ the infinite sequence extending $\boldsymbol{i}$ and satisfying $i_{k+N} = i_k^*$ for $k \in \mathbb{Z}$. Define $\widehat{\beta}^{\boldsymbol{i}}_{s+tN} = ((-1)^t \beta^{\boldsymbol{i}}_s, -t) \in \mathsf{R} \times \mathbb{Z}$ for $1 \leq s \leq N$ and $t \in \mathbb{Z}$. Since the permutation $i \mapsto i^*$ is induced by $w_0$, observe that the first coordinate of $\widehat{\beta}^{\boldsymbol{i}}_k$ is

$$\begin{cases} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k \geq 1, \\ -s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0. \end{cases}$$

Let $\widehat{\mathsf{R}}$ be the subset of $\mathsf{R} \times \mathbb{Z}$ whose elements are the roots $\widehat{\beta}^{\boldsymbol{i}}_k$ for $k \in \mathbb{Z}$. Alternatively, since $\mathsf{R}^+(w_0) = \mathsf{R}^+$, $\widehat{\mathsf{R}}$ is the subset of pairs $(\alpha, k) \in \mathsf{R} \times \mathbb{Z}$ such that $(-1)^k \alpha \in \mathsf{R}^+$.

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. For any root $\alpha \in \mathsf{R}$, we define its residue with respect to $[\boldsymbol{i}]$, denoted by $\mathrm{res}^{[\boldsymbol{i}]}(\alpha)$, to be $i_k \in \Delta_0$ if we have $\widehat{\boldsymbol{i}} = (i_l)_{l \in \mathbb{Z}}$

and $\alpha$ is the first coordinate of $\widehat{\beta}^{\boldsymbol{i}}_k$. It is well defined and extends the previous definition of residue. We remark that $\mathrm{res}^{[\boldsymbol{i}]}(-\alpha) = \mathrm{res}^{[\boldsymbol{i}]}(\alpha)^*$. For $(\alpha, k) \in \widehat{\mathsf{R}}$, we define its residue to be $\mathrm{res}^{[\boldsymbol{i}]}(\alpha)$.

We can enlarge the combinatorial AR quiver $\Upsilon_{[\boldsymbol{i}]}$ to the *(combinatorial) repetition quiver* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ as follows. Its vertex set $(\widehat{\Upsilon}_{[\boldsymbol{i}]})_0$ is the set $\widehat{\mathsf{R}}$ defined above. Given a representative $\boldsymbol{i} \in [\boldsymbol{i}]$, we construct the infinite sequence $\widehat{\boldsymbol{i}} = (i_k)_{k\in\mathbb{Z}}$. We define an arrow in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ from $\widehat{\beta}^{\boldsymbol{i}}_k$ to $\widehat{\beta}^{\boldsymbol{i}}_l$ if $l < k$, $i_k \sim i_l$ and there is no index $l < j < k$ such that $i_j = i_k$ or $i_j = i_l$. As before, $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ depends only on the commutation class. By construction, we can identify $\Upsilon_{[\boldsymbol{i}]}$ with the full subquiver of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ with vertex set $\mathsf{R}^+ \times \{0\} \subset \widehat{\mathsf{R}}$. In particular, we see the injective (resp. projective) positive roots as *injective* (resp. *projective*) *vertices* of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. Similarly, as we did for $\Upsilon_{[\boldsymbol{i}]}$, we can define what is a compatible reading for any finite subset $X$ of $(\widehat{\Upsilon}_{[\boldsymbol{i}]})_0$ and use it to define an element $w[X] \in \mathsf{W}$.

If $\widehat{\beta}^{\boldsymbol{i}}_k$ and $\widehat{\beta}^{\boldsymbol{i}}_l$ have the same residue and $l < k$, then there is a path in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ from $\widehat{\beta}^{\boldsymbol{i}}_k$ to $\widehat{\beta}^{\boldsymbol{i}}_l$ when $\Delta \neq \mathsf{A}_1$. We refer to this as the *segment property of* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. We also point out that $\Upsilon_{[\boldsymbol{i}]}$ is a *convex subquiver* of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, that is, any directed path in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ between vertices of $\Upsilon_{[\boldsymbol{i}]}$ is entirely contained in $\Upsilon_{[\boldsymbol{i}]}$. This follows from the fact that, for an arrow $(\alpha, k) \to (\alpha', k')$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, we must have $k' = k$ or $k' = k + 1$.

**Example 1.2.3.** Suppose $\Delta = \mathsf{A}_3$ and enumerate its vertices as $\Delta_0 = \{1, 2, 3\}$, where 2 is the central vertex. An example of reduced word for $w_0$ is $\boldsymbol{i} = (3, 2, 1, 2, 3, 2)$. In this case, the repetition quiver $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ is the following:

$$\begin{tikzcd}[column sep=tiny]
1 & & & & (-[3],-1) \arrow[dr] & & & & ([1,3],0) \arrow[dr] & & & & (-[1,2],1) \arrow[dr] & \\
2 & \ddots \arrow[dr] & & (-[2,3],-1) \arrow[ur] & & ([2],0) \arrow[dr] & & ([1],0) \arrow[ur] & & ([2,3],0) \arrow[dr] & & (-[2],1) \arrow[ur] & & \ddots \\
3 & & (-[1,3],-1) \arrow[ur] & & & & ([1,2],0) \arrow[ur] & & & & ([3],0) \arrow[ur] & & &
\end{tikzcd}$$

Here, we denote $[i, j] = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \le i \le j \le 3$. In each row, all vertices have the same residue, which is indicated in the column on the left. The subquiver highlighted in a darker color is $\Upsilon_{[\boldsymbol{i}]}$. The injective vertices associated with 1, 2, and 3 are $([1, 3], 0)$, $([2, 3], 0)$ and $([3], 0)$, respectively. The corresponding projective vertices are $([1, 2], 0)$, $([2], 0)$ and $([1, 3], 0)$.

If $\boldsymbol{i}$ is a reduced word for $w_0$ starting with $i \in \Delta_0$, notice that the infinite sequence $\widehat{r_i\boldsymbol{i}}$ is just a shift of $\widehat{\boldsymbol{i}}$. It is thus immediate from the definition that we have an isomorphism of quivers $\widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{[r_i\boldsymbol{i}]}$ that preserves residues. It sends the vertex $\widehat{\beta}^{\boldsymbol{i}}_l$ to the vertex $\widehat{\beta}^{r_i\boldsymbol{i}}_{l-1}$

or, equivalently, the vertex $(\alpha, k) \in \widehat{\mathsf{R}}$ to the vertex

$$\begin{cases} (s_i(\alpha), k) & \text{if } \alpha \neq \pm\alpha_i, \\ (s_i(\alpha), k+1) & \text{if } \alpha = \pm\alpha_i. \end{cases}$$

Consequently, if two commutation classes $[\boldsymbol{i}]$ and $[\boldsymbol{j}]$ are in the same $r$-cluster point, a sequence of reflection functors connecting them defines an isomorphism $\widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{[\boldsymbol{j}]}$ preserving residues.

**Lemma 1.2.4.** *For two commutation classes $[\boldsymbol{i}]$ and $[\boldsymbol{j}]$ in the same $r$-cluster point, let $\varphi : \widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{[\boldsymbol{j}]}$ be an isomorphism as above. Let $x_{[\boldsymbol{i}]}, x^*_{[\boldsymbol{i}]} \in (\widehat{\Upsilon}_{[\boldsymbol{i}]})_0$ be the injective and projective vertices associated with $i \in \Delta_0$. Then $\varphi(x_{[\boldsymbol{i}]})$ is injective if and only if $\varphi(x^*_{[\boldsymbol{i}]})$ is projective.*

*Proof.* We may assume $\Delta \neq \mathsf{A}_1$. Let $[\boldsymbol{i}]$ be an arbitrary commutation class of reduced words for $w_0$ and take $i \in \Delta_0$. Denote by $X_{[\boldsymbol{i}]} \subset \widehat{\mathsf{R}}$ the set of all pairs with residue $i$ with respect to $[\boldsymbol{i}]$. Let $x_{[\boldsymbol{i}]} \in X_{[\boldsymbol{i}]}$ be the injective vertex associated with $i$. By the segment property of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, there is a unique bijective map $d_{[\boldsymbol{i}]} : X_{[\boldsymbol{i}]} \longrightarrow \mathbb{Z}$ such that $d_{[\boldsymbol{i}]}(x_{[\boldsymbol{i}]}) = 0$ and $d_{[\boldsymbol{i}]}(v_1) < d_{[\boldsymbol{i}]}(v_2)$ if and only if there is a nontrivial path from $v_1$ to $v_2$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. If $j \in \Delta_0$ is a source of $[\boldsymbol{i}]$, let $\varphi_j : \widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{r_j[\boldsymbol{i}]}$ be the induced map. Since it preserves residues and is an isomorphism of quivers, we have $\varphi_j(X_{[\boldsymbol{i}]}) = X_{r_j[\boldsymbol{i}]}$ and the difference $d_{r_j[\boldsymbol{i}]} \circ \varphi_j - d_{[\boldsymbol{i}]}$ is always constant and equal to $m = d_{r_j[\boldsymbol{i}]}(\varphi_j(x^{[\boldsymbol{i}]}))$. A simple case-by-case analysis shows that $m = \delta_{ij}$, that is, Kronecker's delta.

Analogously, if $X^*_{[\boldsymbol{i}]} \subset \widehat{\mathsf{R}}$ is the set of all pairs with residue $i^*$ with respect to $[\boldsymbol{i}]$ and $x^*_{[\boldsymbol{i}]} \in X^*_{[\boldsymbol{i}]}$ denotes the projective vertex associated with $i$, there is a unique bijective map $d^*_{[\boldsymbol{i}]} : X^*_{[\boldsymbol{i}]} \longrightarrow \mathbb{Z}$ such that $d^*_{[\boldsymbol{i}]}(x^*_{[\boldsymbol{i}]}) = 0$ and $d^*_{[\boldsymbol{i}]}(v_1) < d^*_{[\boldsymbol{i}]}(v_2)$ if and only if there is a nontrivial path from $v_1$ to $v_2$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. If $j \in \Delta_0$ is a source of $[\boldsymbol{i}]$, one can show as before that $\varphi_j(X^*_{[\boldsymbol{i}]}) = X^*_{r_j[\boldsymbol{i}]}$ and the difference $d^*_{r_j[\boldsymbol{i}]} \circ \varphi_j - d^*_{[\boldsymbol{i}]}$ is a constant function equal to $\delta_{ij}$.

We can now prove the lemma. Applying the argument above multiple times, we can show that $d_{[\boldsymbol{j}]} \circ \varphi - d_{[\boldsymbol{i}]}$ and $d^*_{[\boldsymbol{j}]} \circ \varphi - d^*_{[\boldsymbol{i}]}$ are constant and equal to same value. But we have

$$\varphi(x_{[\boldsymbol{i}]}) \text{ is injective} \iff d_{[\boldsymbol{j}]}(\varphi(x_{[\boldsymbol{i}]})) = 0 \iff d_{[\boldsymbol{j}]} \circ \varphi - d_{[\boldsymbol{i}]} \equiv 0$$

and, similarly, $\varphi(x^*_{[\boldsymbol{i}]})$ is projective if and only if $d^*_{[\boldsymbol{j}]} \circ \varphi - d^*_{[\boldsymbol{i}]}$ is identically zero. This finishes the proof. $\checkmark$

### 1.2.4 Meshes of repetition quivers

Throughout this subsection, we assume that $\Delta \neq \mathsf{A}_1$. Let $\boldsymbol{i}$ be a reduced word for the longest element $w_0$ and write $\widehat{\boldsymbol{i}} = (i_l)_{l \in \mathbb{Z}}$. Given $k \in \mathbb{Z}$, let $k^+ \in \mathbb{Z}$ be the smallest integer such that $k < k^+$ and $i_k = i_{k^+}$. We define the *mesh of* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ *at the vertex* $\widehat{\beta}^{\boldsymbol{i}}_k$ to be the smallest convex subquiver $\mathcal{M} = \mathcal{M}_{[\boldsymbol{i}]}(\widehat{\beta}^{\boldsymbol{i}}_k)$ of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ containing the vertices $\widehat{\beta}^{\boldsymbol{i}}_k$ and $\widehat{\beta}^{\boldsymbol{i}}_{k^+}$. Equivalently, $\mathcal{M}$ is the full subquiver of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ whose set of vertices $\mathcal{M}_0$ consists of those vertices that appear in a path from $\widehat{\beta}^{\boldsymbol{i}}_{k^+}$ to $\widehat{\beta}^{\boldsymbol{i}}_k$. If $x = \widehat{\beta}^{\boldsymbol{i}}_k$, we will denote $\widehat{\beta}^{\boldsymbol{i}}_{k^+}$ by $s_{[\boldsymbol{i}]}(x)$, which is the unique source of $\mathcal{M}$. We define the *set of abutters* $V_{[\boldsymbol{i}]}(x)$ *of* $x$ to be the set of all vertices $y \in \mathcal{M}_0$ such that $\mathrm{res}^{[\boldsymbol{i}]}(y) \sim \mathrm{res}^{[\boldsymbol{i}]}(x)$.

**Lemma 1.2.5.** *Let $\boldsymbol{i}$ be a reduced word for $w_0$. In the notation above, we have $k^+ < k+N$ for any $k \in \mathbb{Z}$.*

*Proof.* Write $\widehat{\boldsymbol{i}} = (i_l)_{l\in\mathbb{Z}}$ and let $\boldsymbol{j} = (i_{k+1}, i_{k+2}, \dots, i_{k+N})$, which is a reduced word for $w_0$. If $k^+ \geq k+N$, then we have $i_l \neq i_k$ for all $k < l < k+N$. Thus, for $1 \leq l < N$, if we write $\beta^{\boldsymbol{j}}_l$ as a linear combination of positive simple roots, the coefficient of $\alpha_{i_k}$ is zero. This implies that there is at most one positive root involving $\alpha_{i_k}$, which contradicts the fact that $\Delta \neq \mathsf{A}_1$. $\surd$

**Lemma 1.2.6.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$ and take a mesh $\mathcal{M}$ of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. If $(x_1, \dots, x_t)$ is a compatible reading of $\mathcal{M}_0$, denote $i_k = \mathrm{res}^{[\boldsymbol{i}]}(x_k)$ for $1 \leq k \leq t$. If $t > 3$, then the sequence $(i_1, \dots, i_{t-1}, i_t, i_{t-1})$ is a reduced word for $s_{i_1} \cdots s_{i_{t-1}} s_{i_t} s_{i_{t-1}}$.*

*Proof.* By Lemma 1.2.5, we may suppose that $\mathcal{M} \subseteq \Upsilon_{[\boldsymbol{i}]}$ after reflecting at sources/sinks of $[\boldsymbol{i}]$. Since $\mathcal{M}$ is convex, Theorem 1.2.2 implies that $(i_1, \dots, i_t)$ is a reduced word for $w[\mathcal{M}_0] = s_{i_1} \cdots s_{i_t}$. Notice that, since $\mathcal{M}$ is a mesh, we have $i_1 = i_t$ (we will denote this common value by $i$), $i_l \neq i$ for $1 < l < t$, and $i_2, i_{t-1} \sim i$.

To prove the lemma, we need to show that $s_{i_1} \cdots s_{i_t}(\alpha_{i_{t-1}})$ is a positive root (see [Hum90, Lemma 1.6]). Since $i_{t-1} \sim i_t = i$, we have

$$s_{i_{t-1}} s_{i_t}(\alpha_{i_{t-1}}) = s_{i_{t-1}}(\alpha_{i_{t-1}} + \alpha_{i_t}) = \alpha_{i_t} = \alpha_i.$$

Since $i_l \neq i$ for $1 < l \leq t-2$, if we write $s_{i_2} s_{i_3} \cdots s_{i_{t-2}}(\alpha_i)$ as a linear combination of the positive simple roots, then the coefficient of $\alpha_i$ must be 1. Thus, $s_{i_2} s_{i_3} \cdots s_{i_{t-2}}(\alpha_i)$ is a positive root. We are finished if we prove that $s_{i_2} s_{i_3} \cdots s_{i_{t-2}}(\alpha_i) \neq \alpha_i$. Suppose by contradiction this is not the case. Then [Hum90, Theorem 1.7] implies

$$s_i s_{i_2} \cdots s_{i_{t-2}} = s_{i_2} \cdots s_{i_{t-2}} s_i.$$

By the previous paragraph, the word on the left is reduced, hence so is the one on the right. By Matsumoto's theorem [Mat64], we can obtain the second word from the first by applying commutation and braid relations. However, one can check that this is impossible because $s_i$ appears only at the beginning of the first word and $i_2 \sim i$ (observe that $s_{i_2}$ appears in the word as $t-2 \geq 2$ by hypothesis). $\surd$

| $\mathfrak{g}$ | $\mathsf{g}$ | $\sigma$ | $r$ | $h^\vee$ | $N$ |
|---|---|---|---|---|---|
| $\mathsf{A}_n$ | $\mathsf{A}_n$ | id | 1 | $n+1$ | $n(n+1)/2$ |
| $\mathsf{D}_n$ | $\mathsf{D}_n$ | id | 1 | $2n-2$ | $n(n-1)$ |
| $\mathsf{E}_{6,7,8}$ | $\mathsf{E}_{6,7,8}$ | id | 1 | $12, 18, 30$ | $36, 63, 120$ |
| $\mathsf{B}_n$ | $\mathsf{A}_{2n-1}$ | $\vee$ | 2 | $2n-1$ | $n(2n-1)$ |
| $\mathsf{C}_n$ | $\mathsf{D}_{n+1}$ | $\vee$ | 2 | $n+1$ | $n(n+1)$ |
| $\mathsf{F}_4$ | $\mathsf{E}_6$ | $\vee$ | 2 | 9 | 36 |
| $\mathsf{G}_2$ | $\mathsf{D}_4$ | $\widetilde{\vee}, \widetilde{\vee}^2$ | 3 | 4 | 12 |

Table 1.1: The Dynkin diagram of $\mathfrak{g}$ is obtained by folding that of $\mathsf{g}$ along the automorphism $\sigma$. The last three columns give the order of $\sigma$, the dual Coxeter number of $\mathfrak{g}$, and the number of positive roots of $\mathsf{g}$.

### 1.2.5 Folding of Dynkin diagrams

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra that is not necessarily simply laced. It is well known that its Dynkin diagram can be obtained by "folding" the Dynkin diagram $\Delta$ of a simple Lie algebra $\mathsf{g}$ of simply laced type along an automorphism $\sigma$. In Table 1.1 (taken from [FO21]), we display which simply laced type and automorphism are needed to obtain each Lie algebra $\mathfrak{g}$. See also Figure 1.1 for the definition of the automorphisms $\vee$ and $\widetilde{\vee}$.

**Remark 1.2.7.** The automorphism $\sigma$ induces an automorphism of the Lie algebra $\mathsf{g}$ and gives rise to the invariant subalgebra $\mathsf{g}^\sigma$, which is again simple. We warn the reader that $\mathsf{g}^\sigma$ is not always isomorphic to $\mathfrak{g}$, but is instead Langlands dual to it. Hence, our folding procedure is dual to the classical folding of Lie algebras. For example, the invariant subalgebra of the special linear Lie algebra $\mathfrak{sl}_{2n}(\mathbb{C})$ of type $\mathsf{A}_{2n-1}$ under the automorphism induced by $\vee$ is the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ of type $\mathsf{C}_n$ (and not $\mathsf{B}_n$).

**Remark 1.2.8.** Whenever we deal with $\mathsf{g}$ and $\mathfrak{g}$ at the same time, we will adopt the convention of [FO21] to notate the indices of their Dynkin diagrams. More specifically, we will mainly use the symbols $\imath$ and $\jmath$ (and their variations) for the vertices of $\Delta$ and save the more common notation $i$ and $j$ for the vertices of the Dynkin diagram of $\mathfrak{g}$. If $\mathfrak{g}$ does not appear at all (as in most of Sections 1.3–1.7), we shall use the usual notation.

From now on, $\sigma$ will always denote the automorphism of $\Delta$ that corresponds to $\mathfrak{g}$. We set $r \in \{1, 2, 3\}$ to be its order. Let $I$ be the set of $\sigma$-orbits of $\Delta_0$, which is the set of vertices of the Dynkin diagram of $\mathfrak{g}$. The quotient map $\Delta_0 \longrightarrow I$ is denoted by $\imath \mapsto \bar{\imath}$. Define $n = |I|$. For each orbit $i \in I$, denote its cardinality by $d_i$, which is either 1 or $r$.

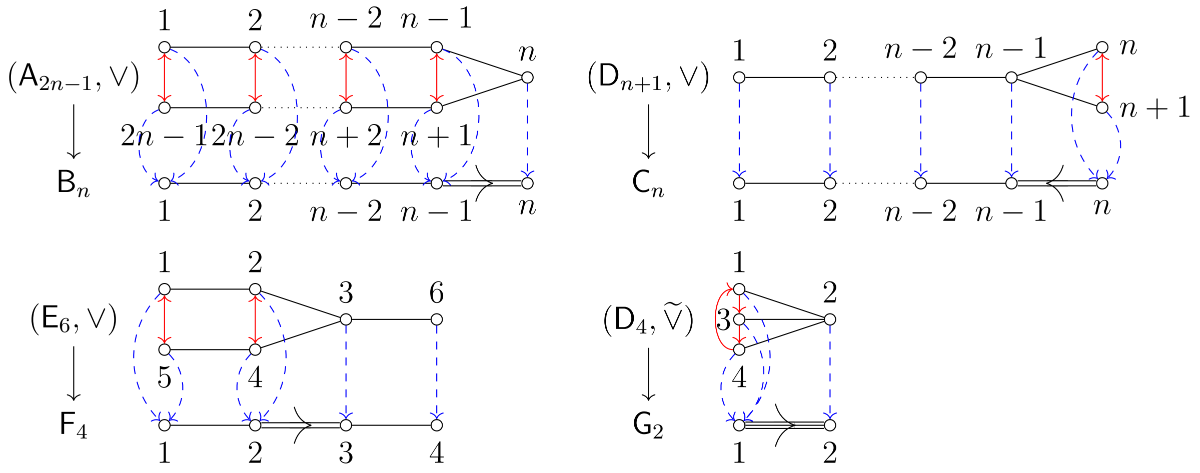



Figure 1.1: The action of the automorphisms $\vee$ and $\widetilde{\vee}$ on the nonfixed vertices is shown in red. After identifying the orbits via the blue arrows, we get the nonsimply laced Dynkin diagrams.

We will write $i \sim j$ for $i, j \in I$ if there are $\imath \in i$ and $\jmath \in j$ such that $\imath \sim \jmath$. We denote by $h^\vee$ the dual Coxeter number of $\mathfrak{g}$, which satisfies the relation $nrh^\vee = 2N = |\mathsf{R}|$.

The automorphism $\sigma$ can be used to turn $\Delta$ into a weighted graph and determine a distance function $d^\sigma_\Delta$ on the set $\Delta_0$ in the following way. We first set the weight of an edge between vertices $\imath, \jmath \in \Delta_0$ to be $\min(d_{\bar\imath}, d_{\bar\jmath})$. Then, for general $\imath, \jmath \in \Delta_0$, $d^\sigma_\Delta(\imath, \jmath)$ is the sum of the weights of the edges in the unique simple path between $\imath$ and $\jmath$.

In addition, $\sigma$ naturally defines an automorphism of the weight lattice $\mathsf{P}$ if we set $\sigma(\varpi_\imath) = \varpi_{\sigma(\imath)}$ for $\imath \in \Delta_0$. We have $\sigma(\alpha_\imath) = \alpha_{\sigma(\imath)}$ and $\sigma s_\imath \sigma^{-1} = s_{\sigma(\imath)}$ for $\imath \in \Delta_0$. In particular, we can consider the coset $\mathsf{W}\sigma$ of $\mathsf{W}$ in $\mathrm{Aut}(\mathsf{P})$.

### 1.2.6 Q-data and twisted AR quivers

We recall the definition of a Q-datum, introduced in [FO21].

**Definition 1.2.9.** A *Q-datum* for $\mathfrak{g}$ is a triple $\mathcal{Q} = (\Delta, \sigma, \xi)$ where $(\Delta, \sigma)$ is the pair corresponding to $\mathfrak{g}$ as in Table 1.1 and $\xi : \Delta_0 \longrightarrow \mathbb{Z}$ is a *height function on* $(\Delta, \sigma)$, that is, a function satisfying the following two conditions (where we write $\xi_\imath = \xi(\imath)$):

(1) For any $\imath, \jmath \in \Delta_0$ such that $\imath \sim \jmath$ and $d_{\bar\imath} = d_{\bar\jmath}$, we have $|\xi_\imath - \xi_\jmath| = d_{\bar\imath} = d_{\bar\jmath}$.

(2) For any $i, j \in I$ with $i \sim j$, $d_i = 1$ and $d_j = r$, there is a unique $\jmath \in j$ such that $|\xi_\imath - \xi_\jmath| = 1$ and $\xi_{\sigma^k(\jmath)} = \xi_\jmath - 2k$ for all $1 \le k < r$, where $\imath$ is the unique element of $i$.

As suggested by [FO21], one should think that these are local conditions at a "non-branching point" and at a "branching point," respectively.

**Remark 1.2.10.** In the simply laced case, when $\sigma = \mathrm{id}$, a height function is just a function $\xi : \Delta_0 \longrightarrow \mathbb{Z}$ satisfying $|\xi_\imath - \xi_\jmath| = 1$ for $\imath, \jmath \in \Delta_0$ such that $\imath \sim \jmath$. Up to the addition of a constant, this is the same as choosing an orientation for the graph $\Delta$: for $\imath \sim \jmath$, we say that there is an arrow from $\imath$ to $\jmath$ if $\xi_\imath > \xi_\jmath$. Hence, in this case, a Q-datum is essentially the same as a quiver whose underlying graph is $\Delta$.

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. A vertex $\imath \in \Delta_0$ is a *source of* $\mathcal{Q}$ if we have $\xi_\imath > \xi_\jmath$ for all $\jmath \in \Delta_0$ such that $\imath \sim \jmath$. In this case, the function $s_\imath \xi : \Delta_0 \longrightarrow \mathbb{Z}$ given by $(s_\imath\xi)_\jmath = \xi_\jmath$ for $\jmath \neq \imath$ and $(s_\imath\xi)_\imath = \xi_\imath - 2d_{\bar\imath}$ is again a height function, and we can define the *reflected* Q-datum $s_\imath\mathcal{Q} = (\Delta, \sigma, s_\imath\xi)$. A sequence of vertices $(\imath_1, \ldots, \imath_l)$ of $\Delta$ is a *source sequence for* $\mathcal{Q}$ if $\imath_k$ is a source of $s_{\imath_{k-1}} s_{\imath_{k-2}} \cdots s_{\imath_1}\mathcal{Q}$ for all $1 \leq k \leq l$.

We define a *sink of* $\mathcal{Q}$ to be a vertex $\imath \in \Delta_0$ satisfying $\xi_\imath + 2d_{\bar\imath} < \xi_\jmath + 2d_{\bar\jmath}$ for all $\jmath \in \Delta_0$ such that $\imath \sim \jmath$. In this case, the function $s_\imath^{-1}\xi : \Delta_0 \longrightarrow \mathbb{Z}$ given by $(s_\imath^{-1}\xi)_\jmath = \xi_\jmath$ for $\jmath \neq \imath$ and $(s_\imath^{-1}\xi)_\imath = \xi_\imath + 2d_{\bar\imath}$ is again a height function, and we can also define the *reflected* Q-datum $s_\imath^{-1}\mathcal{Q} = (\Delta, \sigma, s_\imath^{-1}\xi)$. The definition of a *sink sequence* is analogous to that of a source sequence.

**Remark 1.2.11.** Observe that, if $\imath \in \Delta_0$ is a source of $\mathcal{Q}$, then $\imath$ is a sink of $s_\imath\mathcal{Q}$ and we have $s_\imath^{-1}s_\imath\mathcal{Q} = \mathcal{Q}$. Similarly, if $\imath \in \Delta_0$ is a sink of $\mathcal{Q}$, then $\imath$ is a source of $s_\imath^{-1}\mathcal{Q}$ and we have $s_\imath s_\imath^{-1}\mathcal{Q} = \mathcal{Q}$.

**Example 1.2.12.** Let $\mathfrak{g}$ be of type $\mathsf{B}_3$, so that $(\Delta, \sigma) = (\mathsf{A}_5, \vee)$, and consider the following height functions:

$$
\begin{array}{rl}
\mathcal{Q}^1 = & \begin{array}{ccccccccc} 6 & & 4 & & 5 & & 6 & & 8 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array} \\
\mathcal{Q}^2 = & \begin{array}{ccccccccc} 2 & & 4 & & 5 & & 6 & & 8 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array} \\
\mathcal{Q}^3 = & \begin{array}{ccccccccc} 2 & & 4 & & 7 & & 6 & & 8 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \end{array}
\end{array}
$$

Here we put the value of the height functions above the vertices of $\mathsf{A}_5$. They give examples of Q-data $\mathcal{Q}^1$, $\mathcal{Q}^2$, and $\mathcal{Q}^3$. Notice, for example, that the first vertex is a source of $\mathcal{Q}^1$ and that $s_1\mathcal{Q}^1 = \mathcal{Q}^2$. Similarly, the middle vertex is a source of $\mathcal{Q}^3$ and $s_3\mathcal{Q}^3 = \mathcal{Q}^2$. In particular, the middle vertex is a sink of $\mathcal{Q}^2$, which may not be clear at first glance.

Let $\xi$ be a height function on $(\Delta, \sigma)$. The *(twisted) repetition quiver* associated with $(\Delta, \sigma)$ is the quiver $\widehat{\Delta}^\sigma$ whose set of vertices $\widehat{\Delta}^\sigma_0$ and set of arrows $\widehat{\Delta}^\sigma_1$ are defined as follows:

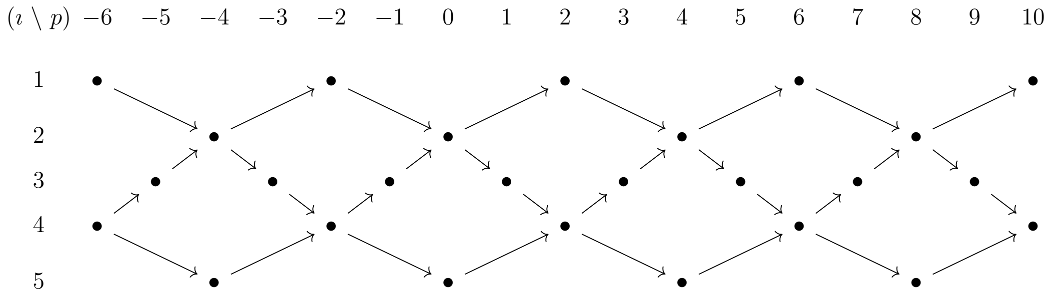
$$
\begin{aligned}
\widehat{\Delta}^\sigma_0 &= \{(\imath, p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_\imath \in 2d_{\bar\imath}\mathbb{Z}\}, \\
\widehat{\Delta}^\sigma_1 &= \{(\imath, p) \longrightarrow (\jmath, s) \mid (\imath, p), (\jmath, s) \in \widehat{\Delta}^\sigma_0, \jmath \sim \imath, s - p = \min(d_{\bar\imath}, d_{\bar\jmath})\}.
\end{aligned}
$$

We observe that if there is a path from $(\imath, p)$ to $(\jmath, s)$ in $\widehat{\Delta}^\sigma$, then $s - p \geq d^\sigma_\Delta(\imath, \jmath)$. We say that $\imath$ is the *residue* of the vertex $(\imath, p)$, while $p$ is its *height*. We denote by $\pi : \widehat{\Delta}^\sigma_0 \longrightarrow \Delta_0$ the projection onto the first coordinate.

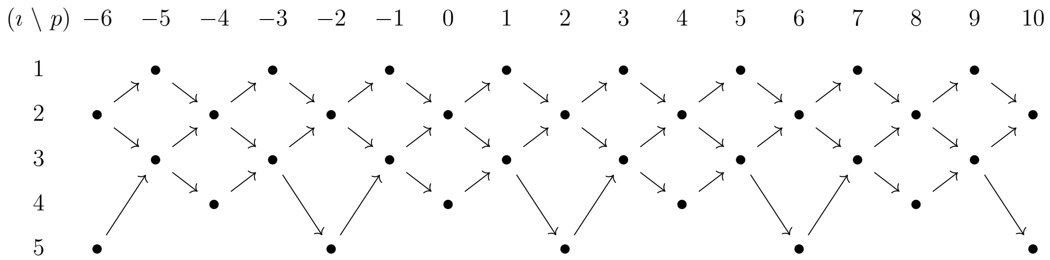
**Remark 1.2.13.** As explained in [FO21], the repetition quiver depends only on the $\sigma$-*parity function* congruent to $\xi$. Therefore, if we vary $\xi$, the only change incurred to $\widehat{\Delta}^\sigma$ is possibly a uniform shift of the height of its vertices by some integer. For this reason, we will omit the height function when referring to $\widehat{\Delta}^\sigma$. Whenever we work with a Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$, it will be implicitly assumed that $\mathcal{Q}$ is such that $\xi$ defines $\widehat{\Delta}^\sigma$. If we want to highlight this fact, we will say that $\mathcal{Q}$ has the same *parity* as $\widehat{\Delta}^\sigma$.

**Example 1.2.14.** Here we have some examples of the repetition quiver $\widehat{\Delta}^\sigma$.

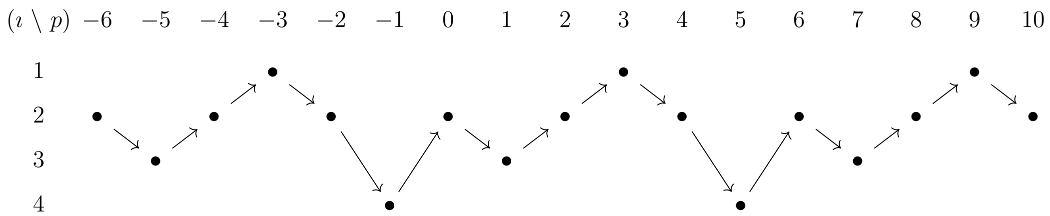
(1) Type $\mathsf{B}_3$, where $(\Delta, \sigma) = (\mathsf{A}_5, \vee)$:




(2) Type $\mathsf{C}_4$, where $(\Delta, \sigma) = (\mathsf{D}_5, \vee)$:




(3) Type $\mathsf{G}_2$, where $(\Delta, \sigma) = (\mathsf{D}_4, \widetilde{\vee})$:




The *twisted AR quiver* $\Gamma_{\mathcal{Q}}$ of a Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$ is the full subquiver of $\widehat{\Delta}^\sigma$ whose vertices are given by the set

$$(\Gamma_{\mathcal{Q}})_0 = \{(\imath, p) \in \widehat{\Delta}^\sigma_0 \mid \xi_{\imath^*} - rh^\vee < p \leq \xi_\imath\}.$$

One can show that it is a convex subquiver of $\widehat{\Delta}^\sigma$.

Let $\imath \in \Delta_0$. The *injective vertex associated with* $\imath$ is the vertex $(\imath, p) \in (\Gamma_{\mathcal{Q}})_0$ with the largest height. We have $p = \xi_\imath$, in this case. Similarly, the *projective vertex associated with* $\imath$ is the vertex $(\imath^*, p) \in (\Gamma_{\mathcal{Q}})_0$ with the smallest height. Its height is given by the lemma below.

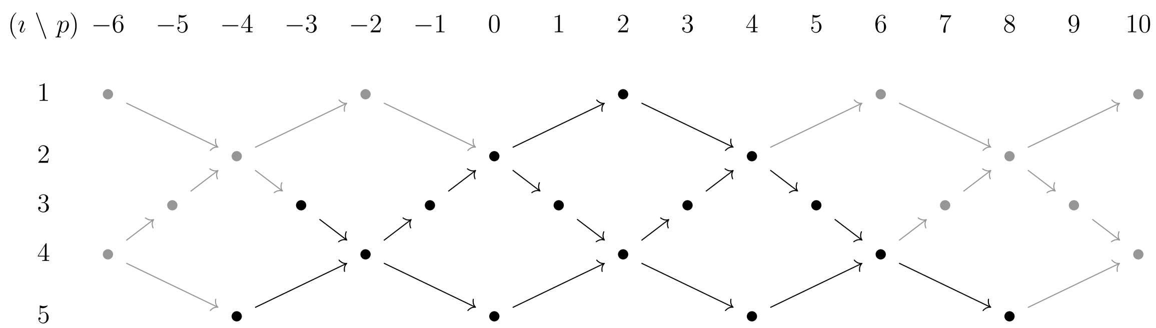
**Lemma 1.2.15.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $\imath \in \Delta_0$. The projective vertex associated with $\imath$ in $\Gamma_{\mathcal{Q}}$ has height given by $\xi_\imath - rh^\vee + 2d_{\bar{\imath}}$.*

*Proof.* We need to show that $\xi_\imath - rh^\vee$ is congruent to $\xi_{\imath^*}$ modulo $2d_{\bar{\imath}}$ in order to prove that $(\imath^*, \xi_\imath - rh^\vee + 2d_{\bar{\imath}})$ is a vertex of $\Gamma_{\mathcal{Q}}$. We will consider two different cases.

First, assume that $h^\vee$ is even. In this case, since $d_{\bar{\imath}}$ divides $r$, we are reduced to the verification that $\xi_\imath$ is congruent to $\xi_{\imath^*}$ modulo $2d_{\bar{\imath}}$. If $r = 1$, then $d_{\bar{\imath}} = 1$ and the congruence can be easily checked by inspecting all types. If $r > 1$, then $\imath^* = \imath$ by [FO21, Remark 3.1] and the congruence is trivial.

Now, assume that $h^\vee$ is odd. We have again two cases: $r = 1$ or $r = 2$. In the first case, we must have $\Delta = \mathsf{A}_n$ with $n$ even. It is not hard to see that we then have $\xi_\imath - \xi_{\imath^*}$ odd, so that $(\xi_\imath - rh^\vee) - \xi_{\imath^*}$ is even, as desired. In the second case, the permutation $\jmath \mapsto \jmath^*$ coincides with $\sigma$ (again by [FO21, Remark 3.1]). Hence, by the statement (2) in [FO21, Lemma 3.9], we have $\xi_\imath - \xi_{\imath^*}$ congruent to 2 modulo 4, and so $(\xi_\imath - rh^\vee) - \xi_{\imath^*}$ is divisible by $4 = 2r$, finishing the proof. $\checkmark$

**Example 1.2.16.** Take $(\Delta, \sigma) = (\mathsf{A}_5, \vee)$, which is of type $\mathsf{B}_3$, and consider $\mathcal{Q}$ to be the second Q-datum from Example 1.2.12. Its twisted AR quiver is the following:




The gray vertices are the vertices of $\widehat{\Delta}^\sigma$ outside of $\Gamma_{\mathcal{Q}}$. The injective vertices are $(1,2)$, $(2,4)$, $(3,5)$, $(4,6)$, and $(5,8)$, while the projective vertices are $(5,-4)$, $(4,-2)$, $(3,-3)$, $(2,0)$, and $(1,2)$. Notice that $\imath \in \Delta_0$ is a source of $\mathcal{Q}$ if and only if the corresponding injective vertex is a sink of $\Gamma_{\mathcal{Q}}$. Hence, for example, 5 is the unique source of $\mathcal{Q}$ because the corresponding injective vertex $(5,8)$ is the unique sink of $\Gamma_{\mathcal{Q}}$. Similarly, $\imath \in \Delta_0$ is a sink of $\mathcal{Q}$ if and only if the corresponding projective vertex is a source of $\Gamma_{\mathcal{Q}}$ (by Lemma 1.2.15). Therefore, 1 and 3 are the sinks of $\mathcal{Q}$, which correspond to the projective vertices $(5,-4)$ and $(3,-3)$, respectively.

For a Q-datum $\mathcal{Q}$, we define $[\mathcal{Q}]$ to be the set of all reduced words for the longest element $w_0 \in \mathsf{W}$ that form a source sequence for $\mathcal{Q}$. The following result shows that we can see the associated twisted AR quiver as a combinatorial AR quiver by means of $[\mathcal{Q}]$.

**Theorem 1.2.17** ([OS19b], [FO21, Theorem 3.24])**.** *For a Q-datum $\mathcal{Q}$, the set $[\mathcal{Q}]$ is a commutation class of reduced words for the longest element $w_0 \in \mathsf{W}$. Moreover, there is a unique isomorphism of quivers $\Gamma_{\mathcal{Q}} \longrightarrow \Upsilon_{[\mathcal{Q}]}$ that intertwines $\pi$ and $\mathrm{res}^{[\mathcal{Q}]}$.*

**Remark 1.2.18.** If $\imath \in \Delta_0$, then $\imath$ is a source of a Q-datum $\mathcal{Q}$ if and only if it is a source of the commutation class $[\mathcal{Q}]$. In this case, one can check that $[s_\imath \mathcal{Q}] = r_\imath[\mathcal{Q}]$. A similar statement holds for sinks. We also remark that the isomorphism of quivers above preserves injective/projective vertices.

**Example 1.2.19.** Let $\mathcal{Q}$ be the second Q-datum of type $\mathsf{B}_3$ from Example 1.2.12. The quiver $\Gamma_{\mathcal{Q}}$ is displayed in Example 1.2.16. Using a convenient compatible reading of $\Gamma_{\mathcal{Q}} \cong \Upsilon_{[\mathcal{Q}]}$, we see that

$$\boldsymbol{i} = (5, 4, 3, 2, 5, 3, 1, 4, 3, 2, 5, 3, 4, 3, 5)$$

is a reduced word for the longest element $w_0$ and forms a source sequence for $\mathcal{Q}$. It allows us to associate a positive root in $\mathsf{R}^+$ (of type $\mathsf{A}_5$ and not $\mathsf{B}_3$) to every vertex of $\Gamma_{\mathcal{Q}}$. The result is the following:

$$\begin{tikzcd}
(\imath \setminus p) & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & & & & & & {[1,5]} \arrow[drr] & & & & & & \\
2 & & & & & {[1,4]} \arrow[urr] \arrow[dr] & & & & {[2,5]} \arrow[dr] & & & & \\
3 & & {[3]} \arrow[dr] & & {[1,2]} \arrow[ur] & & {[3,4]} \arrow[dr] & & {[2]} \arrow[ur] & & {[3,5]} \arrow[dr] & & & \\
4 & & & {[1,3]} \arrow[ur] \arrow[drr] & & & & {[2,4]} \arrow[ur] \arrow[drr] & & & & {[4,5]} \arrow[drr] & & \\
5 & {[1]} \arrow[urr] & & & & {[2,3]} \arrow[urr] & & & & {[4]} \arrow[urr] & & & & {[5]}
\end{tikzcd}$$

Here $[\imath, \jmath]$ denotes the sum of simple roots $\alpha_\imath + \alpha_{\imath+1} \cdots + \alpha_\jmath$ for $1 \leq \imath \leq \jmath \leq 5$. To find the root associated with a vertex $x \in (\Gamma_{\mathcal{Q}})_0$, one should read $\boldsymbol{i}$ from the beginning until the position that corresponds to $x$ (according to the compatible reading chosen before). For example, the vertex $(3, 3)$ corresponds to the sixth residue in $\boldsymbol{i}$; thus, the picture implies that we should have

$$\alpha_2 = s_5 s_4 s_3 s_2 s_5(\alpha_3),$$

which the reader can easily check to be true.

### 1.2.7 Generalized twisted Coxeter elements

We recall the construction of the generalized twisted Coxeter elements, introduced in [FO21, Section 3.6]. Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. For $i \in I$, let $i^\circ \in i$ be the vertex satisfying

$$\xi_{i^\circ} = \max\{\xi_\imath \mid \imath \in i\}$$

(which is unique by [FO21, Lemma 3.9]). Set $I^\circ_{\mathcal{Q}} = \{i^\circ \in \Delta_0 \mid i \in I\}$. Define the set

$$X^\circ_{\mathcal{Q}} = \{(\imath, \xi_\imath) \in \widehat{\Delta}^\sigma_0 \mid \imath \in I^\circ_{\mathcal{Q}}\}.$$

We define $\tau^\circ_{\mathcal{Q}} = w[X^\circ_{\mathcal{Q}}]\sigma \in \mathsf{W}\sigma$. From $\xi$, we can define $\xi^\circ : \Delta_0 \longrightarrow \mathbb{Z}$ by $\xi^\circ_{\sigma^k(\imath)} = \xi_\imath - 2k$ for $\imath \in I^\circ_{\mathcal{Q}}$ and $0 \le k < d_{\bar{\imath}}$. This is a height function on $(\Delta, \sigma)$, and so, we have a Q-datum $\mathcal{Q}^\circ = (\Delta, \sigma, \xi^\circ)$. Define

$$X'_{\mathcal{Q}} = \{(\sigma(i^\circ), p) \in \widehat{\Delta}^\sigma_0 \mid i \in I, \xi_{\sigma(i^\circ)} < p \le \xi_{i^\circ} - 2\},$$

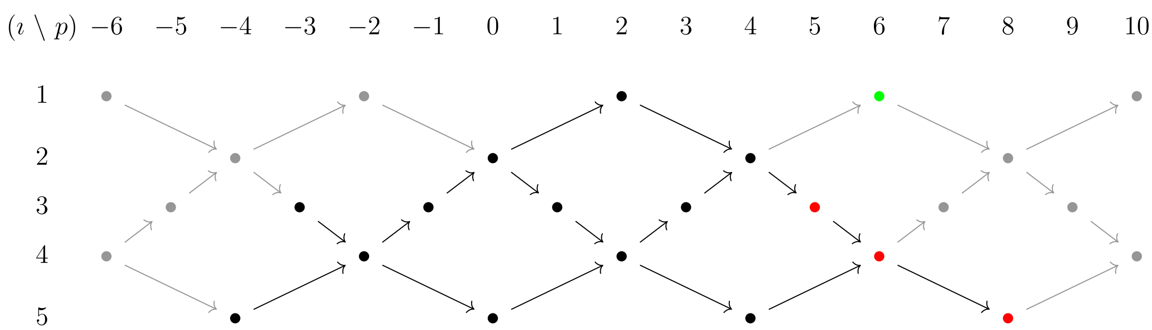
which allows us to construct $w[X'_{\mathcal{Q}}]$.

**Definition 1.2.20.** With the notation above, the *generalized twisted Coxeter element* (or the *generalized $\sigma$-Coxeter element*) $\tau_{\mathcal{Q}} \in \mathsf{W}\sigma$ associated with $\mathcal{Q}$ is defined by

$$\tau_{\mathcal{Q}} = w[X'_{\mathcal{Q}}]^{-1} \cdot \tau^\circ_{\mathcal{Q}} \cdot w[X'_{\mathcal{Q}}].$$

Note that $\tau_{\mathcal{Q}} = \tau^\circ_{\mathcal{Q}}$ when $\mathcal{Q} = \mathcal{Q}^\circ$.

**Example 1.2.21.** Let $\mathcal{Q}$ be the Q-datum of type $\mathsf{B}_3$ from Example 1.2.16. The sets $X^\circ_{\mathcal{Q}}$ and $X'_{\mathcal{Q}}$ defined above are displayed in the colors red and green, respectively, in the picture below.




Therefore, we have $\tau^\circ_{\mathcal{Q}} = s_5 s_4 s_3 \sigma$ and $w[X'_{\mathcal{Q}}] = s_1$, which yields $\tau_{\mathcal{Q}} = s_1 s_5 s_4 s_3 \sigma s_1$.

**Proposition 1.2.22** ([FO21, Proposition 3.34])**.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum.*

*(1) If $\imath \in \Delta_0$ is a source of $\mathcal{Q}$, then $s_\imath^{-1} \tau_{\mathcal{Q}} s_\imath = \tau_{s_\imath \mathcal{Q}}$.*

*(2) The order of* $\tau_{\mathcal{Q}}$ *is* $rh^\vee$.

*(3) If* $\sigma \neq \mathrm{id}$, *then* $\tau_{\mathcal{Q}}^{rh^\vee/2} = -1$.

### 1.2.8 The bijection $\phi_{\mathcal{Q}}$ and the $\mathfrak{g}$-additive property

Following [FO21, Section 3.7], we define a map $\phi_{\mathcal{Q}} : \widehat{\Delta}_0^\sigma \longrightarrow \widehat{\mathsf{R}}$ for a Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$. First, for $\imath \in \Delta_0$, we set $\phi_{\mathcal{Q}}(\imath, \xi_\imath) = (\gamma_\imath^{\mathcal{Q}}, 0)$, where

$$\gamma_\imath^{\mathcal{Q}} = (1 - \tau_{\mathcal{Q}}^{d_{\bar{\imath}}})\varpi_\imath \in \mathsf{R}^+.$$

Now, if $\phi_{\mathcal{Q}}(\imath, p) = (\alpha, k)$, then

$$\phi_{\mathcal{Q}}(\imath, p \pm 2d_{\bar{\imath}}) = \begin{cases} (\tau_{\mathcal{Q}}^{\mp d_{\bar{\imath}}}(\alpha), k) & \text{if } (-1)^k \tau_{\mathcal{Q}}^{\mp d_{\bar{\imath}}}(\alpha) \in \mathsf{R}^+, \\ (\tau_{\mathcal{Q}}^{\mp d_{\bar{\imath}}}(\alpha), k \pm 1) & \text{if } (-1)^{k \pm 1} \tau_{\mathcal{Q}}^{\mp d_{\bar{\imath}}}(\alpha) \in \mathsf{R}^+. \end{cases}$$

This recursively defines $\phi_{\mathcal{Q}}$. Composing $\phi_{\mathcal{Q}}$ with the projection onto the first coordinate, we get the function $\psi_{\mathcal{Q}} : \widehat{\Delta}_0^\sigma \longrightarrow \mathsf{R}$ given by

$$\psi_{\mathcal{Q}}(\imath, p) = \tau_{\mathcal{Q}}^{(\xi_\imath - p)/2}(\gamma_\imath^{\mathcal{Q}})$$

for $(\imath, p) \in \widehat{\Delta}^\sigma$.

**Theorem 1.2.23.** *Let* $\mathcal{Q} = (\Delta, \sigma, \xi)$ *be a Q-datum. The map* $\phi_{\mathcal{Q}} : \widehat{\Delta}_0^\sigma \longrightarrow \widehat{\mathsf{R}}$ *is a bijection. Moreover, we can upgrade it to an isomorphism of quivers* $\widehat{\Delta}^\sigma \longrightarrow \widehat{\Upsilon}_{[\mathcal{Q}]}$ *preserving residues, which restricts to the isomorphism* $\Gamma_{\mathcal{Q}} \longrightarrow \Upsilon_{[\mathcal{Q}]}$ *in Theorem 1.2.17.*

*Proof.* The first statement follows from [FO21, Theorem 3.35]. This theorem also states that, if we identify $\Upsilon_{[\mathcal{Q}]}$ as a subquiver of $\widehat{\Upsilon}_{[\mathcal{Q}]}$, then the restriction of $\phi_{\mathcal{Q}}$ to $(\Gamma_{\mathcal{Q}})_0$ is the underlying bijection of the isomorphism $\Gamma_{\mathcal{Q}} \longrightarrow \Upsilon_{[\mathcal{Q}]}$ in Theorem 1.2.17. The proof in [FO21] shows that, if $\imath \in \Delta_0$ is a source of $\mathcal{Q}$, then the following triangle commutes:

$$\begin{tikzcd} \widehat{\Delta}_0^\sigma \arrow[r, "\phi_{\mathcal{Q}}"] \arrow[dr, "\phi_{s_\imath \mathcal{Q}}"'] & (\widehat{\Upsilon}_{[\mathcal{Q}]})_0 \arrow[d] \\ & (\widehat{\Upsilon}_{[s_\imath \mathcal{Q}]})_0 \end{tikzcd}$$

where the vertical map is the underlying map of the isomorphism $\widehat{\Upsilon}_{[\mathcal{Q}]} \longrightarrow \widehat{\Upsilon}_{[s_\imath \mathcal{Q}]}$ from the end of Section 1.2.3. Now, notice that a Q-datum $\mathcal{Q}'$ can be obtained from $\mathcal{Q}$ by a sequence of reflections precisely if it has the same type and parity as $\mathcal{Q}$. Moreover, if we see $\Upsilon_{[\mathcal{Q}']}$ as a subquiver of $\widehat{\Upsilon}_{[\mathcal{Q}]}$ via the isomorphism $\widehat{\Upsilon}_{[\mathcal{Q}']} \longrightarrow \widehat{\Upsilon}_{[\mathcal{Q}]}$ coming from a sequence of reflections from $\mathcal{Q}'$ to $\mathcal{Q}$, then $\widehat{\Upsilon}_{[\mathcal{Q}]}$ is the union of the subquivers $\Upsilon_{[\mathcal{Q}']}$ as we vary over all such $\mathcal{Q}'$. Similarly, $\widehat{\Delta}^\sigma$ is the union of the subquivers $\Gamma_{\mathcal{Q}'}$ as we vary $\mathcal{Q}'$. Therefore, we conclude the proof by repeatedly applying the commutative triangle above. $\surd$

**Lemma 1.2.24.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $\imath \in \Delta_0$. If $x_\imath \in (\Gamma_{\mathcal{Q}})_0$ is the projective vertex associated with $\imath$, then*

$$\psi_{\mathcal{Q}}(x_\imath) = (1 - \tau_{\mathcal{Q}}^{-d_{\bar{\imath}}})\varpi_\imath.$$

*Proof.* If $\sigma = \mathrm{id}$, we can view $\mathcal{Q}$ as a quiver whose underlying graph is $\Delta$. Note that $\tau_{\mathcal{Q}} = s_{\imath_1} \cdots s_{\imath_n}$ where $(\imath_1, \ldots, \imath_n)$ is a source sequence of $\mathcal{Q}$ where each vertex appears exactly once. One can then compute that $(1 - \tau_{\mathcal{Q}}^{-1})\varpi_\imath$ is the sum of all simple roots $\alpha_\jmath$ such that there is an oriented path from $\imath$ to $\jmath$ in $\mathcal{Q}$. In particular, $(1 - \tau_{\mathcal{Q}}^{-1})\varpi_\imath$ gives the dimension vector of the indecomposable projective representation of $\mathcal{Q}$ corresponding to the vertex $\imath$. The result then follows from [FO21, Theorem 2.5] and the discussion in [FO21, Section 2.3].

If $\sigma \neq \mathrm{id}$, notice that $rh^\vee$ and $\xi_{\imath^*} - \xi_\imath$ are even. By Lemma 1.2.15, we have $x_\imath = (\imath^*, \xi_\imath - rh^\vee + 2d_{\bar{\imath}})$. Hence,

$$\psi_{\mathcal{Q}}(x_\imath) = \tau_{\mathcal{Q}}^{(\xi_{\imath^*} - (\xi_\imath - rh^\vee + 2d_{\bar{\imath}}))/2}(\gamma_{\imath^*}^{\mathcal{Q}}) = \tau_{\mathcal{Q}}^{rh^\vee/2 - d_{\bar{\imath}}}(\tau_{\mathcal{Q}}^{(\xi_{\imath^*} - \xi_\imath)/2}(\gamma_{\imath^*}^{\mathcal{Q}})) = \tau_{\mathcal{Q}}^{rh^\vee/2 - d_{\bar{\imath}}}(\gamma_\imath^{\mathcal{Q}}),$$

where the last equality follows from [FO21, Lemma 3.38]. We conclude using the item (3) of Proposition 1.2.22:

$$\tau_{\mathcal{Q}}^{rh^\vee/2 - d_{\bar{\imath}}}(\gamma_\imath^{\mathcal{Q}}) = -\tau_{\mathcal{Q}}^{-d_{\bar{\imath}}}(\gamma_\imath^{\mathcal{Q}}) = -\tau_{\mathcal{Q}}^{-d_{\bar{\imath}}}(1 - \tau_{\mathcal{Q}}^{d_{\bar{\imath}}})\varpi_\imath = (1 - \tau_{\mathcal{Q}}^{-d_{\bar{\imath}}})\varpi_\imath,$$

as desired. $\surd$

To finish this subsection, we state the $\mathfrak{g}$-*additive property* of [FO21, Theorem 3.41].

**Theorem 1.2.25.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. For $(\imath, p) \in \widehat{\Delta}_0^\sigma$, we have*

$$\psi_{\mathcal{Q}}(\imath, p) + \psi_{\mathcal{Q}}(\imath, p - 2d_{\bar{\imath}}) = \sum_{(\jmath, s) \in V(\imath, p)} \psi_{\mathcal{Q}}(\jmath, s)$$

*where $V(\imath, p) \subset \widehat{\Delta}_0^\sigma$ is the set of vertices $(\jmath, s)$ such that $\jmath \sim \imath$ and $p - 2d_{\bar{\imath}} < s < p$.*

*Proof.* This result is essentially Proposition 8.32 of [OS19b] rewritten with our notation. One can also deduce the formula from [FO21, Theorem 3.41], where they give a different (but equivalent) expression for the sum above. $\surd$

**Remark 1.2.26.** If we identify $\widehat{\Delta}^\sigma$ with $\widehat{\Upsilon}_{[\mathcal{Q}]}$ via Theorem 1.2.23, then we have $(\imath, p - 2d_{\bar{\imath}}) = s_{[\mathcal{Q}]}(\imath, p)$ and the set $V(\imath, p)$ above is the set of abutters of $(\imath, p)$ in $\widehat{\Upsilon}_{[\mathcal{Q}]}$ (see Section 1.2.4).

## 1.3 The ambient category

Let $Q$ be an orientation of the simply laced Dynkin diagram $\Delta$. We will now define and give some properties of the complete 2-dimensional Ginzburg dg algebra $\Pi_Q$ associated with $Q$. Our main goal will be to explain how the perfectly valued derived category $\mathrm{pvd}(\Pi_Q)$ and its spherical twists categorify the root system of $\Delta$ and the action of its Weyl group. This category will serve as an "ambient category" in which we will define the categories $\mathcal{C}([\boldsymbol{i}])$ and $\mathcal{R}([\boldsymbol{i}])$ associated with a commutation class $[\boldsymbol{i}]$ of reduced words (see Section 1.4).

**Remark 1.3.1.** We will assume that the reader is familiar with the theory of differential graded (=dg) algebras and their derived categories (see, e.g., [Kel94] and [KY11]).

**Remark 1.3.2.** We use the cohomological notation when dealing with complexes. All (dg) modules are *left* (dg) modules. In particular, representations of a quiver naturally correspond to modules over its path algebra.

### 1.3.1 The 2-dimensional Ginzburg dg algebra

Let $Q$ be an orientation of the simply laced Dynkin diagram $\Delta$. We denote by $Q_0$ its vertex set and by $Q_1$ its arrow set. Let $Q^*$ be the quiver obtained from $Q$ by adding an arrow $\alpha^* : j \to i$ for every arrow $\alpha : i \to j$ in $Q$, and a loop $t_i : i \to i$ for every vertex $i \in Q_0$. We regard $Q^*$ as a graded quiver where the arrows $\alpha$ and $\alpha^*$ (for $\alpha \in Q_1$) are of degree 0 and the loops $t_i$ (for $i \in Q_0$) are of degree $-1$.

Let $K$ be an algebraically closed field. The *(complete) 2-dimensional Ginzburg dg algebra* $\Pi_Q$ *associated with* $Q$ is defined as the complete differential graded path algebra $\widehat{KQ^*}$ whose differential is continuous and determined by the equations $d(\alpha) = d(\alpha^*) = 0$ for $\alpha \in Q_1$ and

$$d(t_i) = \sum_{\alpha \in Q_1} e_i(\alpha\alpha^* - \alpha^*\alpha)e_i,$$

where $e_i$ denotes the idempotent associated with the vertex $i \in Q_0$. Here, we complete the path algebra $KQ^*$ in the category of graded vector spaces with respect to the ideal generated by the arrows of $Q^*$. Hence, the $n$-th component of the graded algebra $\Pi_Q$ consists of elements of the form $\Sigma_p \lambda_p p$ where $p$ runs over all paths in $Q^*$ of degree $n$. Notice that there are canonical homomorphisms of algebras $KQ \hookrightarrow \Pi_Q$ and $\Pi_Q \twoheadrightarrow KQ$ that compose to the identity on the path algebra $KQ$.

The dg algebra $\Pi_Q$ does not depend, up to isomorphism, on the orientation $Q$, only on $\Delta$. Indeed, if $Q'$ is another orientation of $\Delta$, then we have an isomorphism $\Pi_Q \longrightarrow \Pi_{Q'}$ of dg algebras which sends $e_i$ to $e_i$ and $t_i$ to $t_i$ (for $i \in Q_0$), $\alpha$ to $\beta$ and $\alpha^*$ to $\beta^*$ (if we have arrows $\alpha : i \to j$ in $Q$ and $\beta : i \to j$ in $Q'$), and $\alpha$ to $-\beta^*$ and $\alpha^*$ to $\beta$ (if we have arrows

$\alpha : i \to j$ in $Q$ and $\beta : j \to i$ in $Q'$). In particular, we can canonically identify $KQ'$ as a subalgebra and as a quotient of $\Pi_Q$, and we shall do so without mention from now on.

Any automorphism $\sigma$ of $\Delta$ (as in Section 1.2) induces an automorphism of $\Pi_Q$ which we now define. First, define a new orientation $\sigma Q$ of $\Delta$ as follows: for $i, j \in \Delta_0$, there is an arrow $\sigma(i) \to \sigma(j)$ in $\sigma Q$ if and only if there is an arrow $i \to j$ in $Q$. It is not hard to see that there is an isomorphism $\Pi_Q \longrightarrow \Pi_{\sigma Q}$ sending each $e_i$ to $e_{\sigma(i)}$, each $t_i$ to $t_{\sigma(i)}$, each $\alpha$ to $\sigma(\alpha)$, and each $\alpha^*$ to $\sigma(\alpha)^*$. Composing with the isomorphism $\Pi_{\sigma Q} \longrightarrow \Pi_Q$ of the previous paragraph, we get the desired automorphism of $\Pi_Q$. If we choose $Q$ such that $\sigma Q = Q$ (which we can always do), this last isomorphism is simply the identity.

### 1.3.2 The perfectly valued category and its spherical twists

The *perfectly valued derived category* $\mathrm{pvd}(\Pi_Q)$ of $\Pi_Q$ is defined as the full subcategory of the derived category of $\Pi_Q$ whose objects are the *perfectly valued* dg modules, that is, dg modules that have finite-dimensional total cohomology. By [Kel11a] (see also [Kel18] and [Yeu22]), it is a 2-Calabi–Yau triangulated category, which implies the existence of a natural isomorphism

$$\mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(M, N) \cong D\mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(N, \Sigma^2 M)$$

for $M, N \in \mathrm{pvd}(\Pi_Q)$, where $D$ denotes the $K$-duality functor and $\Sigma$ is the suspension functor from the canonical triangulated structure.

For a vertex $i \in \Delta_0$, let $S_i$ be the corresponding simple $\Pi_Q$-module, viewed as an object in $\mathrm{pvd}(\Pi_Q)$. One can easily compute (using, e.g., [KY11, Lemma 2.15]) that these objects satisfy

$$\mathrm{Ext}^k_{\Pi_Q}(S_i, S_i) \cong \begin{cases} K & \text{if } k = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Together with the 2-Calabi–Yau property, this implies that these simple objects are *2-spherical* in the sense of [ST01]. Hence, by [ST01] (see also [HKP16]), for each $i \in \Delta_0$, the spherical object $S_i$ yields a *spherical twist* $T_i : \mathrm{pvd}(\Pi_Q) \to \mathrm{pvd}(\Pi_Q)$. It is a triangulated autoequivalence, and for every $X \in \mathrm{pvd}(\Pi_Q)$, there is a distinguished triangle

$$\mathbf{R}\mathrm{Hom}_{\Pi_Q}(S_i, X) \otimes_K S_i \longrightarrow X \longrightarrow T_i(X) \longrightarrow \Sigma\mathbf{R}\mathrm{Hom}_{\Pi_Q}(S_i, X) \otimes_K S_i. \qquad (*)$$

Here we can identify $\mathbf{R}\mathrm{Hom}_{\Pi_Q}(S_i, X)$ with the quasi-isomorphic complex with zero differential whose entries are the cohomologies $\mathrm{Ext}^k_{\Pi_Q}(S_i, X) \cong \mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(S_i, \Sigma^k X)$. Under this identification, the first map above is the evident evaluation map.

We list some important properties of these functors below. In particular, by items (2) and (3), they satisfy the same relations that define the generalized braid group associated with $\Delta$.

**Proposition 1.3.3.** *Let $i, j \in \Delta_0$. With the definitions above, the following statements hold.*

*(1) We have $T_i(S_i) \cong \Sigma^{-1} S_i$.*

*(2) If $i \not\sim j$, then $T_i T_j \cong T_j T_i$.*

*(3) If $i \sim j$, then $T_i T_j T_i \cong T_j T_i T_j$.*

*(4) Let $F$ be a triangulated autoequivalence of $\operatorname{pvd}(\Pi_Q)$. If we have $F(S_i) \cong S_j$, then $F T_i F^{-1}(X) \cong T_j(X)$ for all $X \in \operatorname{pvd}(\Pi_Q)$.*

*Proof.* Statements (1) and (4) are Lemma 3.8 and Corollary 3.7 of [Opp26], respectively. The other two statements follow from [ST01, Propositions 2.12 and 2.13] and [KY11, Lemma 2.15]. $\checkmark$

### 1.3.3 Preprojective algebras and reflection functors

To make explicit calculations, we will use some alternative descriptions of $\operatorname{pvd}(\Pi_Q)$ and its spherical twists in terms of preprojective algebras.

The *double quiver* $\overline{Q}$ of $Q$ is the subquiver of $Q^*$ obtained by removing the loops $t_i$ for $i \in Q_0$. The *preprojective algebra* is the quotient of the path algebra $K\overline{Q}$ by the *preprojective relations*

$$\sum_{\alpha \in Q_1} e_i(\alpha\alpha^* - \alpha^*\alpha)e_i = 0$$

for $i \in \Delta_0$. Since $Q$ is a Dynkin quiver, this is a finite-dimensional algebra (see [Rin98] for alternative constructions that make this fact more evident). We can extend the map $\alpha \mapsto \alpha^*$ to an involution on the set $\overline{Q}_1$ of arrows of $\overline{Q}$ if we impose $(\alpha^*)^* = \alpha$ for $\alpha \in Q_1$. With this definition, the preprojective relation at $i \in \Delta_0$ becomes

$$\sum_{\substack{\alpha \in \overline{Q}_1 \\ \alpha : j \to i}} \epsilon(\alpha)\alpha\alpha^* = 0,$$

where $\epsilon(\alpha) = 1$ if $\alpha \in Q_1$, and $\epsilon(\alpha) = -1$ otherwise.

Since $\Pi_Q$ is a connective dg algebra (i.e., its cohomology is concentrated in nonpositive degrees), its derived category has a canonical t-structure whose left (resp. right) aisle consists of the dg modules whose cohomology is concentrated in nonpositive (resp. nonnegative) degrees. Its heart can be identified with the module category of $H^0(\Pi_Q)$ via restriction along the canonical map $\Pi_Q \longrightarrow H^0(\Pi_Q)$ (see [KY11, Section 5.1]). One can check that $H^0(\Pi_Q)$ is isomorphic to the preprojective algebra $\Lambda_Q$. Thus, if we restrict the canonical t-structure to $\mathrm{pvd}(\Pi_Q)$, we deduce that its heart is equivalent to $\mathrm{mod}\,\Lambda_Q$, the category of finite-dimensional $\Lambda_Q$-modules. For this reason, we shall often see an object of $\mathrm{pvd}(\Pi_Q)$ concentrated in degree 0 as a finite-dimensional $\Lambda_Q$-module or, equivalently, as a representation of the double quiver $\overline{Q}$ satisfying the preprojective relations. We also obtain that $\mathrm{pvd}(\Pi_Q)$ is the thick hull of the simple objects $S_i$ for $i \in \Delta_0$.

Let $\widetilde{\Delta}$ denote the affine Dynkin diagram corresponding to $\Delta$. Choose an orientation $\widetilde{Q}$ for it extending $Q$ and similarly define the dg algebra $\Pi_{\widetilde{Q}}$. Let $e_0 \in \Pi_{\widetilde{Q}}$ be the idempotent corresponding to the extended vertex. The canonical quotient map $\Pi_{\widetilde{Q}} \to \Pi_Q$ that sends $e_0$ to zero gives rise to a restriction functor

$$\mathrm{pvd}(\Pi_Q) \longrightarrow \mathrm{pvd}(\Pi_{\widetilde{Q}}).$$

Computing the functors $\mathrm{Ext}^n$ between the simple objects $S_i$ for $i \in \Delta_0$, we see that they are the same whether we compute them in $\mathrm{pvd}(\Pi_Q)$ or in $\mathrm{pvd}(\Pi_{\widetilde{Q}})$. Since $\mathrm{pvd}(\Pi_Q)$ is the thick hull of its simple objects, we deduce by a dévissage argument that the restriction functor above is fully faithful. In particular, we can see $\mathrm{pvd}(\Pi_Q)$ as a full triangulated subcategory of $\mathrm{pvd}(\Pi_{\widetilde{Q}})$.

Since $\widetilde{\Delta}$ is not a Dynkin diagram, $\Pi_{\widetilde{Q}}$ is a stalk dg algebra whose 0-th cohomology is isomorphic to $\Lambda_{\widetilde{Q}}$, the preprojective algebra associated with $\widetilde{Q}$ completed at the ideal generated by the arrows. This follows from the noncomplete case, which is done, for example, in [KW23a, Section 5.2]. In particular, $\Pi_{\widetilde{Q}}$ and $\Lambda_{\widetilde{Q}}$ are quasi-isomorphic dg algebras and have triangle-equivalent derived categories. Therefore, we can identify $\mathrm{pvd}(\Pi_{\widetilde{Q}})$ with $\mathcal{D}^b(\mathrm{mod}\,\Lambda_{\widetilde{Q}})$, the full subcategory of the derived category of $\Lambda_{\widetilde{Q}}$ whose objects have finite-dimensional total cohomology. The simple object $S_i$ for $i \in \Delta_0$ is also 2-spherical in $\mathcal{D}^b(\mathrm{mod}\,\Lambda_{\widetilde{Q}})$, and the inverse of the induced spherical twist is naturally isomorphic to the functor $I_i \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} -$, where $I_i = \Lambda_{\widetilde{Q}}(1-e_i)\Lambda_{\widetilde{Q}}$ is the kernel of the quotient map $\Lambda_{\widetilde{Q}} \longrightarrow S_i$ (see [IR08, Theorem 6.14] and [BIRS09, Section III]). Consequently, the induced spherical twist is given by $\mathbf{R}\mathrm{Hom}_{\Lambda_{\widetilde{Q}}}(I_i, -)$. Viewing $\mathrm{pvd}(\Pi_Q)$ as a full triangulated subcategory of $\mathcal{D}^b(\mathrm{mod}\,\Lambda_{\widetilde{Q}})$, this gives an alternative way for computing $T_i$ and $T_i^{-1}$.

**Lemma 1.3.4.** *Let $(i_1, \ldots, i_t)$ be a reduced word for $w \in \mathsf{W}$. Then $I_{i_1} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{i_2} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{i_t}$ is isomorphic to the product of ideals $I_{i_1} I_{i_2} \cdots I_{i_t}$.*

*Proof.* This result appears in the proof of [BIRS09, Theorem III.1.9]. $\surd$

**Remark 1.3.5.** In the noncompleted case, [MY25] characterizes the inverse spherical twist $T_i^{-1}$ as a derived tensor product with a dg ideal of $\Pi_Q$ defined without referring to the extended preprojective algebra $\Lambda_{\widetilde{Q}}$.

We now recall the notion of reflection functors introduced in [BK12] and independently in [Bol10] (see also [BKT14, Section 5]). Let $M$ be a $\Lambda_Q$-module. We see $M$ as a representation of the double quiver $\overline{Q}$, that is, the data of a finite-dimensional $K$-vector space $M(j)$ for each $j \in Q_0$ and a linear map $M_\alpha : M(j) \to M(j')$ for every arrow $\alpha : j \to j'$ in $\overline{Q}$. For a fixed vertex $i \in \Delta_0$, consider the diagram

$$\bigoplus_{\substack{\alpha\in\overline{Q}_1\\ \alpha:j\to i}} M(j) \xrightarrow{(\epsilon(\alpha)M_\alpha)} M(i) \xrightarrow{(M_{\alpha^*})} \bigoplus_{\substack{\alpha\in\overline{Q}_1\\ \alpha:j\to i}} M(j).$$

To simplify the notation, we will denote it as

$$\widetilde{M}(i) \xrightarrow{M_{\mathrm{in}(i)}} M(i) \xrightarrow{M_{\mathrm{out}(i)}} \widetilde{M}(i).$$

By the preprojective relations, we have $M_{\mathrm{in}(i)}M_{\mathrm{out}(i)} = 0$, so the image of $M_{\mathrm{out}(i)}$ is contained in $\ker M_{\mathrm{in}(i)}$. Therefore, we get a new representation $\Sigma_i(M)$ from $M$ if we replace the data of the previous diagram with

$$\widetilde{M}(i) \xrightarrow{M_{\mathrm{out}(i)}M_{\mathrm{in}(i)}} \ker M_{\mathrm{in}(i)} \hookrightarrow \widetilde{M}(i).$$

Observe that the preprojective relations are still satisfied, hence $\Sigma_i(M)$ is a $\Lambda_Q$-module. One can easily extend $\Sigma_i$ to a functor on $\operatorname{mod}\Lambda_Q$, the *reflection functor* of [BK12].

**Proposition 1.3.6.** *Let $M \in \mathrm{pvd}(\Pi_Q)$ be an object concentrated in degree 0 and view it as a $\Lambda_Q$-module. For $i \in \Delta_0$, $T_i(M)$ is again concentrated in degree 0 if and only if the map $M_{\mathrm{in}(i)}$ defined above is surjective. In this case, we have $T_i(M) \cong \Sigma_i(M)$.*

*Proof.* Let $M \in \mathrm{pvd}(\Pi_Q)$ be an object concentrated in degree 0. Using the identifications above, we have $T_i(M) \cong \mathbf{R}\mathrm{Hom}_{\Lambda_{\widetilde{Q}}}(I_i, M)$. By [BIRS09, Proposition III.1.4], the projective dimension of $I_i$ is at most 1, thus we deduce that $T_i(M)$ is concentrated in degree 0 if and only if $\mathrm{Ext}^1_{\Lambda_{\widetilde{Q}}}(I_i, M) = 0$. By [BKT14, Example 5.6(i)], this happens precisely when the map $M_{\mathrm{in}(i)}$ is surjective. In this case, we conclude that $T_i(M) \cong \mathrm{Hom}_{\Lambda_{\widetilde{Q}}}(I_i, M) \cong \Sigma_i(M)$, where the last isomorphism follows from [BKT14, Proposition 5.1]. $\surd$

**Remark 1.3.7.** Let $Q'$ be another orientation of $\Delta$ and suppose that $i \in Q'_0$ is a source. Denote the reflected quiver $s_iQ'$ by $Q''$. Let $F_i^- : \operatorname{mod} KQ' \longrightarrow \operatorname{mod} KQ''$ be the classical reflection functor of Bernstein–Gelfand–Ponomarev [BGP73]. We claim that we have a commutative diagram of functors (up to natural isomorphism):

$$\begin{tikzcd} \mathcal{D}^b(\operatorname{mod} KQ') \arrow[r] \arrow[d, "\mathbf{L}F_i^-"'] & \mathrm{pvd}(\Pi_{\widetilde{Q}}) \cong \mathcal{D}^b(\operatorname{mod}\Lambda_{\widetilde{Q}}) \arrow[d, "I_i\otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}}-"] \\ \mathcal{D}^b(\operatorname{mod} KQ'') \arrow[r] & \mathrm{pvd}(\Pi_{\widetilde{Q}}) \cong \mathcal{D}^b(\operatorname{mod}\Lambda_{\widetilde{Q}}) \end{tikzcd}$$

where the functor on the left is the left derived functor of $F_i^-$ and the horizontal functors are induced by the natural quotient maps $\Pi_{\widetilde{Q}} \longrightarrow KQ'$ and $\Pi_{\widetilde{Q}} \longrightarrow KQ''$ which factor through $\Pi_Q$. To prove this, notice that all functors above can be seen as left derived functors (as the restriction functors are exact). Therefore, by [Kel94, Lemma 6.4], it is enough to show that $I_i \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} KQ' \cong \mathbf{L}F_i^-(KQ')$ in the derived category of $\Lambda_{\widetilde{Q}}$-$KQ'$-bimodules. This isomorphism can be verified using a dual version of Proposition 1.3.6 after we realize that the reflection functors of [BK12] restrict to those of [BGP73] (see [BK12, Proposition 7.1]). Alternatively, see [AIRT12, Corollary 2.12].

As a consequence, we obtain the following commutative diagram of functors:

$$\begin{tikzcd}
\mathcal{D}^b(\operatorname{mod} KQ') \arrow[r] \arrow[d, "\mathbf{L}F_i^-"'] & \operatorname{pvd}(\Pi_Q) \arrow[d, "T_i^{-1}"] \\
\mathcal{D}^b(\operatorname{mod} KQ'') \arrow[r] & \operatorname{pvd}(\Pi_Q)
\end{tikzcd}$$

Since $\mathbf{L}F_i^-$ is an equivalence with quasi-inverse given by the right derived functor of the other reflection functor $F_i^+ : \operatorname{mod} KQ'' \longrightarrow \operatorname{mod} KQ'$ of [BGP73], we also obtain a similar diagram involving $\mathbf{R}F_i^+$ and $T_i$.

### 1.3.4 A categorification of the root system

Let $K_0(\operatorname{pvd}(\Pi_Q))$ be the Grothendieck group of $\operatorname{pvd}(\Pi_Q)$. One can show that it is isomorphic to the Grothendieck group of the canonical heart $\operatorname{mod}\Lambda_Q$; hence, $K_0(\operatorname{pvd}(\Pi_Q))$ is a finitely generated free abelian group with a canonical basis given by the classes $[S_i]$ ($i \in \Delta_0$) of the simple objects. In particular, we can identify $K_0(\operatorname{pvd}(\Pi_Q))$ with the root lattice $\mathsf{Q}$ associated with $\Delta$ in such a way that the class $[S_i]$ corresponds to the simple root $\alpha_i$.

We can also categorify the weight lattice $\mathsf{P}$ of $\Delta$. Let $\operatorname{per}(\Pi_Q)$ be the *perfect derived category of* $\Pi_Q$, that is, the subcategory of compact objects of the derived category of $\Pi_Q$. Equivalently, $\operatorname{per}(\Pi_Q)$ is given by the thick hull of $\Pi_Q$ viewed as an object in its derived category. Its Grothendieck group $K_0(\operatorname{per}(\Pi_Q))$ is generated by the classes $[P_i]$ for $i \in \Delta_0$, where $P_i = \Pi_Q e_i$ (this follows, e.g., from [Pla11b, Lemma 2.14]). These classes satisfy some orthogonality relations against the classes $[S_j] \in K_0(\operatorname{pvd}(\Pi_Q))$ under the Euler form (see below). Thus, they are also linearly independent and form a basis of $K_0(\operatorname{per}(\Pi_Q))$. We identify $K_0(\operatorname{per}(\Pi_Q))$ with $\mathsf{P}$ by sending $[P_i]$ to the fundamental weight $\varpi_i$.

By [KY11, Theorem 2.19], we have $\operatorname{pvd}(\Pi_Q) \subseteq \operatorname{per}(\Pi_Q)$. This inclusion induces a map

$$\mu : K_0(\operatorname{pvd}(\Pi_Q)) \longrightarrow K_0(\operatorname{per}(\Pi_Q))$$

between the corresponding Grothendieck groups. The next result shows that $\mu$ is injective and that the inclusion $K_0(\mathrm{pvd}(\Pi_Q)) \subseteq K_0(\mathrm{per}(\Pi_Q))$ agrees with the inclusion $\mathsf{Q} \subseteq \mathsf{P}$ under the previous identifications.

**Proposition 1.3.8.** *With the notation above, the matrix of the map $\mu$ in the canonical bases is the Cartan matrix of $\Delta$. In particular, $\mu$ is injective.*

*Proof.* The following argument is based on [KY11, Section 2.14]. Take $i \in \Delta_0$ and let $\pi : P_i \longrightarrow S_i$ be the canonical projection. The kernel of $\pi$ is the closed dg submodule of $P_i$ generated by all nontrivial paths starting at $i$. Hence, we have

$$\ker \pi = \sum_{\substack{\alpha \in Q_1^* \\ \alpha : i \to j}} P_j \alpha = \bigoplus_{\substack{\alpha \in Q_1^* \\ \alpha : i \to j}} P_j \alpha.$$

We remark that the direct sum decomposition above only holds in the category of graded $\Pi_Q$-modules because the summand $P_i t_i$ is not stable under the differential. To fix this, we consider the following filtration of $\ker \pi$ by dg submodules:

$$0 \subset \bigoplus_{\substack{\alpha \in \overline{Q}_1 \\ \alpha : i \to j}} P_j \alpha \subset \ker \pi.$$

Notice that the second term above is isomorphic as a dg module to

$$\bigoplus_{\substack{\alpha \in \overline{Q}_1 \\ \alpha : i \to j}} \Sigma^{-|\alpha|} P_j = \bigoplus_{\substack{\alpha \in \overline{Q}_1 \\ \alpha : i \to j}} P_j,$$

while the quotient of the last two terms is isomorphic to $\Sigma^{-|t_i|} P_i = \Sigma P_i$. Since there is precisely one arrow in $\overline{Q}$ starting at $i$ for each $j \in \Delta_0$ with $j \sim i$, we get the following equality in $K_0(\mathrm{per}(\Pi_Q))$:

$$[\ker \pi] = \sum_{j \sim i} [P_j] - [P_i].$$

Now, since $S_i$ is isomorphic in $\mathrm{per}(\Pi_Q)$ to the cone of the canonical inclusion $\ker \pi \longrightarrow P_i$, we deduce that

$$\mu([S_i]) = [P_i] - [\ker \pi] = 2[P_i] - \sum_{j \sim i} [P_j],$$

proving that the matrix representing $\mu$ is indeed the Cartan matrix of $\Delta$. The last statement is true because the determinant of the Cartan matrix is nonzero. $\surd$

For $i, j \in \Delta_0$, it is easy to see that $\mathbf{R}\mathrm{Hom}_{\Pi_Q}(P_i, S_j)$ is concentrated in degree 0 and its zeroth cohomology has dimension given by Kronecker's delta $\delta_{ij}$. We deduce that $\mathbf{R}\mathrm{Hom}_{\Pi_Q}(P, M)$ has finite-dimensional total cohomology for any $P \in \mathrm{per}(\Pi_Q)$ and

$M \in \operatorname{pvd}(\Pi_Q)$. Therefore, we can define the *Euler form* as the pairing

$$(-,-) : K_0(\operatorname{per}(\Pi_Q)) \times K_0(\operatorname{pvd}(\Pi_Q)) \longrightarrow \mathbb{Z}$$

given by

$$([P],[M]) = \sum_{k\in\mathbb{Z}} (-1)^k \dim_K H^k(\mathbf{R}\mathrm{Hom}_{\Pi_Q}(P,M)) = \sum_{k\in\mathbb{Z}} (-1)^k \dim_K \mathrm{Ext}^k_{\Pi_Q}(P,M)$$

for $P \in \operatorname{per}(\Pi_Q)$ and $M \in \operatorname{pvd}(\Pi_Q)$. Since $([P_i],[S_j]) = \delta_{ij}$ for all $i,j \in \Delta_0$, the Euler form agrees with the symmetric bilinear form defined in Section 1.2.1. Moreover, using the distinguished triangle $(*)$ in the definition of the spherical twist, we have

$$[T_i(X)] = [X] - ([S_i],[X])[S_i]$$

for all $i \in \Delta_0$ and $X \in \operatorname{pvd}(\Pi_Q)$. Thus, the spherical twists categorify the simple reflections $s_i$ of Section 1.2.1.

If $\sigma$ is an automorphism of $\Delta$, we saw in Section 1.3.1 that it induces an automorphism of $\Pi_Q$. Restricting along its inverse induces an autoequivalence $\sigma : \operatorname{per}(\Pi_Q) \longrightarrow \operatorname{per}(\Pi_Q)$. In this setting, notice that the action of $\sigma$ in $K_0(\operatorname{per}(\Pi_Q))$ agrees with its action on the weight lattice $\mathsf{P}$ as defined at the end of Section 1.2.5.

## 1.4 Categories associated with commutation classes

In this section, we introduce the category of representations $\mathcal{C}([\boldsymbol{i}])$ for a commutation class $[\boldsymbol{i}]$ of reduced words for $w \in \mathsf{W}$. When $w = w_0$, we also construct what we call its repetition category $\mathcal{R}([\boldsymbol{i}])$ and its c-derived category $\mathcal{D}([\boldsymbol{i}])$.

We will freely use the notation introduced in Sections 1.2 and 1.3.

**Remark 1.4.1.** We fix an algebraically closed field $K$ and an orientation $Q^\circ$ of $\Delta$ to work with the dg algebra $\Pi_{Q^\circ}$ and its perfectly valued derived category. For simplicity, we will write $Q$ for $Q^\circ$ in general. However, when dealing with other quivers, we will prefer to denote them by $Q$, and the full notation $Q^\circ$ will be used to avoid ambiguity.

### 1.4.1 The category of representations

Let $\boldsymbol{i} = (i_1, i_2, \dots, i_t)$ be a reduced word for $w \in \mathsf{W}$. For $1 \le k \le t$, define

$$M_k^{\boldsymbol{i}} = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(S_{i_k}) \in \operatorname{pvd}(\Pi_Q).$$

It is an indecomposable object whose class in $K_0(\mathrm{pvd}(\Pi_Q))$ corresponds to the positive root $\beta_k^{\boldsymbol{i}} \in \mathsf{R}^+(w)$. For the commutation class $[\boldsymbol{i}]$, define

$$\mathrm{ind}([\boldsymbol{i}]) = \{M_k^{\boldsymbol{i}} \in \mathrm{pvd}(\Pi_Q) \mid 1 \leq k \leq t\},$$

the set of *indecomposable objects associated with* $[\boldsymbol{i}]$. Let

$$\mathcal{C}([\boldsymbol{i}]) = \mathrm{add}\,\mathrm{ind}([\boldsymbol{i}]) \subseteq \mathrm{pvd}(\Pi_Q)$$

be the full additive subcategory generated by these objects and closed under isomorphisms. We call it the *category of representations of* $[\boldsymbol{i}]$. As the notation suggests, we have the following lemma.

**Lemma 1.4.2.** *The set* $\mathrm{ind}([\boldsymbol{i}])$ *does not depend (up to isomorphism of its elements) on the choice of the representative for the commutation class* $[\boldsymbol{i}]$.

*Proof.* Take $\boldsymbol{j} \in [\boldsymbol{i}]$. We may suppose that $\boldsymbol{j}$ is obtained from $\boldsymbol{i}$ by applying a commutation move that exchanges $i_l$ and $i_{l+1}$ for some $1 \leq l < t$ such that $i_l \not\sim i_{l+1}$. Since $T_{i_l}$ and $T_{i_{l+1}}$ commute (Proposition 1.3.3), we have

$$M_k^{\boldsymbol{i}} \cong M_k^{\boldsymbol{j}}$$

for $1 \leq k \leq t$ with $k \neq l, l+1$. Furthermore, we have $M_l^{\boldsymbol{i}} \cong M_{l+1}^{\boldsymbol{j}}$ and $M_{l+1}^{\boldsymbol{i}} \cong M_l^{\boldsymbol{j}}$, because $i_l \not\sim i_{l+1}$ and so $T_{i_{l+1}}(S_{i_l}) \cong S_{i_l}$ and $T_{i_l}(S_{i_{l+1}}) \cong S_{i_{l+1}}$ (by Proposition 1.3.6). Hence, the sets $\mathrm{ind}([\boldsymbol{i}])$ and $\mathrm{ind}([\boldsymbol{j}])$ coincide (up to isomorphism of its elements). √

By the lemma above, we deduce that $M_k^{\boldsymbol{i}} \cong M_l^{\boldsymbol{j}}$ if $\boldsymbol{i}$ and $\boldsymbol{j}$ are commutation-equivalent and $\beta_k^{\boldsymbol{i}} = \beta_l^{\boldsymbol{j}}$. In this way, given a commutation class $[\boldsymbol{i}]$ and a positive root $\alpha \in \mathsf{R}^+(w)$, it is well defined (up to isomorphism) to set $M_\alpha^{[\boldsymbol{i}]}$ as $M_k^{\boldsymbol{i}}$ where $k$ satisfies $\alpha = \beta_k^{\boldsymbol{i}}$. We will adopt this notation from now on.

**Theorem 1.4.3.** *If* $\boldsymbol{i} = (i_1, \ldots, i_t)$ *is any reduced word for* $w \in \mathsf{W}$*, then* $M_k^{\boldsymbol{i}}$ *has cohomology concentrated in degree* $0$ *for all* $1 \leq k \leq t$*.*

*Proof.* Let $\boldsymbol{j} = (j_1, \ldots, j_s)$ be a reduced word for some $w' \in \mathsf{W}$. Let us first show that $M'_k = T_{j_1}^{-1} \cdots T_{j_{k-1}}^{-1}(S_{j_k})$ is concentrated in degree 0 for any $1 \leq k \leq s$. Using the characterization of the spherical twists and the identifications given in Section 1.3.3 (due to [IR08] and [BIRS09]), we have an isomorphism

$$M'_k \cong I_{j_1} \otimes^{\mathbf{L}}_{\Lambda_{\tilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\tilde{Q}}} I_{j_{k-1}} \otimes^{\mathbf{L}}_{\Lambda_{\tilde{Q}}} S_{j_k}$$

in the derived category of the completed preprojective algebra $\Lambda_{\widetilde{Q}}$. Applying the functor $I_{j_1} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{j_{k-1}} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} -$ to the distinguished triangle induced by the exact sequence

$$0 \longrightarrow I_{j_k} \longrightarrow \Lambda_{\widetilde{Q}} \longrightarrow S_{j_k} \longrightarrow 0,$$

we get a distinguished triangle of the form

$$I_{j_1} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{j_k} \longrightarrow I_{j_1} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{j_{k-1}} \longrightarrow M'_k \longrightarrow \Sigma I_{j_1} \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} \cdots \otimes^{\mathbf{L}}_{\Lambda_{\widetilde{Q}}} I_{j_{k-1}}.$$

By Lemma 1.3.4, we deduce that

$$M'_k \cong \frac{I_{j_1} I_{j_2} \cdots I_{j_{k-1}}}{I_{j_1} I_{j_2} \cdots I_{j_k}},$$

hence $M'_k$ is concentrated in degree 0.

Now, let us prove the theorem. Since $\boldsymbol{i}$ can be extended to a reduced word for the longest element $w_0$ (this follows, e.g., from [Bou68, p. 158, Corollaire 3]), we can assume that $t = N$ and $w = w_0$. By the previous paragraph, we will finish the proof once we show that

$$M^{\boldsymbol{i}}_k \cong T^{-1}_{i^*_N} T^{-1}_{i^*_{N-1}} \cdots T^{-1}_{i^*_{k+1}} (S_{i^*_k}).$$

Indeed, by applying $T^{-1}_{i_{k-1}} \cdots T^{-1}_{i_2} T^{-1}_{i_1}$ on both sides, this is equivalent to

$$S_{i_k} \cong T^{-1}_{i_{k-1}} \cdots T^{-1}_{i_2} T^{-1}_{i_1} T^{-1}_{i^*_N} T^{-1}_{i^*_{N-1}} \cdots T^{-1}_{i^*_{k+1}} (S_{i^*_k}).$$

Since $(i_{k-1}, \ldots, i_2, i_1, i^*_N, i^*_{N-1}, \ldots, i^*_k)$ is a reduced word for $w_0$, the previous paragraph implies that the object on the right is concentrated in degree 0. One can check that its class in $K_0(\mathrm{pvd}(\Pi_Q))$ is $[S_{i_k}]$, so we obtain the isomorphism above because $S_{i_k}$ is a simple object of the canonical heart of $\mathrm{pvd}(\Pi_Q)$. $\surd$

**Remark 1.4.4.** When $w = w_0$, the proof above shows that $\mathrm{ind}([\boldsymbol{i}])$ coincides with the set of *layers* of the preprojective algebra $\Lambda_Q$ associated with the reduced word $(i^*_N, i^*_{N-1}, \ldots, i^*_1)$ in the sense of [AIRT12]. This alternative description of the layers using the spherical twist functors has already been shown in [AIRT12, Proposition 2.2].

**Remark 1.4.5.** Let $\boldsymbol{i} = (i_1, \ldots, i_N)$ be a reduced word for $w_0$. Consider the composition of functors $T = T_{i^*_1} \cdots T_{i^*_N}$. The end of the proof above shows that $T(S_{i_1}) \cong \Sigma^{-1} S_{i^*_1}$. But notice that $T$ does not depend on the choice of reduced word by Proposition 1.3.3. Thus, more generally, $T(S_j) \cong \Sigma^{-1} S_{j^*}$ for all $j \in \Delta_0$. As a consequence, if $\boldsymbol{j}$ is the reduced word $(i^*_1, \ldots, i^*_N)$, then the objects of $\mathcal{C}([\boldsymbol{j}])$ can be obtained from the objects of $\mathcal{C}([\boldsymbol{i}])$ by applying $\Sigma \circ T$.

**Corollary 1.4.6.** *If $[\boldsymbol{i}]$ is a commutation class of reduced words for the longest element $w_0$, then $\mathcal{C}([\boldsymbol{i}])$ contains $S_i$ for all $i \in \Delta_0$.*

*Proof.* Since $M^{[\boldsymbol{i}]}_{\alpha_i}$ is concentrated in degree 0 and has the same class in $K_0(\mathrm{pvd}(\Pi_Q))$ as the simple $S_i$, we must have $M^{[\boldsymbol{i}]}_{\alpha_i} \cong S_i$. $\checkmark$

The next result shows that we can see $\mathcal{C}([\boldsymbol{i}])$ as a generalization of the category of representations of a Dynkin quiver. See also [AIRT12, Theorem 3.3].

**Proposition 1.4.7.** *If $\boldsymbol{i} = (i_1, \dots, i_N)$ is a reduced word for $w_0$ that is a source sequence for some orientation $Q$ of $\Delta$, then $M^{\boldsymbol{i}}_k$ is the image of an indecomposable $KQ$-module via the restriction functor*

$$\mathrm{mod}\, KQ \longrightarrow \mathrm{pvd}(\Pi_{Q^\circ})$$

*for all $1 \leq k \leq N$. In particular, $\mathcal{C}([\boldsymbol{i}])$ and $\mathrm{mod}\, KQ$ are equivalent as $K$-linear categories.*

*Proof.* Observe that the restriction functor above is fully faithful. Assuming the first statement above and considering the map induced on Grothendieck groups, this restriction must send any indecomposable $KQ$-module to an object isomorphic to an indecomposable object in $\mathcal{C}([\boldsymbol{i}])$. This proves the equivalence of categories claimed above.

Let us focus on the first part. Take $Q' = s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1} Q$, which is well defined by the hypothesis on $Q$. Observe that the sequence $(i_{k-1}, i_{k-2}, \dots, i_1)$ is a sink sequence for $Q'$. Recall that we have the restriction functor

$$\mathrm{mod}\, KQ'' \longrightarrow \mathrm{pvd}(\Pi_{Q^\circ})$$

induced by the quotient map $\Pi_{Q^\circ} \longrightarrow KQ''$ for every Dynkin quiver $Q''$ of type $\Delta$. We will prove by backward induction on $1 \leq l \leq k$ that

$$N_l = T_{i_l} T_{i_{l+1}} \cdots T_{i_{k-1}} (S_{i_k})$$

is the image of an indecomposable $KQ''$-module via the functor above, where $Q''$ is given by $s_{i_l}^{-1} s_{i_{l+1}}^{-1} \cdots s_{i_{k-1}}^{-1} Q'$. This will prove the lemma since $N_1 = M^{\boldsymbol{i}}_k$ and $Q = s_{i_1}^{-1} \cdots s_{i_{k-1}}^{-1} Q'$.

One important observation for the induction to work is the following: if $l > 1$, we cannot have $N_l \cong S_{i_{l-1}}$. Indeed, the classes of $N_{l-1}$ and $N_l$ in $K_0(\mathrm{pvd}(\Pi_{Q^\circ}))$ correspond to the roots $\beta = s_{i_{l-1}} s_{i_l} \cdots s_{i_{k-1}} (\alpha_{i_k})$ and $s_{i_{l-1}}(\beta)$, respectively. Note that $\beta$ is a positive root since $(i_{l-1}, i_l, \dots, i_k)$ represents a reduced word for some element in $\mathsf{W}$. However, if we had $N_l \cong S_{i_{l-1}}$, we would get $\alpha_{i_{l-1}} = s_{i_{l-1}}(\beta)$ and hence $\beta = -\alpha_{i_{l-1}}$, a contradiction.

Our claim is immediate for $l = k$. Take $1 < l \leq k$ and suppose that $N_l$ is an indecomposable $KQ''$-module for $Q'' = s_{i_l}^{-1} \cdots s_{i_{k-1}}^{-1} Q'$. Note that the vertex $i_{l-1}$ is a sink for $Q''$. Seeing $N_l$ as a representation of the quiver $Q''$, the natural map from the direct sum of the vector spaces at the neighbors of $i_{l-1}$ to the vector space at $i_{l-1}$ must be surjective, otherwise $N_l$ would have a direct summand isomorphic to $S_{i_{l-1}}$, contradicting the observation from the previous paragraph. Hence, by Proposition 1.3.6, if we see $N_l$ as a $\Lambda_{Q^\circ}$-module, we have $N_{l-1} = T_{i_{l-1}}(N_l) \cong \Sigma_{i_{l-1}}(N_l)$. Since $N_l$ is a $KQ''$-module and $i_l$ is a sink of $Q''$, the reflection functor $\Sigma_{i_{l-1}}$ acts on $N_l$ as the classical reflection functor of Bernstein–Gelfand–Ponomarev [BGP73]. Therefore, we conclude that $N_{l-1}$ is an indecomposable representation of $s_{i_{l-1}}^{-1}Q''$, finishing the induction and the proof. $\surd$

**Proposition 1.4.8.** *Let $\boldsymbol{i} = (i_1, \dots, i_N)$ be a reduced word for $w_0$. For a source $i \in \Delta_0$ of $[\boldsymbol{i}]$, the equivalence $T_i^{-1} : \operatorname{pvd}(\Pi_Q) \longrightarrow \operatorname{pvd}(\Pi_Q)$ restricts to an equivalence from the full additive subcategory of $\mathcal{C}([\boldsymbol{i}])$ generated by $\operatorname{ind}([\boldsymbol{i}]) \setminus \{S_i\}$ to the full additive subcategory of $\mathcal{C}(r_i[\boldsymbol{i}])$ generated by $\operatorname{ind}(r_i[\boldsymbol{i}]) \setminus \{S_i\}$.*

*Proof.* We can suppose $i_1 = i$. Thus, $r_i\boldsymbol{i} = (i_2, \cdots, i_N, i_1^*)$ is in $r_i[\boldsymbol{i}]$. It is immediate that $T_i^{-1}M_k^{\boldsymbol{i}} \cong M_{k-1}^{r_i\boldsymbol{i}}$ for $1 < k \leq N$, so we get the equivalence once we note that $M_1^{\boldsymbol{i}} \cong S_i \cong M_N^{r_i\boldsymbol{i}}$. $\surd$

We conclude this subsection with two concrete examples involving commutation classes associated with Q-data. More examples can be found in Section 1.9. By Theorem 1.4.3, the indecomposable objects of $\mathcal{C}([\boldsymbol{i}])$ are objects of $\operatorname{pvd}(\Pi_Q)$ whose cohomology is concentrated in degree 0. As discussed in Section 1.3.3, one can thus view these objects as $\Lambda_Q$-modules or, equivalently, as representations of the double quiver $\overline{Q}$ satisfying the preprojective relations. We will represent these objects in the examples in this way.

We will display representations graphically, following the usual practice. For example, the diagram

$$\begin{tikzcd}
& & K \arrow[dl, bend left, "0"] \\
K \arrow[r, bend left, "\mathrm{id}"] & K \arrow[l, bend left, "0"] \arrow[ur, bend left, "\mathrm{id}"] \arrow[dr, bend left, "0"] & \\
& & K \arrow[ul, bend left, "\mathrm{id}"]
\end{tikzcd}$$

depicts a representation of the double quiver in type $\mathsf{D}_4$. Most of our representations will have vector spaces of dimension at most one on the vertices. In this case, the linear maps between the vertices will always be zero or the identity. To simplify the notation, we will omit the arrow for zero maps and the label for identity maps. For instance, we would denote the representation above as

$$K \overset{\rightarrow}{K} \begin{smallmatrix} \nearrow K \\ \nwarrow K \end{smallmatrix}$$

Whenever there is a vertex of dimension greater than 1, we write the representation completely. If $m > 1$, we denote by $i_k : K \to K^m$ and $\pi_k : K^m \to K$ the canonical inclusion and the canonical projection on the $k$-th coordinate, respectively.

The choice of the orientation $Q$ of $\Delta$ matters when deciding if a given representation of $\overline{Q}$ comes from a $\Lambda_Q$-module. Indeed, the signs appearing in the preprojective relations come from the orientation. Hence, we will choose a particular orientation to fix the signs when needed. For convenience, our choices will satisfy $\sigma Q = Q$ (see Section 1.3.1).

**Example 1.4.9.** Suppose $(\Delta, \sigma) = (\mathsf{A}_5, \vee)$ is of type $\mathsf{B}_3$. Let $\mathcal{Q}$ be the Q-datum of Example 1.2.16. The indecomposable objects of $\mathcal{C}([\mathcal{Q}])$ are presented below. We place the object $M_\alpha^{[\mathcal{Q}]}$ at the vertex of $\Gamma_{\mathcal{Q}}$ corresponding to the positive root $\alpha$ via the isomorphism $\Gamma_{\mathcal{Q}} \cong \Upsilon_{[\mathcal{Q}]}$ of Theorem 1.2.17.

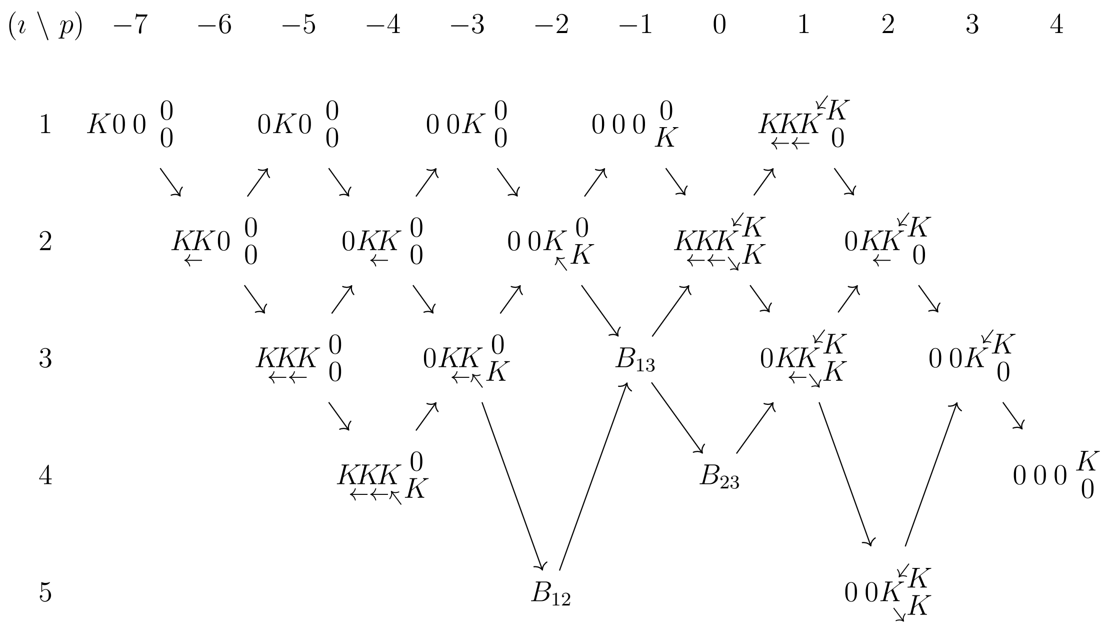
$$
\begin{tikzcd}[column sep=tiny]
(\imath \setminus p) & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & & & & & & \underleftarrow{KKKKK} \arrow[drr] & & & & & & \\
2 & & & & & \underleftarrow{KKKK}0 \arrow[urr] \arrow[dr] & & & & 0\underleftarrow{KKKK} \arrow[dr] & & & & \\
3 & & 0\,0K0\,0 \arrow[dr] & & \underleftarrow{KK}0\,0\,0 \arrow[ur] & & 0\,0\underleftarrow{KK}0 \arrow[dr] & & 0K0\,0\,0 \arrow[ur] & & 0\,0\underleftarrow{KKK} \arrow[dr] & & & \\
4 & & & \overrightarrow{\underleftarrow{KK}}K0\,0 \arrow[ur] \arrow[drr] & & & & 0\overrightarrow{\underleftarrow{KK}}K0 \arrow[ur] \arrow[drr] & & & & 0\,0\,0\underleftarrow{KK} \arrow[drr] & & \\
5 & K0\,0\,0\,0 \arrow[urr] & & & & 0\overrightarrow{KK}0\,0 \arrow[urr] & & & & 0\,0\,0K0 \arrow[urr] & & & & 0\,0\,0\,0K
\end{tikzcd}
$$

This picture agrees with Example 1.2.19. To compute the objects above, we used Proposition 1.3.6. See also Proposition 1.9.2. We remark that the arrows above can be seen as nonzero morphisms in $\mathcal{C}([\mathcal{Q}])$, as it will be explained in Section 1.5.

**Example 1.4.10.** Suppose $(\Delta, \sigma) = (\mathsf{D}_5, \vee)$ is of type $\mathsf{C}_4$. Let $Q$ be the following orientation for $\Delta$:

$$
\begin{tikzcd}
& & & 4 \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\
& & & 5
\end{tikzcd}
$$

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be the Q-datum defined by $\xi_k = k$ for $1 \le k \le 4$ and $\xi_5 = 2$. The indecomposable objects of $\mathcal{C}([\mathcal{Q}])$ are shown below.



The objects $B_{12}$, $B_{13}$ and $B_{23}$ are given by the following representations:

$$B_{12} = \begin{tikzcd} & & & K \arrow[dl, bend left, "i_2"] \\ K \arrow[r, bend left, "i_1"] & K^2 \arrow[l, bend left, "\pi_2"] \arrow[r, bend left, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & K^2 \arrow[l, bend left, "\mathrm{id}"] \arrow[ur, bend left, "0"] \arrow[dr, bend left, "\pi_2"] & \\ & & & K \arrow[ul, bend left, "i_1"] \end{tikzcd}$$

$$B_{13} = \begin{tikzcd} & & & K \arrow[dl, bend left, "i_2"] \\ K \arrow[r, bend left, "0"] & K \arrow[l, bend left, "\mathrm{id}"] \arrow[r, bend left, "i_1"] & K^2 \arrow[l, bend left, "\pi_2"] \arrow[ur, bend left, "0"] \arrow[dr, bend left, "\pi_2"] & \\ & & & K \arrow[ul, bend left, "i_1"] \end{tikzcd}$$

$$B_{23} = \begin{tikzcd} & & & K \arrow[dl, bend left, "i_2"] \\ 0 \arrow[r, bend left] & K \arrow[l, bend left] \arrow[r, bend left, "i_1"] & K^2 \arrow[l, bend left, "\pi_2"] \arrow[ur, bend left, "0"] \arrow[dr, bend left, "\pi_2"] & \\ & & & K \arrow[ul, bend left, "i_1"] \end{tikzcd}$$

## 1.4.2 The repetition category and tilting functors

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for the longest element $w_0$. We define the *repetition category* $\mathcal{R}([\boldsymbol{i}])$ of $[\boldsymbol{i}]$ as the full additive subcategory of $\mathrm{pvd}(\Pi_Q)$ generated by the objects of the form $\Sigma^k M$ for $k \in \mathbb{Z}$ and $M \in \mathcal{C}([\boldsymbol{i}])$. Denote by $\operatorname{ind}(\mathcal{R}([\boldsymbol{i}]))$ a set of representatives for the isomorphism classes of indecomposable objects of $\mathcal{R}([\boldsymbol{i}])$.

If $M \in \operatorname{ind}(\mathcal{R}([\boldsymbol{i}]))$ has cohomology concentrated in degree $k$, we set $c_{[\boldsymbol{i}]}(M) = ([M], -k) \in \widehat{\mathsf{R}}$, where $[M]$ denotes the class of $M$ in the Grothendieck group $K_0(\operatorname{pvd}(\Pi_Q))$. Here, we use the identification of $K_0(\operatorname{pvd}(\Pi_Q))$ with the root lattice of $\Delta$, and observe that $(-1)^{-k}[M]$ is indeed a positive root. We refer to $c_{[\boldsymbol{i}]}(M)$ as the *coordinate of* $M$ *in* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ and its residue as the *residue of* $M$. We remark that the function $c_{[\boldsymbol{i}]} : \operatorname{ind}(\mathcal{R}([\boldsymbol{i}])) \longrightarrow \widehat{\mathsf{R}}$ is a bijection. Moreover, the suspension functor $\Sigma$ sends the indecomposable object whose coordinate is $(\alpha, k) \in \widehat{\mathsf{R}}$ to the indecomposable object whose coordinate is $(-\alpha, k+1)$.

**Proposition 1.4.11.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. If $(i_1, \ldots, i_k)$ is a source sequence for $[\boldsymbol{i}]$, then the equivalence*

$$T_{i_k}^{-1} T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1} : \operatorname{pvd}(\Pi_Q) \longrightarrow \operatorname{pvd}(\Pi_Q)$$

*restricts to an equivalence from $\mathcal{R}([\boldsymbol{i}])$ to $\mathcal{R}([\boldsymbol{j}])$, where $[\boldsymbol{j}] = r_{i_k} r_{i_{k-1}} \cdots r_{i_1}[\boldsymbol{i}]$. Additionally, the following square is commutative:*

$$\begin{tikzcd}
\operatorname{ind}(\mathcal{R}([\boldsymbol{i}])) \arrow[r, "T_{i_k}^{-1} T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1}"] \arrow[d, "c_{[\boldsymbol{i}]}"'] & \operatorname{ind}(\mathcal{R}([\boldsymbol{j}])) \arrow[d, "c_{[\boldsymbol{j}]}"] \\
(\widehat{\Upsilon}_{[\boldsymbol{i}]})_0 \arrow[r] & (\widehat{\Upsilon}_{[\boldsymbol{j}]})_0
\end{tikzcd}$$

*where the bottom map comes from the isomorphism $\widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{[\boldsymbol{j}]}$ induced by the source sequence.*

*Proof.* Without loss of generality, suppose that $k = 1$. The first part can be obtained by applying Proposition 1.4.8 and using that $T_{i_1}^{-1} S_{i_1} \cong \Sigma S_{i_1}$. The second part follows from the explicit description of the bottom map given at the end of Section 1.2.3 and the fact that $T_{i_1}^{-1}$ acts as the simple reflection $s_{i_1}$ on $K_0(\operatorname{pvd}(\Pi_Q))$. $\checkmark$

An equivalence $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ is a *tilting functor* if it is a composition of equivalences as the ones appearing in Proposition 1.4.11 or their inverses. Note that a tilting functor is compatible with the coordinate functions in the sense of Proposition 1.4.11.

### 1.4.3 The (combinatorially) derived category

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for the longest element $w_0$. We will now define an ideal $\mathcal{I}_{[\boldsymbol{i}]}$ of the repetition category $\mathcal{R}([\boldsymbol{i}])$. For indecomposable objects $M, N \in \mathcal{R}([\boldsymbol{i}])$, we set $\mathcal{I}_{[\boldsymbol{i}]}(M, N) = 0$ if there is a tilting functor $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$

such that $T(M)$ and $T(N)$ are both concentrated in degree 0, that is, $T(M)$ and $T(N)$ belong to $\mathcal{C}([\boldsymbol{j}])$. Otherwise, we define

$$\mathcal{I}_{[\boldsymbol{i}]}(M,N) = \mathrm{Hom}_{\mathcal{R}([\boldsymbol{i}])}(M,N).$$

We extend $\mathcal{I}_{[\boldsymbol{i}]}$ to all objects by additivity. Let us prove that $\mathcal{I}_{[\boldsymbol{i}]}$ is indeed an ideal.

**Lemma 1.4.12.** *Suppose $\Delta \neq \mathsf{A}_1$. Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$ and take $M, M' \in \mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$. If there is a nonzero morphism $M \longrightarrow M'$, then there is a path from $\mathsf{c}_{[\boldsymbol{i}]}(M)$ to $\mathsf{c}_{[\boldsymbol{i}]}(M')$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$.*

*Proof.* By Proposition 1.4.11, after applying a suitable tilting functor, we may replace $[\boldsymbol{i}]$ by any other commutation class in the same $r$-cluster point. Hence, if we write $\boldsymbol{i} = (i_1, \ldots, i_N)$, we can assume that $c_{[\boldsymbol{i}]}(M') = \widehat{\beta}^{\boldsymbol{i}}_1 = (\alpha_{i_1}, 0)$ and that $i_1$ is the unique source of $[\boldsymbol{i}]$. In this case, $\alpha_{i_1}$ is the unique sink of $\Upsilon_{[\boldsymbol{i}]}$. Since $M'$ is concentrated in degree 0, the existence of a nonzero morphism $M \longrightarrow M'$ implies that $M$ is concentrated in a nonnegative degree, that is, $c_{[\boldsymbol{i}]}(M) = (\alpha, k)$ for some $k \leq 0$. Thus, by the segment property of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, there is a path from $c_{[\boldsymbol{i}]}(M)$ to $(\beta, 0)$ for some $\beta \in \mathsf{R}^+$. Since there is a path from $\beta$ to $\alpha_{i_1}$ in $\Upsilon_{[\boldsymbol{i}]}$, there is a path from $(\beta, 0)$ to $c_{[\boldsymbol{i}]}(M') = (\alpha_{i_1}, 0)$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, concluding the proof. $\surd$

**Proposition 1.4.13.** *For every commutation class $[\boldsymbol{i}]$ of reduced words for $w_0$, $\mathcal{I}_{[\boldsymbol{i}]}$ is an ideal of the category $\mathcal{R}([\boldsymbol{i}])$. Moreover, it is preserved by the functor $\Sigma$, and a tilting functor $\mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ sends $\mathcal{I}_{[\boldsymbol{i}]}$ to $\mathcal{I}_{[\boldsymbol{j}]}$.*

*Proof.* We give a proof assuming $\Delta \neq \mathsf{A}_1$, the other case being easy. Let $\varphi : M_1 \longrightarrow M_2$ and $\psi : M_2 \longrightarrow M_3$ be morphisms between indecomposable objects of $\mathcal{R}([\boldsymbol{i}])$. If $\psi\varphi \notin \mathcal{I}_{[\boldsymbol{i}]}(M_1, M_3)$, then we must have $\psi\varphi \neq 0$ and there must be a tilting functor $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ such that $T(M_1), T(M_3) \in \mathcal{C}([\boldsymbol{j}])$. In particular, $\varphi$ and $\psi$ are nonzero and, by Lemma 1.4.12, there is a path in $\widehat{\Upsilon}_{[\boldsymbol{j}]}$ from $c_{[\boldsymbol{j}]}(T(M_1))$ to $c_{[\boldsymbol{j}]}(T(M_3))$ passing by $c_{[\boldsymbol{j}]}(T(M_2))$. Since $\Upsilon_{[\boldsymbol{j}]}$ is convex in $\widehat{\Upsilon}_{[\boldsymbol{j}]}$, we deduce that $T(M_2) \in \mathcal{C}([\boldsymbol{j}])$. Therefore, we conclude that $\varphi \notin \mathcal{I}_{[\boldsymbol{i}]}(M_1, M_2) = 0$ and $\psi \notin \mathcal{I}_{[\boldsymbol{i}]}(M_2, M_3) = 0$. This proves that $\mathcal{I}_{[\boldsymbol{i}]}$ is an ideal of $\mathcal{R}([\boldsymbol{i}])$. The second part of the proposition is clear from the definition of $\mathcal{I}_{[\boldsymbol{i}]}$ and Remark 1.4.5. $\surd$

For a commutation class $[\boldsymbol{i}]$ of reduced words for $w_0$, we define its *combinatorially derived category* (or *c-derived category* for short) as the quotient $\mathcal{D}([\boldsymbol{i}])$ of $\mathcal{R}([\boldsymbol{i}])$ by the ideal $\mathcal{I}_{[\boldsymbol{i}]}$. It is still an additive $K$-linear category and has the same objects as $\mathcal{R}([\boldsymbol{i}])$. One can check that $\mathcal{D}([\boldsymbol{i}])$ and $\mathcal{R}([\boldsymbol{i}])$ also have the same indecomposable objects, so we will alternatively write $\mathrm{ind}(\mathcal{D}([\boldsymbol{i}]))$ for $\mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$. For $M, N \in \mathrm{ind}(\mathcal{D}([\boldsymbol{i}]))$, we have

$$\mathrm{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M,N) \cong \mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(M,N)$$

if there is a tilting functor $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ such that $T(M)$ and $T(N)$ belong to $\mathcal{C}([\boldsymbol{j}])$. Otherwise, the Hom-space from $M$ to $N$ in $\mathcal{D}([\boldsymbol{i}])$ is zero.

By the definition of $\mathcal{I}_{[\boldsymbol{i}]}$, the composition of the inclusion $\mathcal{C}([\boldsymbol{i}]) \subset \mathcal{R}([\boldsymbol{i}])$ with the quotient functor to $\mathcal{D}([\boldsymbol{i}])$ is a fully faithful functor. In this way, we shall identify $\mathcal{C}([\boldsymbol{i}])$ as a full subcategory of $\mathcal{D}([\boldsymbol{i}])$. By Proposition 1.4.13, the functor $\Sigma$ descends to an autoequivalence of $\mathcal{D}([\boldsymbol{i}])$, and any tilting functor $\mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ induces an equivalence $\mathcal{D}([\boldsymbol{i}]) \longrightarrow \mathcal{D}([\boldsymbol{j}])$ which we will also call a tilting functor.

The following result justifies the chosen construction for the c-derived category.

**Proposition 1.4.14.** *Let $\boldsymbol{i}$ be a reduced word for $w_0$ that is a source sequence for some orientation $Q$ of $\Delta$. We have an equivalence of $K$-linear categories*

$$\mathcal{D}^b(\operatorname{mod} KQ) \longrightarrow \mathcal{D}([\boldsymbol{i}])$$

*that commutes with the suspension functors. Moreover, tilting functors on $\mathcal{D}([\boldsymbol{i}])$ naturally correspond to compositions of derived reflection functors on $\mathcal{D}^b(\operatorname{mod} KQ)$.*

*Proof.* Since every indecomposable object of $\mathcal{D}^b(\operatorname{mod} KQ)$ is a shift of some indecomposable $KQ$-module, the essential image of the restriction functor $\mathcal{D}^b(\operatorname{mod} KQ) \longrightarrow \operatorname{pvd}(\Pi_{Q^\circ})$ is the repetition category $\mathcal{R}([\boldsymbol{i}])$ by Proposition 1.4.7. If $\mathcal{D}^b(\operatorname{mod} KQ) \longrightarrow \mathcal{R}([\boldsymbol{i}])$ is the functor obtained by restricting the codomain, it commutes with the suspension functors and, by Remark 1.3.7, it also has the last property stated above (if we replace $\mathcal{D}([\boldsymbol{i}])$ by $\mathcal{R}([\boldsymbol{i}])$). Hence, we only need to prove that its composition with the quotient functor $\mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{D}([\boldsymbol{i}])$ yields an equivalence.

Let $M$ and $N$ be two indecomposable objects of $\mathcal{D}^b(\operatorname{mod} KQ)$. We have to show that the map

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} KQ)}(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{R}([\boldsymbol{i}])}(M, N) \tag{$*$}$$

is a bijection if there is a tilting functor $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ such that $T(M)$ and $T(N)$ are concentrated in degree 0, and that

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{mod} KQ)}(M, N) = 0$$

otherwise. We argue in each case separately.

In the first case, Remark 1.3.7 gives the following diagram of functors (which is commutative up to a natural isomorphism):

$$\begin{tikzcd}
\mathcal{D}^b(\operatorname{mod} KQ) \arrow[r] \arrow[d, "T'"'] & \mathcal{R}([\boldsymbol{i}]) \arrow[d, "T"] \\
\mathcal{D}^b(\operatorname{mod} KQ') \arrow[r] & \mathcal{R}([\boldsymbol{j}])
\end{tikzcd}$$

The vertical functor $T'$ on the left is the composition of the derived reflection functors (as the ones appearing in Remark 1.3.7) corresponding to the sequence of sources/sinks that defines the tilting functor $T$. By Proposition 1.4.7, the horizontal functor on the bottom is fully faithful on the canonical heart of $\mathcal{D}^b(\operatorname{mod} KQ')$. Since $T'(M)$ and $T'(N)$ belong to this heart and the vertical functors are equivalences, we conclude that the map ($*$) is indeed a bijection.

In the second case, it is possible to find a composition of reflection functors $T'$ from $\mathcal{D}^b(\operatorname{mod} KQ)$ to $\mathcal{D}^b(\operatorname{mod} KQ')$ such that $T'(M)$ is an indecomposable projective $KQ'$-module. By hypothesis, $T'(N)$ is not concentrated in degree 0, so there are no morphisms from $T'(M)$ to $T'(N)$ in $\mathcal{D}^b(\operatorname{mod} KQ')$. Since $T'$ is an equivalence, there are no morphisms from $M$ to $N$ in $\mathcal{D}^b(\operatorname{mod} KQ)$, as desired. $\surd$

## 1.5 Irreducible morphisms and Gabriel quivers

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w \in \mathsf{W}$. By construction, the set of indecomposable objects of $\mathcal{C}([\boldsymbol{i}])$ is in bijection with $\mathsf{R}^+(w)$, which is the vertex set of the combinatorial AR quiver $\Upsilon_{[\boldsymbol{i}]}$. We will show that each arrow of $\Upsilon_{[\boldsymbol{i}]}$ gives rise to an essentially unique irreducible morphism in $\mathcal{C}([\boldsymbol{i}])$; hence, $\Upsilon_{[\boldsymbol{i}]}$ can be seen as a subquiver of the Gabriel quiver of $\mathcal{C}([\boldsymbol{i}])$. At the end of the section, we will consider the case $w = w_0$ and prove similar results for the categories $\mathcal{R}([\boldsymbol{i}])$ and $\mathcal{D}([\boldsymbol{i}])$.

If there is an arrow $\beta \longrightarrow \alpha$ in $\Upsilon_{[\boldsymbol{i}]}$ between two positive roots in $\mathsf{R}^+(w)$, then there is $\boldsymbol{j} \in [\boldsymbol{i}]$ such that $\alpha$ and $\beta$ are consecutive with respect to the order $<_{\boldsymbol{j}}$ (this follows from Theorem 1.2.2). In particular, if $\alpha$ and $\beta$ have residues $i$ and $j$ in $\Delta_0$, then

$$M_\alpha^{[\boldsymbol{i}]} \cong T(S_i) \quad \text{and} \quad M_\beta^{[\boldsymbol{i}]} \cong T(T_i(S_j)),$$

where $T$ is a composition of spherical twists. Since $i \sim j$, $\operatorname{Ext}^n_{\Pi_Q}(S_i, S_j)$ vanishes for all $n \neq 1$ and $\operatorname{Ext}^1_{\Pi_Q}(S_i, S_j)$ is one-dimensional. Thus, the triangle ($*$) in the definition of the spherical twist $T_i(S_j)$ becomes

$$\Sigma^{-1} S_i \longrightarrow S_j \longrightarrow T_i(S_j) \longrightarrow S_i$$

and $T_i(S_j)$ must be the unique nonsplit extension of $S_i$ by $S_j$. We deduce that the space $\operatorname{Hom}_{\operatorname{pvd}(\Pi_Q)}(T_i(S_j), S_i)$ is one-dimensional, and so is $\operatorname{Hom}_{\operatorname{pvd}(\Pi_Q)}(M_\beta^{[\boldsymbol{i}]}, M_\alpha^{[\boldsymbol{i}]})$ as $T$ is an equivalence of categories. In other words, given an arrow $\beta \longrightarrow \alpha$ in $\Upsilon_{[\boldsymbol{i}]}$, there is a unique nonzero map (up to multiplication by a nonzero scalar) from $M_\beta^{[\boldsymbol{i}]}$ to $M_\alpha^{[\boldsymbol{i}]}$ in $\mathcal{C}([\boldsymbol{i}])$.

This discussion motivates the definition of the following relation on $\operatorname{ind}([\boldsymbol{i}])$. For two indecomposable objects $M, N \in \operatorname{ind}([\boldsymbol{i}])$, consider the smallest transitive relation $\preceq_{\mathcal{C}([\boldsymbol{i}])}$

satisfying

$$\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(N, M) \neq 0 \implies M \preceq_{\mathcal{C}([\boldsymbol{i}])} N.$$

We call it the Hom-*order of* $\mathcal{C}([\boldsymbol{i}])$. Our next goal is to prove that $\preceq_{\mathcal{C}([\boldsymbol{i}])}$ is indeed a partial order and that the map $\alpha \mapsto M_\alpha^{[\boldsymbol{i}]}$ induces an isomorphism between the posets $(\mathsf{R}^+(w), \preceq_{[\boldsymbol{i}]})$ and $(\mathrm{ind}([\boldsymbol{i}]), \preceq_{\mathcal{C}([\boldsymbol{i}])})$.

**Lemma 1.5.1.** *Let* $\alpha, \beta \in \mathsf{R}^+(w)$*. If* $\alpha \not\preceq_{[\boldsymbol{i}]} \beta$*, then*

$$\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M_\beta^{[\boldsymbol{i}]}, M_\alpha^{[\boldsymbol{i}]}) = 0.$$

*Proof.* By the definition of $\preceq_{[\boldsymbol{i}]}$, there is $\boldsymbol{j} = (j_1, \dots, j_t)$ in $[\boldsymbol{i}]$ such that $\beta <_{\boldsymbol{j}} \alpha$. Let $1 \leq k < l \leq t$ be such that $\beta = \beta_k^{\boldsymbol{j}}$ and $\alpha = \beta_l^{\boldsymbol{j}}$, so that we have $M_\beta^{[\boldsymbol{i}]} \cong M_k^{\boldsymbol{j}}$ and $M_\alpha^{[\boldsymbol{i}]} \cong M_l^{\boldsymbol{j}}$. Thus, $\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M_\beta^{[\boldsymbol{i}]}, M_\alpha^{[\boldsymbol{i}]})$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M_k^{\boldsymbol{j}}, M_l^{\boldsymbol{j}}) = \mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(T_{j_1} \cdots T_{j_{k-1}}(S_{j_k}), T_{j_1} \cdots T_{j_{l-1}}(S_{j_l})).$$

Applying the functor $T_{j_k}^{-1} T_{j_{k-1}}^{-1} \cdots T_{j_1}^{-1}$ and recalling that $k < l$, the Hom-space above is isomorphic to

$$\mathrm{Hom}_{\mathrm{pvd}(\Pi_Q)}(T_{j_k}^{-1}(S_{j_k}), T_{j_{k+1}} \cdots T_{j_{l-1}}(S_{j_l})).$$

But $T_{j_k}^{-1}(S_{j_k}) \cong \Sigma S_{j_k}$ is concentrated in degree $-1$, while $T_{j_{k+1}} \cdots T_{j_{l-1}}(S_{j_l})$ is concentrated in degree 0 by Theorem 1.4.3. We deduce that the Hom-space above is zero. $\surd$

**Proposition 1.5.2.** *For* $\alpha, \beta \in \mathsf{R}^+(w)$*, we have*

$$\alpha \preceq_{[\boldsymbol{i}]} \beta \iff M_\alpha^{[\boldsymbol{i}]} \preceq_{\mathcal{C}([\boldsymbol{i}])} M_\beta^{[\boldsymbol{i}]}.$$

*In particular, the relation* $\preceq_{\mathcal{C}([\boldsymbol{i}])}$ *is a partial order on the set* $\mathrm{ind}([\boldsymbol{i}])$.

*Proof.* If $\alpha \preceq_{[\boldsymbol{i}]} \beta$, then Theorem 1.2.1 gives a path in $\Upsilon_{[\boldsymbol{i}]}$ from $\beta$ to $\alpha$. As argued before, each arrow on this path gives rise to a nonzero morphism in $\mathcal{C}([\boldsymbol{i}])$, so we have $M_\alpha^{[\boldsymbol{i}]} \preceq_{\mathcal{C}([\boldsymbol{i}])} M_\beta^{[\boldsymbol{i}]}$. Conversely, if $M_\alpha^{[\boldsymbol{i}]} \preceq_{\mathcal{C}([\boldsymbol{i}])} M_\beta^{[\boldsymbol{i}]}$, then there is a sequence $\alpha_1, \dots, \alpha_k \in \mathsf{R}^+(w)$ such that $\alpha_1 = \alpha$, $\alpha_k = \beta$ and

$$\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M_{\alpha_{l+1}}^{[\boldsymbol{i}]}, M_{\alpha_l}^{[\boldsymbol{i}]}) \neq 0$$

for all $1 \leq l < k$. By Lemma 1.5.1, we have $\alpha_l \preceq_{[\boldsymbol{i}]} \alpha_{l+1}$ for all $1 \leq l < k$, which gives $\alpha \preceq_{[\boldsymbol{i}]} \beta$ by transitivity. $\surd$

Together with Theorem 1.2.1, this result implies the following corollary.

**Corollary 1.5.3.** *The combinatorial AR quiver* $\Upsilon_{[\boldsymbol{i}]}$ *is isomorphic to the Hasse quiver of the* Hom-*order of* $\mathcal{C}([\boldsymbol{i}])$.

We now describe the relationship between $\Upsilon_{[\boldsymbol{i}]}$ and the Gabriel quiver of $\mathcal{C}([\boldsymbol{i}])$. Let us first recall some definitions. Let $\mathcal{C}$ be an additive $K$-linear Krull–Schmidt category and assume for simplicity that the radical quotient of the endomorphism algebra of any indecomposable object is isomorphic to $K$. We say that a morphism $\varphi : M \longrightarrow N$ in $\mathcal{C}$ is *irreducible* if $\varphi$ is neither a section nor a retraction and, for any factorization $\varphi = \psi_2\psi_1$, we have that $\psi_1$ is a section or $\psi_2$ is a retraction. If $M$ and $N$ are indecomposable, then $\varphi$ is irreducible if and only if $\varphi \in \operatorname{rad}_{\mathcal{C}}(M,N) \setminus \operatorname{rad}^2_{\mathcal{C}}(M,N)$, where $\operatorname{rad}_{\mathcal{C}}$ is the radical ideal of $\mathcal{C}$ (see [ARS97, Section V.7]). The *Gabriel quiver* $\Gamma(\mathcal{C})$ *of* $\mathcal{C}$ has as a set of vertices a set of representatives for the isomorphism classes of indecomposable objects in $\mathcal{C}$. The number of arrows in $\Gamma(\mathcal{C})$ from an indecomposable object $M$ to another indecomposable object $N$ is the dimension of the $K$-vector space $\operatorname{rad}_{\mathcal{C}}(M,N)/\operatorname{rad}^2_{\mathcal{C}}(M,N)$.

**Lemma 1.5.4.** *Let $\alpha, \beta \in \mathsf{R}^+(w)$. If there is an arrow $\beta \longrightarrow \alpha$ in $\Upsilon_{[\boldsymbol{i}]}$, then any nonzero morphism $\varphi : M_\beta^{[\boldsymbol{i}]} \longrightarrow M_\alpha^{[\boldsymbol{i}]}$ is irreducible in $\mathcal{C}([\boldsymbol{i}])$.*

*Proof.* Take a factorization $\varphi = \psi_2\psi_1$, where $\psi_1 : M_\beta^{[\boldsymbol{i}]} \longrightarrow X$ and $\psi_2 : X \longrightarrow M_\alpha^{[\boldsymbol{i}]}$ are morphisms in $\mathcal{C}([\boldsymbol{i}])$. Since $\varphi$ is nonzero, there is at least one indecomposable summand $Y$ of $X$ such that the corresponding components $\psi_1' : M_\beta^{[\boldsymbol{i}]} \longrightarrow Y$ and $\psi_2' : Y \longrightarrow M_\alpha^{[\boldsymbol{i}]}$ of $\psi_1$ and $\psi_2$ are both nonzero. We have $Y \cong M_\gamma^{[\boldsymbol{i}]}$ for some $\gamma \in \mathsf{R}^+(w)$, and the existence of these nonzero morphisms gives $\alpha \preceq_{[\boldsymbol{i}]} \gamma \preceq_{[\boldsymbol{i}]} \beta$ by Proposition 1.5.2. Since there is an arrow $\beta \longrightarrow \alpha$ in $\Upsilon_{[\boldsymbol{i}]}$, we must have $\gamma = \alpha$ or $\gamma = \beta$. This implies that either $\psi_1'$ or $\psi_2'$ is an isomorphism because the endomorphism algebra of any indecomposable object in $\mathcal{C}([\boldsymbol{i}])$ is one-dimensional. We deduce that $\psi_1$ is a section or $\psi_2$ is a retraction. Finally, since $M_\alpha^{[\boldsymbol{i}]}$ and $M_\beta^{[\boldsymbol{i}]}$ are nonisomorphic indecomposable objects, $\varphi$ is neither a section nor a retraction. $\surd$

For the next result, we will say that an arrow $\alpha : x \longrightarrow y$ in an acyclic quiver is *superfluous* if there is a path from $x$ to $y$ with length strictly greater than one.

**Theorem 1.5.5.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w \in \mathsf{W}$. The combinatorial AR quiver $\Upsilon_{[\boldsymbol{i}]}$ is isomorphic to the quiver obtained from the Gabriel quiver $\Gamma(\mathcal{C}([\boldsymbol{i}]))$ by removing all superfluous arrows.*

*Proof.* By Lemma 1.5.4, $\Upsilon_{[\boldsymbol{i}]}$ can be seen as a subquiver of $\Gamma(\mathcal{C}([\boldsymbol{i}]))$. It contains all vertices. On the other hand, since $\Delta$ is an acyclic graph, $\Upsilon_{[\boldsymbol{i}]}$ does not have superfluous arrows by construction. Given an arrow $\varphi : M_\alpha^{[\boldsymbol{i}]} \longrightarrow M_\beta^{[\boldsymbol{i}]}$ of $\Gamma(\mathcal{C}([\boldsymbol{i}]))$ which is not superfluous, let us show that it comes from an arrow in $\Upsilon_{[\boldsymbol{i}]}$. First, we have $\alpha \neq \beta$ as any nontrivial endomorphism of an indecomposable object in $\mathcal{C}([\boldsymbol{i}])$ is invertible. By Proposition 1.5.2, we have $\beta \preceq_{[\boldsymbol{i}]} \alpha$ and, by Theorem 1.2.1, there is a path from $\alpha$ to $\beta$ in $\Upsilon_{[\boldsymbol{i}]}$. Any such path has to be of length one because, otherwise, we would get by Lemma 1.5.4 a path in $\Gamma(\mathcal{C}([\boldsymbol{i}]))$ of length greater than one from $M_\alpha^{[\boldsymbol{i}]}$ to $M_\beta^{[\boldsymbol{i}]}$, contradicting the hypothesis that $\varphi$ is not superfluous. In other words, there is an edge in $\Upsilon_{[\boldsymbol{i}]}$ from $\alpha$ to $\beta$. It corresponds to an edge $\psi : M_\alpha^{[\boldsymbol{i}]} \longrightarrow M_\beta^{[\boldsymbol{i}]}$ in $\Gamma(\mathcal{C}([\boldsymbol{i}]))$. Since $\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M_\alpha^{[\boldsymbol{i}]}, M_\beta^{[\boldsymbol{i}]})$ is one-dimensional, there is at most one arrow from $M_\alpha^{[\boldsymbol{i}]}$ to $M_\beta^{[\boldsymbol{i}]}$ in $\Gamma(\mathcal{C}([\boldsymbol{i}]))$; thus, $\varphi = \psi$ and the proof is completed. $\surd$

Suppose now $w = w_0$ is the longest element. Let $\varphi : (\alpha, k) \longrightarrow (\beta, l)$ be an arrow in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. By applying combinatorial reflection functors to $[\boldsymbol{i}]$, we can find another commutation class $[\boldsymbol{j}]$ and an isomorphism $\widehat{\Upsilon}_{[\boldsymbol{i}]} \longrightarrow \widehat{\Upsilon}_{[\boldsymbol{j}]}$ that sends $\varphi$ to an arrow in $\Upsilon_{[\boldsymbol{j}]}$. Using the previous discussion and Proposition 1.4.11, we deduce that

$$\dim_K \mathrm{Hom}_{\mathcal{R}([\boldsymbol{i}])}(M, N) = \dim_K \mathrm{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M, N) = 1,$$

where $M$ and $N$ are the indecomposable objects in $\mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$ with coordinates $(\alpha, k)$ and $(\beta, l)$, respectively. Therefore, as before, any arrow in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ gives rise to a unique nonzero map in $\mathcal{R}([\boldsymbol{i}])$ and in $\mathcal{D}([\boldsymbol{i}])$ (up to multiplication by a nonzero scalar) between the corresponding indecomposable objects.

We define the Hom-order of $\mathcal{R}([\boldsymbol{i}])$ and $\mathcal{D}([\boldsymbol{i}])$ in the same way as we did it for $\mathcal{C}([\boldsymbol{i}])$. These are relations $\preceq_{\mathcal{R}([\boldsymbol{i}])}$ and $\preceq_{\mathcal{D}([\boldsymbol{i}])}$ on $\mathrm{ind}(\mathcal{R}(\boldsymbol{i}))$.

**Proposition 1.5.6.** *Suppose $\Delta \neq \mathsf{A}_1$. If $M, N \in \mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$, then*

$$N \preceq_{\mathcal{R}([\boldsymbol{i}])} M \iff \textit{there is a path from } \mathsf{c}_{[\boldsymbol{i}]}(M) \textit{ to } \mathsf{c}_{[\boldsymbol{i}]}(N) \textit{ in } \widehat{\Upsilon}_{[\boldsymbol{i}]} \iff N \preceq_{\mathcal{D}([\boldsymbol{i}])} M.$$

*In particular, the relations $\preceq_{\mathcal{R}([\boldsymbol{i}])}$ and $\preceq_{\mathcal{D}([\boldsymbol{i}])}$ on the set $\mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$ coincide and are partial orders.*

*Proof.* The reasoning in the proof of Proposition 1.5.2 works here after replacing Lemma 1.5.1 by Lemma 1.4.12. $\surd$

After adapting the proofs of Lemma 1.5.4 and Theorem 1.5.5, we obtain the following result.

**Theorem 1.5.7.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for the longest element $w_0$. The combinatorial repetition quiver $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ is isomorphic to the quiver obtained from the Gabriel quiver $\Gamma(\mathcal{D}([\boldsymbol{i}]))$ by removing all superfluous arrows. If $\Delta \neq \mathsf{A}_1$, the same result holds if we replace $\mathcal{D}([\boldsymbol{i}])$ by $\mathcal{R}([\boldsymbol{i}])$.*

## 1.6 Meshes and distinguished triangles

We will now investigate the relationship between the triangulated structure of $\mathrm{pvd}(\Pi_Q)$ and the categories $\mathcal{R}([\boldsymbol{i}])$ and $\mathcal{D}([\boldsymbol{i}])$ for a commutation class $[\boldsymbol{i}]$ of reduced words for the longest element $w_0$. The main result of this section is Theorem 1.6.5, which states that certain distinguished triangles with corners in $\mathcal{R}([\boldsymbol{i}])$ can be obtained by looking at the meshes of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. As a result, we give an alternative proof for the $\mathfrak{g}$-additive property of [FO21] (Theorem 1.2.25) and generalize it to any commutation class.

As in Section 1.2.4, we assume that $\Delta \neq \mathsf{A}_1$.

**Lemma 1.6.1.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. Suppose $\varphi : M' \longrightarrow M$ is an irreducible morphism of $\mathcal{R}([\boldsymbol{i}])$ corresponding to an arrow in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. Then, the cocone of $\varphi$ in* $\mathrm{pvd}(\Pi_Q)$ *is indecomposable and belongs to $\mathcal{R}([\boldsymbol{i}])$.*

*Proof.* Let $\boldsymbol{i} = (i_1, \ldots, i_N)$ be a reduced word in $[\boldsymbol{i}]$. After applying a tilting functor, we can suppose that $M \cong M_1^{\boldsymbol{i}} = S_{i_1}$, $M' \cong M_2^{\boldsymbol{i}} = T_{i_1}(S_{i_2})$ and $i_1 \sim i_2$. As argued in Section 1.5, the distinguished triangle $(*)$ in the definition of the spherical twist $T_{i_1}(S_{i_2})$ becomes

$$\Sigma^{-1} S_{i_1} \longrightarrow S_{i_2} \longrightarrow T_{i_1}(S_{i_2}) \longrightarrow S_{i_1}.$$

Up to multiplication by a nonzero scalar, the map $T_{i_1}(S_{i_2}) \longrightarrow S_{i_1}$ in the triangle above is $\varphi$. We conclude that the cocone of $\varphi$ in $\mathrm{pvd}(\Pi_Q)$ is isomorphic to $S_{i_2}$, which is indeed in $\mathcal{R}([\boldsymbol{i}])$ by Corollary 1.4.6. $\surd$

**Remark 1.6.2.** With the notation above, suppose we have $M \cong M_k^{\boldsymbol{i}}$ and $M' \cong M_{k+1}^{\boldsymbol{i}}$ for some $1 \leq k < N$. By keeping track of the tilting functor used to reduce the proof above, one can easily check that the cocone of any nonzero morphism $M' \longrightarrow M$ is isomorphic to $X = T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(S_{i_{k+1}})$. In particular, $X \in \mathcal{R}([\boldsymbol{j}])$ for any reduced word $\boldsymbol{j}$ which starts with the sequence $(i_1, \ldots, i_{k-1})$ since $T_{i_1} T_{i_2} \cdots T_{i_{k-1}}$ is a tilting functor from $\mathcal{R}(r_{i_{k-1}} \cdots r_{i_1}[\boldsymbol{j}])$ to $\mathcal{R}([\boldsymbol{j}])$ and $S_{i_{k+1}} \in \mathcal{R}(r_{i_{k-1}} \cdots r_{i_1}[\boldsymbol{j}])$.

**Lemma 1.6.3.** *Let $\boldsymbol{i} = (i_1, \ldots, i_N)$ be a reduced word for $w_0$ and suppose that there is $1 < k < N$ such that $i_{k-1} = i_{k+1} \sim i_k$. Then there is a nonsplit distinguished triangle*

$$M_{k+1}^{\boldsymbol{i}} \longrightarrow M_k^{\boldsymbol{i}} \longrightarrow M_{k-1}^{\boldsymbol{i}} \longrightarrow \Sigma M_{k+1}^{\boldsymbol{i}}$$

*in* $\mathrm{pvd}(\Pi_Q)$.

*Proof.* By applying a tilting functor, we may assume $k = 2$. In this case, $M_3^{\boldsymbol{i}} = T_{i_1} T_{i_2}(S_{i_1}) \cong S_{i_2}$, $M_2^{\boldsymbol{i}} = T_{i_1}(S_{i_2})$, and $M_1^{\boldsymbol{i}} = S_{i_1}$. Therefore, we obtain the triangle in the statement by rotating the triangle in the proof of Lemma 1.6.1. $\surd$

**Lemma 1.6.4.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. Suppose that there is a distinguished triangle*

$$M \xrightarrow{\varphi} M' \xrightarrow{\varphi'} M'' \xrightarrow{\varphi''} \Sigma M$$

*in* $\mathrm{pvd}(\Pi_Q)$ *such that* $M, M', M'' \in \mathrm{ind}([\boldsymbol{i}])$*. If the three maps above are nonzero, then they do not belong to the ideal* $\mathcal{I}_{[\boldsymbol{i}]}$ *of* $\mathcal{R}([\boldsymbol{i}])$*.*

*Proof.* By the definition of $\mathcal{I}_{[\boldsymbol{i}]}$ and our hypotheses, we have $\mathcal{I}_{[\boldsymbol{i}]}(M, M') = \mathcal{I}_{[\boldsymbol{i}]}(M', M'') = 0$, so $\varphi$ and $\varphi'$ do not belong to $\mathcal{I}_{[\boldsymbol{i}]}$. By Lemma 1.4.12, there is a path in $\Upsilon_{[\boldsymbol{i}]}$ from $c_{[\boldsymbol{i}]}(M)$ to $c_{[\boldsymbol{i}]}(M'')$. Observe that these coordinates are distinct since no indecomposable object in $\mathcal{R}([\boldsymbol{i}])$ has a nonsplit self-extension. Consequently, there is a source sequence of $[\boldsymbol{i}]$ such that the induced tilting functor $T : \mathcal{R}([\boldsymbol{i}]) \longrightarrow \mathcal{R}([\boldsymbol{j}])$ satisfies $\Sigma^{-1}T(M''), T(M) \in \mathcal{C}([\boldsymbol{j}])$. We conclude that $\mathcal{I}_{[\boldsymbol{i}]}(M'', \Sigma M) = 0$, finishing the proof. $\surd$

Recall from Section 1.2.4 the notion of a mesh and the related definitions. For $M \in \mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$, denote by $\mathcal{M}_{[\boldsymbol{i}]}(M)$ the mesh of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ at the vertex $x = c_{[\boldsymbol{i}]}(M)$. Let $s_{[\boldsymbol{i}]}(M) \in \mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$ be the indecomposable object whose coordinate is $s_{[\boldsymbol{i}]}(x)$. We also define the *set of abutters* $V_{[\boldsymbol{i}]}(M)$ *of* $M$ to be the set of all $X \in \mathrm{ind}(\mathcal{R}([\boldsymbol{i}]))$ whose coordinate is an abutter of $x$, that is, $c_{[\boldsymbol{i}]}(X) \in V_{[\boldsymbol{i}]}(x)$. An ordering $X_1, \ldots, X_t$ of the elements in $V_{[\boldsymbol{i}]}(M)$ is *anticompatible* if $(c_{[\boldsymbol{i}]}(X_t), \ldots, c_{[\boldsymbol{i}]}(X_2), c_{[\boldsymbol{i}]}(X_1))$ is a compatible reading of $V_{[\boldsymbol{i}]}(x) \subset \widehat{\Upsilon}_{[\boldsymbol{i}]}$.

**Theorem 1.6.5.** *Let* $[\boldsymbol{i}]$ *be a commutation class of reduced words for* $w_0$ *and take an indecomposable object* $M \in \mathcal{R}([\boldsymbol{i}])$*. Let* $X_1, \ldots, X_t$ *be an anticompatible ordering of the set of abutters* $V_{[\boldsymbol{i}]}(M)$ *of* $M$*. Then, there are indecomposable objects* $Y_1 = s_{[\boldsymbol{i}]}(M)$*,* $Y_2$*,* $\ldots$*,* $Y_t$*,* $Y_{t+1} = \Sigma^{-1}M$ *in* $\mathcal{R}([\boldsymbol{i}])$ *and distinguished triangles in* $\mathrm{pvd}(\Pi_Q)$ *of the form*

$$Y_{k+1} \longrightarrow Y_k \longrightarrow X_k \longrightarrow \Sigma Y_{k+1}$$

*for all* $1 \leq k \leq t$*. Moreover, none of the morphisms in the triangles above belongs to the ideal* $\mathcal{I}_{[\boldsymbol{i}]}$ *of* $\mathcal{R}([\boldsymbol{i}])$*.*

*Proof.* Let $\mathcal{M} = \mathcal{M}_{[\boldsymbol{i}]}(M)$. By Lemma 1.2.5, we can apply a tilting functor and assume that $\mathcal{M} \subseteq \Upsilon_{[\boldsymbol{i}]}$. Moreover, since $\mathcal{M}$ is convex in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, we can also assume there is a representative $\boldsymbol{i} = (i_1, \ldots, i_N)$ of $[\boldsymbol{i}]$ that starts with the residues $(i_1, \ldots, i_k)$ of a compatible reading of the vertex set $\mathcal{M}_0$. By the hypothesis on the ordering of $V_{[\boldsymbol{i}]}(M)$, we can further assume that, if $c_{[\boldsymbol{i}]}(X_l) = \widehat{\beta}^{\boldsymbol{i}}_m$ and $c_{[\boldsymbol{i}]}(X_{l'}) = \widehat{\beta}^{\boldsymbol{i}}_{m'}$, then $l < l'$ implies $m > m'$. In particular, since $i_2 \sim i_1 = i_k \sim i_{k-1}$, we have $X_t \cong M^{\boldsymbol{i}}_2$ and $X_1 \cong M^{\boldsymbol{i}}_{k-1}$. Since $M^{\boldsymbol{i}}_k = s_{[\boldsymbol{i}]}(M) = Y_1$, we have an irreducible morphism $Y_1 \longrightarrow X_1$ corresponding to the arrow $\beta^{\boldsymbol{i}}_k \longrightarrow \beta^{\boldsymbol{i}}_{k-1}$ in $\Upsilon_{[\boldsymbol{i}]}$. By Lemma 1.6.1, it is part of a distinguished triangle

$$Y_2 \longrightarrow Y_1 \longrightarrow X_1 \longrightarrow \Sigma Y_2$$

where $Y_2 \in \mathcal{R}([\boldsymbol{i}])$ is indecomposable. By Lemma 1.2.6 (and assuming that $k > 3$), the sequence $(i_1, \ldots, i_{k-1}, i_k, i_{k-1})$ represents a reduced word, which we can extend to a reduced word $\boldsymbol{j}$ for $w_0$. By Lemma 1.6.3, we have $Y_2 \cong M^{\boldsymbol{j}}_{k+1}$. This implies that $Y_2$ is concentrated in degree 0 and, by Lemma 1.6.4, the three morphisms in the triangle above do not belong to $\mathcal{I}_{[\boldsymbol{i}]}$. This argument completes the construction of the first triangle.

By applying a braid relation, we can transform $\boldsymbol{j}$ into a new reduced word $\boldsymbol{i'}$ that starts with the sequence $(i_1, \ldots, i_{k-2}, i_k, i_{k-1}, i_k)$. We still have $M = S_{i_1} \in \mathcal{C}([\boldsymbol{i'}])$, and so, we can work with the mesh $\mathcal{M}' = \mathcal{M}_{[\boldsymbol{i'}]}(M)$. It is not hard to see that, as before, there is a reduced word in $[\boldsymbol{i'}]$ that starts with the residues of a compatible reading of $\mathcal{M}'_0$. Notice that $V_{[\boldsymbol{i'}]}(M) = V_{[\boldsymbol{i}]}(M) \setminus \{X_1\}$ and that $X_2, \ldots, X_t$ is still an anticompatible ordering of $V_{[\boldsymbol{i'}]}(M)$. Additionally, we have

$$s_{[\boldsymbol{i'}]}(M) = M^{\boldsymbol{i'}}_{k-1} = T_{i_1} \cdots T_{i_{k-2}}(S_{i_k}) \cong T_{i_1} \cdots T_{i_{k-2}}(T_{i_{k-1}} T_{i_k}(S_{i_{k-1}})) = M^{\boldsymbol{j}}_{k+1} \cong Y_2.$$

Therefore, we have essentially the same conditions as in the previous paragraph, and a similar argument gives an indecomposable object $Y_3 \in \mathcal{C}([\boldsymbol{i'}])$ and a distinguished triangle

$$Y_3 \longrightarrow Y_2 \longrightarrow X_2 \longrightarrow \Sigma Y_3.$$

Using Remark 1.6.2, one can show that $Y_3$ also belongs to $\mathcal{C}([\boldsymbol{i}])$. Again, by Lemma 1.6.4, the three morphisms in this new triangle do not belong to $\mathcal{I}_{[\boldsymbol{i}]}$.

In this way, we can continue by finding new reduced words for $w_0$ for which the set of abutters of $M$ has one element less at each step. We obtain the first $t-1$ triangles from the statement along the way. At the end, we find a reduced word $\boldsymbol{i''}$ starting with the residues of a compatible reading of the mesh of $M$ such that $V_{[\boldsymbol{i''}]}(M) = \{X_t\}$ and $s_{[\boldsymbol{i''}]}(M) \cong Y_t$. In this case, $\boldsymbol{i''}$ has to start with the sequence $(i_1, i_2, i_1)$, otherwise $M$ would have more abutters. By Lemma 1.6.3, we get the distinguished triangle

$$Y_t \longrightarrow X_t \longrightarrow M \longrightarrow \Sigma Y_t,$$

which gives the final triangle of the statement after a rotation. Since the last three terms of this final triangle are concentrated in degree 0, an analogous result to Lemma 1.6.4 implies that its morphisms do not lie in $\mathcal{I}_{[\boldsymbol{i}]}$. $\surd$

**Example 1.6.6.** Let $\mathcal{Q} = (\mathsf{A}_5, \vee, \xi)$ be the Q-datum of type $\mathsf{B}_3$ from Example 1.4.9. Let $M \in \mathcal{C}([\mathcal{Q}])$ be the indecomposable object with coordinate $(4, 6)$. We highlight in the diagram below the mesh determined by $M$ and color in red its abutters.

$$\begin{tikzcd}[column sep=tiny]
(\imath \setminus p) & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & & & & & & \underset{\leftarrow\leftarrow\leftarrow\leftarrow}{KKKKK} \arrow[drr] & & & & & & \\
2 & & & & & \underset{\leftarrow\leftarrow\leftarrow}{KKKK}0 \arrow[urr] \arrow[dr] & & & & 0\underset{\leftarrow\leftarrow\leftarrow}{KKKK} \arrow[dr] & & & & \\
3 & & 00K00 \arrow[dr] & & \underset{\leftarrow}{KK}000 \arrow[ur] & & 00\underset{\leftarrow}{KK}0 \arrow[dr] & & \color{red}{0K000} \arrow[ur] & & \color{red}{00\underset{\leftarrow\leftarrow}{KKK}} \arrow[dr] & & & \\
4 & & & \underset{\leftarrow}{\overset{\rightarrow}{KKK}}00 \arrow[ur] \arrow[drr] & & & & 0\underset{\leftarrow}{\overset{\rightarrow}{KKK}}0 \arrow[ur] \arrow[drr] & & & & 000\underset{\leftarrow}{KK} \arrow[drr] & & \\
5 & K0000 \arrow[urr] & & & & 0\overset{\rightarrow}{KK}00 \arrow[urr] & & & & \color{red}{000K0} \arrow[urr] & & & & 0000K
\end{tikzcd}$$

An example of anticompatible ordering for $V_{[\mathcal{Q}]}(M)$ is

$$000K0\,, \quad 0K000\,, \quad 00\underset{\leftarrow\leftarrow}{KKK}\,.$$

The distinguished triangles in $\mathrm{pvd}(\Pi_Q)$ given by Theorem 1.6.5 that correspond to this anticompatible ordering are of the following form:

$$0\overset{\rightarrow}{KK}00 \longrightarrow 0\underset{\leftarrow}{\overset{\rightarrow}{KKK}}0 \longrightarrow 000K0 \longrightarrow \Sigma 0\overset{\rightarrow}{KK}00\,,$$

$$00K00 \longrightarrow 0\overset{\rightarrow}{KK}00 \longrightarrow 0K000 \longrightarrow \Sigma 00K00\,,$$

$$\Sigma^{-1}000\underset{\leftarrow}{KK} \longrightarrow 00K00 \longrightarrow 00\underset{\leftarrow\leftarrow}{KKK} \longrightarrow 000\underset{\leftarrow}{KK}\,.$$

**Remark 1.6.7.** When $\boldsymbol{i}$ is a source sequence for an orientation $Q$ of $\Delta$, we know from Propositions 1.4.7 and 1.4.14 that $\mathcal{C}([\boldsymbol{i}]) \cong \operatorname{mod} KQ$ and $\mathcal{D}([\boldsymbol{i}]) \cong \mathcal{D}^b(\operatorname{mod} KQ)$. Hence, $\mathcal{C}([\boldsymbol{i}])$ is an abelian category and $\mathcal{D}([\boldsymbol{i}])$ has a triangulated structure. The latter has AR triangles (in the sense of [Hap88]), which we can obtain by assembling the distinguished triangles of Theorem 1.6.5 into a single one.

One may wonder if we have similar results for general commutation classes. A quick inspection of examples shows that $\mathcal{C}([\boldsymbol{i}])$ is not abelian in general. However, one may still search for a nontrivial extriangulated structure (in the sense of [NP19]) on $\mathcal{D}([\boldsymbol{i}])$ for which $\mathcal{C}([\boldsymbol{i}])$ is an extension-closed subcategory. It would be desirable for this structure to take into account the triangles of Theorem 1.6.5, which can be naturally seen as triangles in $\mathcal{D}([\boldsymbol{i}])$. Unfortunately, we were not able to find such a structure. A sign against its existence is given in Remark 1.7.5.

**Corollary 1.6.8.** *Let* $[\boldsymbol{i}]$ *be a commutation class of reduced words for* $w_0$. *For* $x \in$

$(\widehat{\Upsilon}_{[\boldsymbol{i}]})_0 = \widehat{\mathsf{R}}$, *we have*

$$\pi(x) + \pi(s_{[\boldsymbol{i}]}(x)) = \sum_{y \in V_{[\boldsymbol{i}]}(x)} \pi(y),$$

*where* $\pi : \widehat{\mathsf{R}} \longrightarrow \mathsf{R}$ *denotes the projection onto the first coordinate.*

*Proof.* Let $M \in \mathcal{R}([\boldsymbol{i}])$ be the indecomposable object of coordinate $x$. After choosing an anticompatible reading for $V_{[\boldsymbol{i}]}(M)$, we obtain the formula by summing the relations in $K_0(\mathrm{pvd}(\Pi_Q))$ induced by the distinguished triangles from Theorem 1.6.5. $\surd$

**Remark 1.6.9.** When $[\boldsymbol{i}]$ is a commutation class coming from a Q-datum for $\mathfrak{g}$, the formula above specializes to the $\mathfrak{g}$-additive property of [FO21] (see Theorem 1.2.25). For general $[\boldsymbol{i}]$, this result was already implicit in the literature. Indeed, by considering the subword of $\boldsymbol{i}$ corresponding to a given mesh, one obtains from [KKOP24a, Theorem 4.25] a nonsplit exact sequence between certain modules over a quantum affine algebra, known as a generalized T-system. These modules must belong to the same block in the block decomposition of the category of finite-dimensional integrable modules over this algebra (see [KKOP22]), which yields the equality in Corollary 1.6.8. Similarly, one can use the recent paper [KKOP25] to extend the above result to words that are not necessarily reduced.

We end this section with an application to frieze patterns. We say that a function $f : (\widehat{\Upsilon}_{[\boldsymbol{i}]})_0 \to \mathbb{Z}$ is an *additive frieze on* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ if it satisfies the relation

$$f(x) + f(s_{[\boldsymbol{i}]}(x)) = \sum_{y \in V_{[\boldsymbol{i}]}(x)} f(y)$$

for all $x \in (\widehat{\Upsilon}_{[\boldsymbol{i}]})_0$. When $[\boldsymbol{i}]$ is the commutation class associated with an orientation $Q$ of $\Delta$, this definition specializes to that of an additive function on the repetition quiver of $Q$ (in the sense of [Gab80]). Thus, our definition can be seen as an additive analog of Coxeter's frieze patterns [Cox71]. We refer the reader to [Mor15] for a recent survey on other frieze patterns appearing in representation theory.

For $i \in \Delta_0$, let $\rho_i : \mathsf{R} \to \mathbb{Z}$ be the function that returns the coefficient of $\alpha_i$ in a root $\alpha \in \mathsf{R}$ when written as a linear combination of the simple roots. By Corollary 1.6.8, the composition $\rho_i \circ \pi$ is an additive frieze on $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. We have the following result.

**Proposition 1.6.10.** *Let* $\boldsymbol{i} = (i_1, \ldots, i_N)$ *be a reduced word for* $w_0$. *The functions* $\rho_i \circ \pi$ *for* $i \in \Delta_0$ *form a* $\mathbb{Z}$*-basis of the space of additive friezes on* $\widehat{\Upsilon}_{[\boldsymbol{i}]}$. *Consequently, any additive frieze* $f : (\widehat{\Upsilon}_{[\boldsymbol{i}]})_0 \to \mathbb{Z}$ *satisfies* $f(\widehat{\beta}^{\boldsymbol{i}}_{k+N}) = -f(\widehat{\beta}^{\boldsymbol{i}}_k)$ *for* $k \in \mathbb{Z}$ *and is thus periodic.*

*Proof.* Denote by $x_j$ the injective vertex of $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ associated with $j \in \Delta_0$. Let $M$ be the square matrix indexed by $\Delta_0$ whose $(i, j)$-entry is $\rho_i(\pi(x_j))$. It is not hard to see that any additive frieze $f : (\widehat{\Upsilon}_{[\boldsymbol{i}]})_0 \to \mathbb{Z}$ is determined by the values $f(x_j)$ for $j \in \Delta_0$, which can be chosen to be arbitrary. Therefore, to prove the first statement, it suffices to show that $M$ is invertible over the integers. This is simple: if we order $\Delta_0$ so that $i < j$ whenever there is a path from $x_j$ to $x_i$ in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, then $M$ becomes upper unitriangular hence invertible. The second statement holds for the functions $\rho_i \circ \pi$ since $\pi(\widehat{\beta}^{\boldsymbol{i}}_{k+N}) = -\pi(\widehat{\beta}^{\boldsymbol{i}}_k)$ for all $k \in \mathbb{Z}$, so it must hold for all additive friezes. $\surd$

## 1.7 Extension groups and the Euler form

In this section, we define extension groups and use them to construct a sort of Euler form on the c-derived category of a commutation class. The main result is that the symmetrization of this Euler form agrees with the symmetric bilinear form of the root lattice (Theorem 1.7.6). Along the way, we also introduce the concept of projective and injective objects.

### 1.7.1 Projective/injective objects and extension groups

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for the longest element $w_0$. The *injective indecomposable object* $I_i^{[\boldsymbol{i}]}$ *associated with* $i \in \Delta_0$ is the indecomposable object of $\mathcal{C}([\boldsymbol{i}])$ whose coordinate is the injective vertex associated with $i$ (see Sections 1.2.2 and 1.2.3). Dually, the *projective indecomposable object* $P_i^{[\boldsymbol{i}]}$ *associated with* $i \in \Delta_0$ is the indecomposable object of $\mathcal{C}([\boldsymbol{i}])$ whose coordinate is the projective vertex associated with $i$. Notice that $I_i^{[\boldsymbol{i}]}$ has residue $i$, while $P_i^{[\boldsymbol{i}]}$ has residue $i^*$.

**Remark 1.7.1.** Let $\boldsymbol{i}$ be a reduced word for $w_0$ that is a source sequence for an orientation $Q$ of $\Delta$. By Proposition 1.4.7, $\mathcal{C}([\boldsymbol{i}])$ and mod $KQ$ are equivalent categories. In this case, the usual notion of projective/injective indecomposable objects in mod $KQ$ coincides with our definition. This follows by directly computing the projective/injective positive roots with respect to $[\boldsymbol{i}]$.

By Theorem 1.4.3, we can see any object of $\mathcal{C}([\boldsymbol{i}])$ as a module over the preprojective algebra $\Lambda_Q$. In particular, we can see it as a representation of the double quiver $\overline{Q}$ of $Q$. For $M \in \mathcal{C}([\boldsymbol{i}])$ and $i \in \Delta_0$, we denote by $M(i)$ the $K$-vector space at the vertex $i$ if we see $M$ as a representation of $\overline{Q}$. Recall from Section 1.3.3 (when we defined the reflection functor $\Sigma_i$) that we also have a diagram

$$\widetilde{M}(i) \xrightarrow{M_{\mathrm{in}(i)}} M(i) \xrightarrow{M_{\mathrm{out}(i)}} \widetilde{M}(i)$$

describing the representation around the vertex $i$.

The next result follows from [AIRT12, Proposition 2.4]. We provide a different proof.

**Lemma 1.7.2.** *If $i \in \Delta_0$, then the space* $\mathrm{Hom}_{\mathcal{C}([\boldsymbol{i}])}(P_i^{[\boldsymbol{i}]}, I_i^{[\boldsymbol{i}]})$ *is one-dimensional.*

*Proof.* After applying a tilting functor not involving a spherical twist at the vertex $i$, we can assume that $i$ is a sink of $[\boldsymbol{i}]$ so that $P_i^{[\boldsymbol{i}]} \cong S_i$. If we view $M = I_i^{[\boldsymbol{i}]}$ as a representation of $\overline{Q}$, we need to show that $\dim_K \ker M_{\mathrm{out}(i)} = 1$ in order to prove the lemma. We claim that we have a stronger property: $M_{\mathrm{out}(i)} = 0$ and $\dim_K M(i) = 1$.

Since $M$ is injective, we can write $M \cong T_{i_1} \cdots T_{i_k}(S_i)$ where the indices $i_1, \ldots, i_k$ are different from $i$. By Theorem 1.4.3 and Proposition 1.3.6, we may replace each spherical twist $T_{i_l}$ above by the corresponding reflection functor $\Sigma_{i_l}$. As a representation, $S_i$ satisfies $\dim_K S_i(i) = 1$ and $(S_i)_{\mathrm{out}(i)} = 0$. Thus, it suffices to show that, if $N$ is a representation of $\Lambda_Q$ satisfying $\dim_K N(i) = 1$ and $N_{\mathrm{out}(i)} = 0$, then $\Sigma_j(N)$ has the same property whenever $j \neq i$. One can easily check this claim using the definition of $\Sigma_j$. $\surd$

We define the *$k$-th extension group* ($k \in \mathbb{Z}$) between two objects $M, N \in \mathcal{D}([\boldsymbol{i}])$ as

$$\mathrm{Ext}^k_{[\boldsymbol{i}]}(M, N) = \mathrm{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M, \Sigma^k N).$$

Observe that this vector space is zero if $M, N \in \mathcal{C}([\boldsymbol{i}])$ and $k < 0$.

**Proposition 1.7.3.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$ and take $M \in \mathrm{ind}([\boldsymbol{i}])$. Then $M$ is projective if and only if $\mathrm{Ext}^k_{[\boldsymbol{i}]}(M, M') = 0$ for all $M' \in \mathcal{C}([\boldsymbol{i}])$ and $k > 0$. Dually, $M$ is injective if and only if $\mathrm{Ext}^k_{[\boldsymbol{i}]}(M', M) = 0$ for all $M' \in \mathcal{C}([\boldsymbol{i}])$ and $k > 0$.*

*Proof.* We will prove the first statement. The injective case is similar.

Assume $M = P_i^{[\boldsymbol{i}]}$ is projective and let $I = I_i^{[\boldsymbol{i}]}$ be the corresponding injective indecomposable object. Suppose by contradiction that there exists $M' \in \mathcal{C}([\boldsymbol{i}])$ and $k > 0$ such that $\mathrm{Ext}^k_{[\boldsymbol{i}]}(M, M') \neq 0$. We can assume that $M'$ is indecomposable and that $\Delta \neq \mathsf{A}_1$. By Lemma 1.4.12, there is a path in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ from $c_{[\boldsymbol{i}]}(M)$ to $c_{[\boldsymbol{i}]}(\Sigma^k M')$. Since $\Sigma^k M' \notin \mathcal{C}([\boldsymbol{i}])$ and $\Upsilon_{[\boldsymbol{i}]}$ is convex in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$, there is no path in $\widehat{\Upsilon}_{[\boldsymbol{i}]}$ from $c_{[\boldsymbol{i}]}(\Sigma^k M')$ to $c_{[\boldsymbol{i}]}(I)$. By our hypothesis and by the definition of the c-derived category, there is a tilting functor $T : \mathcal{D}([\boldsymbol{i}]) \longrightarrow \mathcal{D}([\boldsymbol{j}])$ such that $T(M), T(\Sigma^k M') \in \mathcal{C}([\boldsymbol{j}])$. Since there is a path from $c_{[\boldsymbol{i}]}(M)$ to $c_{[\boldsymbol{i}]}(\Sigma^k M')$, we can assume that $T(M)$ is still projective and that $T(M) \cong S_i$. By Lemma 1.2.4, $T(I) \in \mathcal{C}([\boldsymbol{j}])$ and it is still injective of residue $i$. Since there is no path from the coordinate of $T(\Sigma^k M')$ to the coordinate of $T(I)$, we conclude from Theorem 1.2.1 that $T(\Sigma^k M') \cong T_{i_1} \cdots T_{i_{l-1}}(S_{i_l})$ where $(i_1, \ldots, i_l)$ is a source sequence for $[\boldsymbol{j}]$ that does not contain $i$. But then $T(\Sigma^k M')(i) = 0$ by Proposition 1.3.6 and there cannot exist a nonzero morphism from $T(M) \cong S_i$ to $T(\Sigma^k M')$, a contradiction.

Let us prove the converse. Take a representative $\boldsymbol{i} = (i_1, \ldots, i_N)$ for $[\boldsymbol{i}]$ and let $1 \leq k \leq N$ be such that $M \cong M_k^{\boldsymbol{i}}$. If $M$ is not projective, there is $k < l \leq N$ such that $i_l = i_k$. Suppose that $l$ is the smallest such index. Since $(i_N^*, i_{N-1}^*, \ldots, i_l^*)$ is a sink sequence for $[\boldsymbol{i}]$, it induces a tilting functor $T : \mathcal{D}([\boldsymbol{i}]) \longrightarrow \mathcal{D}([\boldsymbol{j}])$, where $\boldsymbol{j} = (i_l^*, \ldots, i_N^*, i_1, \ldots, i_{l-1})$. Notice that $T(M)$ is now projective in $\mathcal{D}([\boldsymbol{j}])$ and the corresponding injective indecomposable object is $I = M_1^{\boldsymbol{j}}$. We deduce that $c_{[\boldsymbol{i}]}(T^{-1}(I)) = \widehat{\beta}_{l-N}^{\boldsymbol{i}}$, so $\Sigma^{-1}T^{-1}(I) \in \mathcal{C}([\boldsymbol{i}])$ since $-N < l - N \leq 0$. Finally, we have

$$\operatorname{Ext}^1_{[\boldsymbol{i}]}(M, \Sigma^{-1}T^{-1}(I)) = \operatorname{Hom}_{\mathcal{D}([\boldsymbol{i}])}(M, T^{-1}(I)) \cong \operatorname{Hom}_{\mathcal{D}([\boldsymbol{j}])}(T(M), I),$$

which is nonzero by Lemma 1.7.2. $\surd$

**Proposition 1.7.4.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. If $M, M' \in \mathcal{C}([\boldsymbol{i}])$, then $\operatorname{Ext}^k_{[\boldsymbol{i}]}(M, M') = 0$ for all $k > 1$.*

*Proof.* Take a representative $\boldsymbol{i} = (i_1, \ldots, i_N)$ for $[\boldsymbol{i}]$ and let $1 \leq l \leq N$ be such that $M \cong M_l^{\boldsymbol{i}}$. Since $(i_N^*, i_{N-1}^*, \ldots, i_{l+1}^*)$ is a sink sequence for $[\boldsymbol{i}]$, it induces a tilting functor $T : \mathcal{D}([\boldsymbol{i}]) \longrightarrow \mathcal{D}([\boldsymbol{j}])$, where $\boldsymbol{j} = (i_{l+1}^*, \ldots, i_N^*, i_1, \ldots, i_l)$. Note that $T(M) \cong M_N^{\boldsymbol{j}}$ is now projective. On the other hand, if $M' = M_{l'}^{\boldsymbol{i}}$ for $1 \leq l' \leq N$, then $c_{[\boldsymbol{j}]}(T(M')) = \widehat{\beta}_{l'+N-l}^{\boldsymbol{j}}$. Since $1 \leq l' + N - l < 2N$, $T(M')$ is concentrated in degree $m$ for $m = 0$ or $m = 1$. Thus, if $k > 1$, we have $k - m > 0$, and so,

$$\operatorname{Ext}^k_{[\boldsymbol{i}]}(M, M') \cong \operatorname{Ext}^k_{[\boldsymbol{j}]}(T(M), T(M')) = \operatorname{Ext}^{k-m}_{[\boldsymbol{j}]}(T(M), \Sigma^m T(M')) = 0$$

by Proposition 1.7.3 as $T(M)$ is projective and $\Sigma^m T(M') \in \mathcal{C}([\boldsymbol{j}])$. $\surd$

### 1.7.2 The Grothendieck group and the Euler form

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. The split Grothendieck group $K_0^{\mathrm{sp}}(\mathcal{D}([\boldsymbol{i}]))$ of the additive category $\mathcal{D}([\boldsymbol{i}])$ is the free abelian group generated by isomorphism classes $[M]$ of objects $M$ in $\mathcal{D}([\boldsymbol{i}])$ modulo the relation $[M \oplus N] = [M] + [N]$. We define the *Grothendieck group* $K_0(\mathcal{D}([\boldsymbol{i}]))$ *of* $\mathcal{D}([\boldsymbol{i}])$ to be the quotient of $K_0^{\mathrm{sp}}(\mathcal{D}([\boldsymbol{i}]))$ where we impose the additional relation $[M] + [\Sigma M] = 0$. Note that $K_0(\mathcal{D}([\boldsymbol{i}]))$ is isomorphic to the split Grothendieck group of $\mathcal{C}([\boldsymbol{i}])$.

The *Euler form of* $\mathcal{D}([\boldsymbol{i}])$ is the pairing

$$\langle -, - \rangle_{[\boldsymbol{i}]} : K_0(\mathcal{D}([\boldsymbol{i}])) \times K_0(\mathcal{D}([\boldsymbol{i}])) \longrightarrow \mathbb{Z}$$

defined by

$$\langle [M], [N] \rangle_{[\boldsymbol{i}]} = \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \operatorname{Ext}^k_{[\boldsymbol{i}]}(M, N)$$

for objects $M, N \in \mathcal{D}([\boldsymbol{i}])$. The sum above is finite by Proposition 1.7.4. Moreover, it is compatible with direct sums and the suspension functor, so we indeed get a well-defined pairing on $K_0(\mathcal{D}([\boldsymbol{i}]))$. For simplicity, we will denote $\langle [M], [N] \rangle_{[\boldsymbol{i}]}$ by $\langle M, N \rangle_{[\boldsymbol{i}]}$.

**Remark 1.7.5.** The Euler form defined above does not always respect the triangulated structure of $\mathrm{pvd}(\Pi_Q)$. For example, let $\mathcal{Q} = (\mathsf{A}_3, \vee, \xi)$ be the Q-datum of type $\mathsf{B}_2$ with height function given by $\xi_1 = 3$, $\xi_2 = 4$ and $\xi_3 = 5$. We can describe $\mathrm{ind}([\mathcal{Q}])$ with the following picture:

$$
\begin{array}{ccccccccccccc}
(\imath \setminus p) & 0 & & 1 & & 2 & & 3 & & 4 & & 5 \\
1 & & & & & & & \underset{\leftarrow\leftarrow}{KKK} & & & & \\
 & & & & & & \nearrow & & \searrow & & & \\
2 & 0K0 & & & & K00 & & & & 0\underset{\leftarrow}{KK} & & \\
 & & \searrow & & \nearrow & & & & & & \searrow & \\
3 & & & \overset{\rightarrow}{KK}0 & & & & & & & & 00K
\end{array}
$$

If we consider the mesh determined by $I_2^{[\mathcal{Q}]}$, we obtain the distinguished triangle

$$K00 \longrightarrow \underset{\leftarrow\leftarrow}{KKK} \longrightarrow 0\underset{\leftarrow}{KK} \longrightarrow \Sigma K00$$

in $\mathrm{pvd}(\Pi_Q)$ with corners in $\mathcal{R}([\mathcal{Q}])$. One can compute that

$$\langle\, K00\,,\ 0K0\,\rangle_{[\mathcal{Q}]} = -1, \quad \langle\, \underset{\leftarrow\leftarrow}{KKK}\,,\ 0K0\,\rangle_{[\mathcal{Q}]} = 0 \quad \text{and} \quad \langle\, 0\underset{\leftarrow}{KK}\,,\ 0K0\,\rangle_{[\mathcal{Q}]} = 0.$$

Hence, the equality

$$\langle\, K00\,,\ 0K0\,\rangle_{[\mathcal{Q}]} - \langle\, \underset{\leftarrow\leftarrow}{KKK}\,,\ 0K0\,\rangle_{[\mathcal{Q}]} + \langle\, 0\underset{\leftarrow}{KK}\,,\ 0K0\,\rangle_{[\mathcal{Q}]} = 0$$

is false. More strongly, one can check that the square matrix indexed by $\mathrm{ind}([\mathcal{Q}])$ whose $(M,N)$-entry is $\langle M, N\rangle_{[\mathcal{Q}]}$ is invertible. Therefore, we cannot impose more relations in the definition of $K_0(\mathcal{D}([\mathcal{Q}]))$ if we want to work with the Euler form defined above.

When $Q$ is a Dynkin quiver of simply laced type $\Delta$, the Grothendieck group of $\mathcal{D}^b(\mathrm{mod}\, KQ)$ can be naturally identified with the root lattice $\mathsf{Q}$ of $\Delta$. With this identification, the symmetrization of the Euler form coincides with the usual symmetric bilinear form $(-,-)$ on $\mathsf{Q}$. We generalize this result to our setting.

**Theorem 1.7.6.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. If $M$ and $N$ are objects in $\mathcal{D}([\boldsymbol{i}])$, then*

$$\langle M, N\rangle_{[\boldsymbol{i}]} + \langle N, M\rangle_{[\boldsymbol{i}]} = ([M]_{\mathrm{pvd}(\Pi_Q)}, [N]_{\mathrm{pvd}(\Pi_Q)})$$

*where $[M]_{\mathrm{pvd}(\Pi_Q)}$ and $[N]_{\mathrm{pvd}(\Pi_Q)}$ denote the classes of $M$ and $N$ in $K_0(\mathrm{pvd}(\Pi_Q))$.*

*Proof.* By additivity, we may assume that $M$ and $N$ are indecomposable. Tilting functors preserve the formula above, so we can also suppose that $N = S_i$ for a source $i \in \Delta_0$ of $[\boldsymbol{i}]$. Finally, by replacing $M$ with a suitable shift, we may take $M \in \mathcal{C}([\boldsymbol{i}])$.

Since $S_i$ is injective, we have

$$\langle M, S_i\rangle_{[\boldsymbol{i}]} = \dim_K \operatorname{Hom}_{\mathcal{C}([\boldsymbol{i}])}(M, S_i)$$

by Proposition 1.7.3. If $M \cong S_i$, we thus obtain $\langle S_i, S_i\rangle_{[\boldsymbol{i}]} = 1$ and

$$([S_i]_{\operatorname{pvd}(\Pi_Q)}, [S_i]_{\operatorname{pvd}(\Pi_Q)}) = (\alpha_i, \alpha_i) = 2 = 2\langle S_i, S_i\rangle_{[\boldsymbol{i}]},$$

as needed. Hence, we may suppose $M \not\cong S_i$. In this case, the tilting functor $T_i^{-1} : \mathcal{D}([\boldsymbol{i}]) \longrightarrow \mathcal{D}(r_i[\boldsymbol{i}])$ takes $M$ to an object of $\mathcal{C}(r_i[\boldsymbol{i}])$ by Proposition 1.4.8. Since $S_i$ is projective with respect to $r_i[\boldsymbol{i}]$, we get

$$\begin{aligned}\langle S_i, M\rangle_{[\boldsymbol{i}]} = \langle T_i^{-1}(S_i), T_i^{-1}(M)\rangle_{r_i[\boldsymbol{i}]} &= -\langle S_i, T_i^{-1}(M)\rangle_{r_i[\boldsymbol{i}]} \\ &= -\dim_K \operatorname{Hom}_{\mathcal{C}(r_i[\boldsymbol{i}])}(S_i, T_i^{-1}(M)),\end{aligned}$$

where we used item (1) of Proposition 1.3.3 and Proposition 1.7.3. Therefore, by Lemma 1.7.7 below,

$$\langle M, S_i\rangle_{[\boldsymbol{i}]} + \langle S_i, M\rangle_{[\boldsymbol{i}]} = \dim_K M(i) - \dim_K T_i^{-1}(M)(i),$$

which we can write as

$$\begin{aligned}((1 - s_i)[M]_{\operatorname{pvd}(\Pi_Q)}, \varpi_i) = ([M]_{\operatorname{pvd}(\Pi_Q)}, (1 - s_i)\varpi_i) &= ([M]_{\operatorname{pvd}(\Pi_Q)}, \alpha_i) \\ &= ([M]_{\operatorname{pvd}(\Pi_Q)}, [S_i]_{\operatorname{pvd}(\Pi_Q)}),\end{aligned}$$

finishing the proof. $\checkmark$

**Lemma 1.7.7.** *Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$ and let $i \in \Delta_0$ be a source of $[\boldsymbol{i}]$. If $M \in \operatorname{ind}(r_i[\boldsymbol{i}])$ and $M \not\cong S_i$, then we have an equality:*

$$\dim_K M(i) - \dim_K \operatorname{Hom}_{\mathcal{C}(r_i[\boldsymbol{i}])}(S_i, M) = \dim_K T_i(M)(i) - \dim_K \operatorname{Hom}_{\mathcal{C}([\boldsymbol{i}])}(T_i(M), S_i).$$

Note that we indeed have $T_i(M) \in \operatorname{ind}([\boldsymbol{i}])$ by Proposition 1.4.8.

*Proof.* Viewing the objects above as representations of $\overline{Q}$, observe that the dimension of the Hom-space from $S_i$ to $M$ is precisely $\dim_K \ker M_{\operatorname{out}(i)}$. Therefore, the left-hand side in the statement equals $\dim_K \operatorname{im} M_{\operatorname{out}(i)}$. Since $T_i(M)$ is concentrated in degree 0 by Proposition 1.4.8, we know that $M_{\operatorname{in}(i)}$ is surjective and that $T_i(M) \cong \Sigma_i(M)$ by

Proposition 1.3.6. Consequently, in the diagram

$$\widetilde{T_i(M)}(i) \xrightarrow{(T_i(M))_{\mathrm{in}(i)}} T_i(M)(i) \xrightarrow{(T_i(M))_{\mathrm{out}(i)}} \widetilde{T_i(M)}(i)$$

defining the representation associated with $T_i(M)$ around the vertex $i$, we have

$$T_i(M)(i) = \ker M_{\mathrm{in}(i)}, \quad \widetilde{T_i(M)}(i) = \widetilde{M}(i) \quad \text{and} \quad (T_i(M))_{\mathrm{in}(i)} = M_{\mathrm{out}(i)} M_{\mathrm{in}(i)}.$$

The dimension of the Hom-space from $T_i(M)$ to $S_i$ equals the dimension of the cokernel of $(T_i(M))_{\mathrm{in}(i)}$; hence, the right-hand side in the statement is $\dim_K \operatorname{im} M_{\mathrm{out}(i)} M_{\mathrm{in}(i)}$. The lemma then follows because $M_{\mathrm{in}(i)}$ is a surjective map. $\surd$

## 1.8 The case of Q-data

We now specialize to the case of a commutation class $[\mathcal{Q}]$ coming from a Q-datum $\mathcal{Q}$. In the first subsection, we show how to see the generalized twisted Coxeter element $\tau_{\mathcal{Q}}$ as an autoequivalence of $\mathcal{R}([\mathcal{Q}])$ and $\mathcal{D}([\mathcal{Q}])$. In the subsequent section, we prove that $\mathcal{D}([\mathcal{Q}])$ has a kind of partial Serre duality (Theorem 1.8.10). Finally, we apply our results to reinterpret a formula for computing inverse quantum Cartan matrices due to [FO21].

**Remark 1.8.1.** To simplify the notation, we will replace $[\mathcal{Q}]$ for $\mathcal{Q}$ in most of the symbols introduced before. For example, we shall write $\mathcal{C}([\mathcal{Q}])$, $M_\alpha^{[\mathcal{Q}]}$, and $P_i^{[\mathcal{Q}]}$ as $\mathcal{C}(\mathcal{Q})$, $M_\alpha^{\mathcal{Q}}$, and $P_\imath^{\mathcal{Q}}$. We will also adopt the convention from Remark 1.2.8 for notating vertices of $\Delta$.

Moreover, instead of seeing the coordinate function $c_{\mathcal{Q}}$ as a map from $\operatorname{ind}(\mathcal{R}(\mathcal{Q}))$ to $\widehat{\mathsf{R}}$, we will use the bijection $\phi_{\mathcal{Q}}$ from Section 1.2.8 to view it as a function $c_{\mathcal{Q}} : \operatorname{ind}(\mathcal{R}(\mathcal{Q})) \longrightarrow \widehat{\Delta}_0^\sigma$. In this case, we remark that the suspension functor sends the indecomposable object with coordinate $(\imath, p) \in \widehat{\Delta}_0^\sigma$ to the indecomposable object with coordinate $(\imath^*, p + rh^\vee)$ (see [FO21, Corollary 3.40]). We also extend the terminology from Section 1.2 and define the *height of* $M \in \operatorname{ind}(\mathcal{R}(\mathcal{Q}))$ as the height of $c_{\mathcal{Q}}(M)$.

### 1.8.1 A categorification of the generalized twisted Coxeter element

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. Recall from Section 1.2.7 the definition of $X_{\mathcal{Q}}^\circ$ and $X'_{\mathcal{Q}}$. Choose compatible readings $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ for them and define

$$\tau_{\mathcal{Q}}^\circ = T_{\imath_1} T_{\imath_2} \cdots T_{\imath_n} \sigma,$$

and

$$T[X'_{\mathcal{Q}}] = T_{\jmath_1} T_{\jmath_2} \cdots T_{\jmath_m},$$

where $\imath_k = \pi(x_k)$ and $\jmath_l = \pi(y_l)$ (we remind that $\pi : \widehat{\Delta}_0^\sigma \to \Delta_0$ is the projection onto the first coordinate). By the same reasoning as in [FO21, Lemma 3.18], $\tau_{\mathcal{Q}}^\circ$ and $T[X'_{\mathcal{Q}}]$ are independent of the choice of compatible reading. We set $\tau_{\mathcal{Q}} : \mathrm{pvd}(\Pi_Q) \longrightarrow \mathrm{pvd}(\Pi_Q)$ as the composition

$$\tau_{\mathcal{Q}} = T[X'_{\mathcal{Q}}]^{-1} \circ \tau_{\mathcal{Q}}^\circ \circ T[X'_{\mathcal{Q}}],$$

which is an autoequivalence of $\mathrm{pvd}(\Pi_Q)$.

**Remark 1.8.2.** Note that $\tau_{\mathcal{Q}}^\circ$ and $\tau_{\mathcal{Q}}$ can either denote an element of the coset $\mathsf{W}\sigma$ or an autoequivalence of $\mathrm{pvd}(\Pi_Q)$. The context will always allow us to resolve this ambiguity.

**Lemma 1.8.3.** *If $\imath \in \Delta_0$ is a source of $\mathcal{Q}$, then $T_\imath^{-1}\tau_{\mathcal{Q}}T_\imath \cong \tau_{s_\imath \mathcal{Q}}$. Consequently, we have $T\tau_{\mathcal{Q}}T^{-1} \cong \tau_{\mathcal{Q}'}$ for any tilting functor $T : \mathcal{R}(\mathcal{Q}) \longrightarrow \mathcal{R}(\mathcal{Q}')$.*

*Proof.* Since the spherical twists satisfy the same commutation relations as the reflections $s_\jmath$ ($\jmath \in \Delta_0$), the proof is essentially the same as the one in [FO21, Proposition 3.34]. The only care that we should take into consideration is that, in [FO21], all appearances of $s_\jmath^{-1}$ are replaced by $s_\jmath$ (we cannot do this on the level of spherical twists), but this replacement is not necessary for the proof. $\surd$

**Proposition 1.8.4.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. Then $\tau_{\mathcal{Q}} : \mathrm{pvd}(\Pi_Q) \longrightarrow \mathrm{pvd}(\Pi_Q)$ can be restricted to an autoequivalence of the repetition category $\mathcal{R}(\mathcal{Q})$. It also preserves the ideal $\mathcal{I}_{\mathcal{Q}}$ and induces an autoequivalence of $\mathcal{D}(\mathcal{Q})$. Moreover, it sends the indecomposable object with coordinate $(\imath, p) \in \widehat{\Delta}^\sigma$ to the indecomposable object with coordinate $(\sigma(\imath), p-2)$.*

*Proof.* Suppose first that $\sigma = \mathrm{id}$. In this case, $\tau_{\mathcal{Q}} = T_{\imath_1} \cdots T_{\imath_n}$ where $(\imath_1, \dots, \imath_n)$ is a source sequence for $\mathcal{Q}$ where each vertex appears exactly once. Thus, $\tau_{\mathcal{Q}}$ restricts to a tilting functor from $\mathcal{R}(\mathcal{Q}')$ to $\mathcal{R}(\mathcal{Q})$, where $\mathcal{Q}' = s_{\imath_n} \cdots s_{\imath_1}\mathcal{Q}$. The height function defining $\mathcal{Q}'$ is a translation by $-2$ of the height function defining $\mathcal{Q}$. In particular, $\mathcal{Q}$ and $\mathcal{Q}'$ represent the same Dynkin quiver and we have $\mathcal{R}(\mathcal{Q}) = \mathcal{R}(\mathcal{Q}')$ (but the coordinate functions differ by $-2$). Therefore, all three statements above are proved.

From now on, suppose $\sigma \neq \mathrm{id}$. After applying a suitable tilting functor and using Lemma 1.8.3, we can assume that $\mathcal{Q} = \mathcal{Q}^\circ$. Let $(x_1, \dots, x_n)$ be a compatible reading of $X_{\mathcal{Q}}^\circ$ and denote $\imath_k = \pi(x_k)$ for $1 \leq k \leq n$. By [FO21, Proposition 3.31], we have $\boldsymbol{i} \in [\mathcal{Q}]$ where

$$\boldsymbol{i} = (\imath_1, \dots, \imath_n, \sigma(\imath_1), \dots, \sigma(\imath_n), \dots\dots, \sigma^{rh^\vee/2-1}(\imath_1), \dots, \sigma^{rh^\vee/2-1}(\imath_n)).$$

By the definition of $[\mathcal{Q}]$, we deduce that $(\imath_1, \dots, \imath_n)$ is a source sequence for $\mathcal{Q}$. Denote $\mathcal{Q}^\sigma = s_{\imath_n} \cdots s_{\imath_1}\mathcal{Q}$. Hence, we have $\boldsymbol{i}^\sigma \in [\mathcal{Q}^\sigma]$ for

$$\boldsymbol{i}^\sigma = (\sigma(\imath_1), \dots, \sigma(\imath_n), \dots\dots, \sigma^{rh^\vee/2-1}(\imath_1), \dots, \sigma^{rh^\vee/2-1}(\imath_n), \imath_1^*, \dots, \imath_n^*).$$

By [FO21, Remark 3.1], we have $\sigma^{rh^\vee/2}(\imath_k) = \imath_k^*$ for $1 \leq k \leq n$, so $\boldsymbol{i}^\sigma$ is obtained from $\boldsymbol{i}$ by applying $\sigma$ to its terms. We deduce from item (4) in Proposition 1.3.3 that $\sigma$, now viewed as an autoequivalence of $\mathrm{pvd}(\Pi_Q)$, restricts to a functor $\mathcal{R}(\mathcal{Q}) \longrightarrow \mathcal{R}(\mathcal{Q}^\sigma)$. On the other hand, by Proposition 1.4.11, the composition $T_{\imath_1} \cdots T_{\imath_n}$ restricts to a tilting functor $\mathcal{R}(\mathcal{Q}^\sigma) \longrightarrow \mathcal{R}(\mathcal{Q})$. Since $\tau_{\mathcal{Q}} = T_{\imath_1} \cdots T_{\imath_n}\sigma$, this proves the first statement. For the second statement, by Proposition 1.4.13, it suffices to prove that $\sigma : \mathcal{R}(\mathcal{Q}) \longrightarrow \mathcal{R}(\mathcal{Q}^\sigma)$ sends $\mathcal{I}_{\mathcal{Q}}$ to $\mathcal{I}_{\mathcal{Q}^\sigma}$, which follows from item (4) in Proposition 1.3.3 and the fact that $\sigma$ sends objects concentrated in degree 0 to objects concentrated in degree 0.

For the last part, since tilting functors preserve coordinates, we may assume that $\mathcal{Q} = \mathcal{Q}^\circ$, $p = \xi_\imath$, and $\imath$ starts the reduced word $\boldsymbol{i}$ from the previous paragraph. In this case, the indecomposable object of $\mathcal{R}(\mathcal{Q})$ with coordinate $(\imath, p)$ is $S_\imath$, and the functor $\sigma$ sends it to $S_{\sigma(\imath)}$. Since $S_{\sigma(\imath)} \cong M_1^{\boldsymbol{i}^\sigma}$, this object is injective in $\mathcal{C}(\mathcal{Q}^\sigma)$ and its coordinate in $\mathcal{R}(\mathcal{Q}^\sigma)$ is $(\sigma(\imath), \xi^\sigma_{\sigma(\imath)})$, where $\xi^\sigma$ denotes the height function defining $\mathcal{Q}^\sigma$. Since $\mathcal{Q} = \mathcal{Q}^\circ$, one can easily check that $\xi^\sigma_{\sigma(\imath)} = \xi_\imath - 2 = p - 2$. Since $\tau_{\mathcal{Q}}$ is the composition of $\sigma$ with a tilting functor, we conclude that the coordinate of $\tau_{\mathcal{Q}}(S_\imath)$ is also $(\sigma(\imath), p-2)$, as desired. $\surd$

**Corollary 1.8.5.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $M \in \mathrm{ind}(\mathcal{Q})$ of residue $\imath \in \Delta_0$. Then $\Sigma^k \tau_{\mathcal{Q}}^{d_{\bar{\imath}}}(M) \in \mathcal{C}(\mathcal{Q})$ for some $k \in \{0,1\}$. Moreover, we have $k = 1$ if and only if $M$ is projective and, in this case,*

$$\Sigma \tau_{\mathcal{Q}}^{d_{\bar{\imath}}}(M) \cong I_{\imath^*}^{\mathcal{Q}}.$$

*Proof.* Let $(\imath, p) \in \widehat{\Delta}^\sigma$ be the coordinate of $M$. By Proposition 1.8.4, the coordinate of $\tau_{\mathcal{Q}}^{d_{\bar{\imath}}}(M)$ is $(\imath, p - 2d_{\bar{\imath}})$. We deduce that $\tau_{\mathcal{Q}}^{d_{\bar{\imath}}}(M) \in \mathcal{C}(\mathcal{Q})$ if $M$ is not projective. On the other hand, if $M$ is projective, then $M = P_{\imath^*}^{\mathcal{Q}}$ and we have $p = \xi_{\imath^*} - rh^\vee + 2d_{\bar{\imath}}$ by Lemma 1.2.15. Thus, $\tau_{\mathcal{Q}}^{d_{\bar{\imath}}}(M)$ has coordinate $(\imath, \xi_{\imath^*} - rh^\vee)$, which is also the coordinate of $\Sigma^{-1} I_{\imath^*}^{\mathcal{Q}}$. $\surd$

**Corollary 1.8.6.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. If $M \in \mathrm{ind}(\mathcal{R}(\mathcal{Q}))$ has coordinates $(\imath, p) \in \widehat{\Delta}_0^\sigma$, then*

$$M \cong \tau_{\mathcal{Q}}^{(\xi_\imath - p)/2}(I_\imath^{\mathcal{Q}}).$$

*Proof.* It is immediate since both sides have the same coordinate (see Proposition 1.8.4). $\surd$

### 1.8.2 Partial Serre duality

Let $[\boldsymbol{i}]$ be a commutation class of reduced words for $w_0$. If $M \in \mathcal{C}([\boldsymbol{i}])$, recall that $M$ can be seen as a representation of the double quiver $\overline{Q}$ by Theorem 1.4.3 and we describe it around the vertex $i \in \Delta_0$ by a diagram

$$\widetilde{M}(i) \xrightarrow{M_{\mathrm{in}(i)}} M(i) \xrightarrow{M_{\mathrm{out}(i)}} \widetilde{M}(i)$$

(see Sections 1.3.3 and 1.7.1). We say that $i$ is a *source in* $M$ if $M_{\mathrm{in}(i)} = 0$. Dually, $i$ is a *sink in* $M$ if $M_{\mathrm{out}(i)} = 0$.

**Lemma 1.8.7.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. Take $\imath \in \Delta_0$ and let $M \in \mathrm{ind}(\mathcal{Q})$ be an indecomposable object of residue $\jmath \in \Delta_0$. If $\imath$ is a source (resp. sink) of $\mathcal{Q}$, then $\imath$ is a source (resp. sink) in $M$ if one of the following conditions hold:*

*(1) $d_{\bar{\imath}} = r$;*

*(2) $\mathcal{Q}$ is of type $\mathsf{B}_n$ and $\imath = \jmath = n$;*

*(3) $\mathcal{Q}$ is of type $\mathsf{C}_n$ and $\imath + \jmath \leq n$;*

*(4) $\mathcal{Q}$ is of type $\mathsf{F}_4$ and $\imath = \jmath = 6$.*

*Proof.* If we know the result for the case when $\imath$ is a source, then we can deduce it for when $\imath$ is a sink, and conversely. By Proposition 1.3.6, one statement follows from the other one after applying a reflection as in Proposition 1.4.8. With this in mind, let us consider each of the cases above separately.

(1) Assume $d_{\bar{\imath}} = r$. For this case, we suppose that $\imath$ is a source of $\mathcal{Q}$. We argue for each type separately, the simply laced case being immediate by Proposition 1.4.7.

Suppose that $\mathcal{Q}$ is of type $\mathsf{B}_n$. By Proposition 1.9.2, there are two quivers $Q^{\leq n}$ and $Q^{\geq n}$ orienting $\Delta = \mathsf{A}_{2n-1}$ such that each indecomposable object of $\mathcal{C}(\mathcal{Q})$ is a representation of either $Q^{\leq n}$ or $Q^{\geq n}$. Since $d_{\bar{\imath}} = r$, $\imath$ is not the central vertex of $\Delta$. Hence, if $\imath$ is a source of $\mathcal{Q}$, it is not hard to see that it is also a source of $Q^{\leq n}$ and $Q^{\geq n}$, implying our claim.

Now, suppose that $\mathcal{Q}$ is of type $\mathsf{C}_n$. Assume first that $\imath$ is the vertex $n$ in $\Delta = \mathsf{D}_{n+1}$. By applying reflections, we may assume that $\mathcal{Q}$ is the Q-datum of Example 1.9.3. A quick inspection of the list of objects given by this example shows that $\imath$ is a source in any indecomposable object of $\mathcal{C}(\mathcal{Q})$, as desired. If $\imath \neq n$, since $d_{\bar{\imath}} = r$, we must have $\imath = n+1$. But $\sigma(n+1) = n$, therefore, by applying the induced functor $\sigma : \mathrm{pvd}(\Pi_Q) \longrightarrow \mathrm{pvd}(\Pi_Q)$, we are reduced to the previous case.

Finally, we can check the lemma for types $\mathsf{F}_4$ and $\mathsf{G}_2$ using Examples 1.9.4 and 1.9.5, respectively. Although these examples do not cover all possibilities for the source $\imath$, we can obtain the remaining cases by applying $\sigma$ as we did in the previous paragraph.

(2) If $\mathcal{Q}$ is of type $\mathsf{B}_n$ and $M$ is of residue $n$, then Proposition 1.9.2 says that we can view $M$ as a representation of both quivers $Q^{\leq n}$ and $Q^{\geq n}$. There is a unique edge of $\Delta$ where these orientations differ. Hence, the map in the representation that

corresponds to such an edge must be zero. This fact implies without difficulty that if $n$ is a source (resp. sink) of $\mathcal{Q}$, then $n$ is a source (resp. sink) in any indecomposable object of $\mathcal{C}(\mathcal{Q})$ that has residue $n$.

(3) Suppose that $\mathcal{Q}$ is of type $\mathsf{C}_n$ and that $\imath$ is a sink of $\mathcal{Q}$. Assume $\imath + \jmath \leq n$, so that, in particular, we have $\imath, \jmath \leq n-1$ and $d_{\bar{\imath}} = d_{\bar{\jmath}} = 1$. If $\mathcal{Q}'$ denotes the Q-datum of Example 1.9.3, then we may assume that $\mathcal{Q} = s_{\imath-1}^{-1} s_{\imath-2}^{-2} \cdots s_1^{-1} \mathcal{Q}'$ after possibly applying some reflections and the automorphism $\sigma$ to $\mathcal{Q}$. Thus, by Proposition 1.4.11, we can compute the objects of residue $\jmath$ in $\operatorname{ind}(\mathcal{Q})$ by applying the functor $T_{\imath-1}T_{\imath-2}\cdots T_1$ to those in $\operatorname{ind}(\mathcal{Q}')$ and then a shift if necessary. By the description of the action of $\tau_{\mathcal{Q}'}$ in Example 1.9.3, we see that the objects of residue $\jmath$ in $\operatorname{ind}(\mathcal{Q}')$ are precisely

$$A_{\jmath}^{1,0}, A_{\jmath-1}^{1,1}, B_{\jmath-2,n-1}, B_{\jmath-3,n-2}, \ldots, B_{1,n-\jmath+2}, A_{n-\jmath+1}^{0,1}, A_{n-\jmath,n-1}^{0,0}, A_{n-\jmath-1,n-2}^{0,0}, \ldots, A_{1,\jmath}^{0,0},$$

where the objects of the form $B_{\imath',\jmath'}$ do not appear if $\jmath \leq 2$ and the object $A_{\jmath-1}^{1,1}$ does not appear if $\jmath = 1$. Computing the image of the objects above under $T_{\imath-1}T_{\imath-2}\cdots T_1$ using Proposition 1.3.6, it is not hard to see that $\imath$ is a sink in any object of residue $\jmath$ in $\operatorname{ind}(\mathcal{Q})$. The crucial observation here is that the inequality $\imath+\jmath \leq n$ guarantees that we have $\jmath' \geq \imath + 2$ in all objects of the form $B_{\imath',\jmath'}$ above. In particular, when computing the spherical twists and checking whether $\imath$ is a sink, we may ignore the vertices where $B_{\imath',\jmath'}$ has dimension 2. The inequality also forces $\jmath = 1$ when $\imath = n-1$, so that $A_{\jmath-1}^{1,1}$ is not taken into account in this case.

(4) Suppose $\mathcal{Q}$ is of type $\mathsf{F}_4$. If 6 is a sink of $\mathcal{Q}$, then we may assume that $\mathcal{Q}$ is the Q-datum of Example 1.9.4 after applying reflections. One can then directly check that 6 is a sink in any indecomposable object of residue 6.

$\checkmark$

**Remark 1.8.8.** The lemma is not true if we remove the conditions (1)–(4). Counterexamples can be easily found using the explicit descriptions in Section 1.9. For instance, in Example 1.9.5, the vertex $\imath = 2$ is a sink of the given Q-datum, but it is not a sink in the indecomposable object with coordinate $(2, 0)$. Note that $d_{\bar{\imath}} = 1$ in this case.

**Proposition 1.8.9.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $\imath \in \Delta_0$. If $d_{\bar{\imath}} = r$, then there are isomorphisms*

$$\operatorname{Hom}_{\mathcal{C}(\mathcal{Q})}(P_\imath^{\mathcal{Q}}, M) \xrightarrow{\sim} M(\imath) \xleftarrow{\sim} D\operatorname{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}})$$

*natural in the variable $M \in \mathcal{C}(\mathcal{Q})$, where $D$ denotes the duality functor for $K$-vector spaces.*

*Proof.* We consider only the second natural isomorphism above. The proof for the first is similar.

Let $X \subset \mathrm{ind}(\mathcal{Q})$ be the set of indecomposable objects $N$ satisfying $N \preceq_{\mathcal{C}(\mathcal{Q})} I_\imath^{\mathcal{Q}}$ and $I_\imath^{\mathcal{Q}} \ncong N$. Let $(\imath_1, \dots, \imath_l)$ be the residues of a compatible reading of the coordinates of the objects in $X$. Notice that $\imath_k \neq \imath$ for all $1 \leq k \leq l$ since $I_\imath^{\mathcal{Q}}$ is injective. By Proposition 1.4.8, the autoequivalence $T = T_{\imath_l}^{-1} \cdots T_{\imath_2}^{-1} T_{\imath_1}^{-1}$ of $\mathrm{pvd}(\Pi_Q)$ restricts to a fully faithful functor from the full additive subcategory of $\mathcal{C}(\mathcal{Q})$ generated by $\mathrm{ind}(\mathcal{Q}) \setminus X$ to $\mathcal{C}(\mathcal{Q}')$, where $\mathcal{Q}' = s_{\imath_l} \cdots s_{\imath_2} s_{\imath_1} \mathcal{Q}$. It is easy to see that $\imath$ is a source for $\mathcal{Q}'$ and $T(I_\imath^{\mathcal{Q}}) \cong S_\imath \in \mathcal{C}(\mathcal{Q}')$.

We can assume that $M \in \mathrm{ind}(\mathcal{Q})$ since both sides are additive on the variable $M$. If $M \notin X$, then $T$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}(\mathcal{Q}')}(T(M), S_\imath).$$

By Lemma 1.8.7, $\imath$ is a source in $T(M)$ so, if we regard $T(M)$ and $S_\imath$ as representations of the double quiver $\overline{Q}$, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{Q}')}(T(M), S_\imath) \xrightarrow{\sim} \mathrm{Hom}_K(T(M)(\imath), K) = D(T(M)(\imath)).$$

Finally, since $\imath_k \neq \imath$ for all $k$, the spherical twists $T_{\imath_k}^{-1}$ do not change $M$ at the vertex $\imath$ (see Proposition 1.3.6), hence we have an isomorphism $D(T(M)(\imath)) \xrightarrow{\sim} D(M(\imath))$. Composing the inverse of these isomorphisms and applying the duality functor, we get an isomorphism from $D\mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}})$ to $M(\imath)$.

On the other hand, if $M \in X$, then $\mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}}) = 0$ since $I_\imath^{\mathcal{Q}} \npreceq_{\mathcal{C}(\mathcal{Q})} M$. There also exists $1 \leq k \leq l$ such that

$$M \cong T_{\imath_1} T_{\imath_2} \cdots T_{\imath_{k-1}}(S_{\imath_k}),$$

which implies that $M(\imath) = 0$ because $\imath_{k'} \neq \imath$ for all $1 \leq k' \leq l$. Consequently, we conclude that $D\mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}})$ is trivially isomorphic to $M(\imath)$.

We have thus defined the isomorphism in the statement. It remains to check that it is natural in the variable $M$. For this, it is enough to show that the square induced by a morphism $M \longrightarrow M'$ between indecomposable objects is commutative. One can do this on a case-by-case analysis by considering whether $M$ and $M'$ are in $X$ or not. $\surd$

To extend the previous result to the whole c-derived category $\mathcal{D}(\mathcal{Q})$, we define $S_{\mathcal{Q}} : \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q})$ to be the equivalence $S_{\mathcal{Q}} = \Sigma \circ \tau_{\mathcal{Q}}^r$. It behaves like a Serre functor:

**Theorem 1.8.10.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $M \in \mathrm{ind}(\mathcal{D}(\mathcal{Q}))$ of residue $\imath \in \Delta_0$ satisfying $d_{\bar{\imath}} = r$. There is an isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{Q})}(M, N) \xrightarrow{\sim} D\mathrm{Hom}_{\mathcal{D}(\mathcal{Q})}(N, S_{\mathcal{Q}} M)$$

*natural in the variable $N \in \mathcal{D}(\mathcal{Q})$, where $D$ denotes the duality functor for $K$-vector spaces.*

*Proof.* By applying a tilting functor and using Lemma 1.8.3, we may assume that $M$ is projective. In this case, by Corollary 1.8.5, $S_{\mathcal{Q}}M$ is the injective object corresponding to the same vertex. By Proposition 1.7.3, the Hom-spaces above are zero if $N$ is indecomposable and is not in $\mathcal{C}(\mathcal{Q})$. Thus, we may assume that $N$ is in $\mathcal{C}(\mathcal{Q})$, and then the theorem follows from Proposition 1.8.9. $\surd$

**Remark 1.8.11.** If $N \in \operatorname{ind}(\mathcal{D}(\mathcal{Q}))$ has residue $\imath \in \Delta_0$, then $S_{\mathcal{Q}}^{-1}N$ has residue $\imath^*$. Thus, if $d_{\bar{\imath}} = d_{\overline{\imath^*}} = r$, we have isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(M,N) \cong \operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(M, S_{\mathcal{Q}}(S_{\mathcal{Q}}^{-1}N)) &\cong D\operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(S_{\mathcal{Q}}^{-1}N, M)\\ &\cong D\operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(N, S_{\mathcal{Q}}M),\end{aligned}$$

natural in the variable $M \in \mathcal{D}(\mathcal{Q})$. This gives another form of Theorem 1.8.10.

**Remark 1.8.12.** In the nonsimply laced case, one may also consider the autoequivalence $S'_{\mathcal{Q}} : \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q})$ given by $S'_{\mathcal{Q}} = \Sigma \circ \tau_{\mathcal{Q}}$. We can adapt the previous proofs to construct an isomorphism

$$\operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(M,N) \xrightarrow{\ \sim\ } D\operatorname{Hom}_{\mathcal{D}(\mathcal{Q})}(N, S'_{\mathcal{Q}}M)$$

for $M, N \in \operatorname{ind}(\mathcal{Q})$ with residues $\imath, \jmath \in \Delta_0$ satisfying one of the conditions (2), (3), or (4) in Lemma 1.8.7. Notice that in these cases, we have $d_{\bar{\imath}} = d_{\bar{\jmath}} = 1$.

For the next result, we identify $K_0(\operatorname{pvd}(\Pi_Q))$ with the root lattice $\mathsf{Q}$ of $\Delta$ and denote the class of $M \in \operatorname{pvd}(\Pi_Q)$ in $K_0(\operatorname{pvd}(\Pi_Q))$ as $[M]_{\operatorname{pvd}(\Pi_Q)}$ (see Section 1.3.4).

**Proposition 1.8.13.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum and take $M \in \operatorname{ind}(\mathcal{D}(\mathcal{Q}))$ of residue $\jmath \in \Delta_0$. For $\imath \in \Delta_0$, if we have $\max\{d_{\bar{\imath}}, d_{\bar{\jmath}}\} = r$ or if one of the conditions (2), (3), or (4) in Lemma 1.8.7 holds, then*

$$([M]_{\operatorname{pvd}(\Pi_Q)}, \varpi_\imath) = \left\langle M, \bigoplus_{k=0}^{\lceil d_{\bar{\jmath}}/d_{\bar{\imath}}\rceil - 1} \tau_{\mathcal{Q}}^{k}(I_\imath^{\mathcal{Q}}) \right\rangle_{\mathcal{Q}} = \left\langle \bigoplus_{k=0}^{\lceil d_{\bar{\jmath}}/d_{\bar{\imath}}\rceil - 1} \tau_{\mathcal{Q}}^{-k}(P_\imath^{\mathcal{Q}}), M \right\rangle_{\mathcal{Q}}.$$

*Proof.* By shifting $M$, we can suppose that $M \in \mathcal{C}(\mathcal{Q})$. If $d_{\bar{\imath}} = r$, the result follows from Propositions 1.7.3 and 1.8.9. We can similarly prove the result if one of the conditions (2), (3), or (4) in Lemma 1.8.7 holds by modifying the proof of the latter proposition. From now on, we additionally suppose that $d_{\bar{\imath}} = 1$ and $d_{\bar{\jmath}} = r > 1$. Once we prove the first equality, the second will follow from Theorem 1.8.10 and the fact that $P_\imath^{\mathcal{Q}} \cong \Sigma^{-1}\tau_{\mathcal{Q}}^{-1}(I_\imath^{\mathcal{Q}})$ (see Corollary 1.8.5).

We claim that $\mathrm{Ext}^l_{\mathcal{Q}}(M, \tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)) = 0$ for $0 \le k < r$ and $l > 0$. By Proposition 1.7.4, we only need to check the case $l = 1$. By Theorem 1.8.10, we have

$$\mathrm{Ext}^1_{\mathcal{Q}}(M, \tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{Q})}(\Sigma^{-1}M, \tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)) \cong D\mathrm{Hom}_{\mathcal{D}(\mathcal{Q})}(\tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath), \tau^r_{\mathcal{Q}}(M)).$$

Hence, by Lemma 1.4.12, it is enough to show that there is no path from the coordinate of $\tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)$ to the coordinate of $\tau^r_{\mathcal{Q}}(M)$ in $\widehat{\Delta}^\sigma$. If $p$ is the height of $M$, it is not hard to check that $p \le \xi_\imath + d^\sigma_\Delta(\imath, \jmath)$ by the definition of a height function and the fact that $d_{\bar{\imath}} = 1$. Keeping in mind that the height of $\tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)$ is $\xi_\imath - 2k$ and that the height of $\tau^r_{\mathcal{Q}}(M)$ is $p - 2r$, the inequality

$$(p - 2r) - (\xi_\imath - 2k) = (p - \xi_\imath) + 2(k - r) < d^\sigma_\Delta(\imath, \jmath)$$

ensures what we need. Therefore, the first equality of the statement simplifies to

$$\dim_K M(\imath) = \dim_K \mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}\left(M, \bigoplus_{k=0}^{r-1} \tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)\right).$$

Assume first that $M \not\preceq_{\mathcal{C}(\mathcal{Q})} \tau^{r-1}_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)$. In this case, there is a sink sequence $(\imath_1, \dots, \imath_l)$ of $\mathcal{Q}$ such that the induced tilting functor $T : \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q}')$ satisfies $T(M) \cong P^{\mathcal{Q}'}_{\jmath^*}$ and $T(\tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)) \in \mathcal{C}(\mathcal{Q}')$ for $0 \le k < r$. Hence, if we denote $w = s_{\imath_l} \cdots s_{\imath_2} s_{\imath_1} \in \mathsf{W}$, we have

$$\begin{aligned}\dim_K \mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}\left(M, \bigoplus_{k=0}^{r-1} \tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath)\right) &= \dim_K \mathrm{Hom}_{\mathcal{C}(\mathcal{Q}')}\left(P^{\mathcal{Q}'}_{\jmath^*}, \bigoplus_{k=0}^{r-1} T(\tau^k_{\mathcal{Q}}(I^{\mathcal{Q}}_\imath))\right) \\ &= \left(\varpi_{\jmath^*}, \sum_{k=0}^{r-1} w\tau^k_{\mathcal{Q}}(\gamma^{\mathcal{Q}}_\imath)\right),\end{aligned}$$

where we used Proposition 1.8.9 for the last equality. Since $d_{\bar{\imath}} = 1$, we have $\gamma^{\mathcal{Q}}_\imath = (1 - \tau_{\mathcal{Q}})\varpi_\imath$, which gives

$$\begin{aligned}\left(\varpi_{\jmath^*}, \sum_{k=0}^{r-1} w\tau^k_{\mathcal{Q}}(\gamma^{\mathcal{Q}}_\imath)\right) = \left(\varpi_{\jmath^*}, w(1 - \tau^r_{\mathcal{Q}})\varpi_\imath\right) &= \left((1 - \tau^{-r}_{\mathcal{Q}})w^{-1}\varpi_{\jmath^*}, \varpi_\imath\right) \\ &= (w^{-1}(1 - \tau^{-r}_{\mathcal{Q}'})\varpi_{\jmath^*}, \varpi_\imath),\end{aligned}$$

where the last equality follows from Proposition 1.2.22. By Lemma 1.2.24, $(1 - \tau^{-r}_{\mathcal{Q}'})\varpi_{\jmath^*}$ represents the class of $P^{\mathcal{Q}'}_{\jmath^*}$ in $K_0(\mathrm{pvd}(\Pi_Q))$, so the isomorphism $M \cong T^{-1}(P^{\mathcal{Q}'}_{\jmath^*})$ tells us that the left term inside the last bilinear form above is the class of $M$ in $K_0(\mathrm{pvd}(\Pi_Q))$, as desired.

Suppose now that $M \preceq_{\mathcal{C}(\mathcal{Q})} \tau_{\mathcal{Q}}(I_\imath^{\mathcal{Q}})$. By our hypotheses, we have

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, \tau_{\mathcal{Q}}^k(I_\imath^{\mathcal{Q}})) = 0$$

for $1 \leq k < r$, so we need to prove that

$$\dim_K M(\imath) = \dim_K \mathrm{Hom}_{\mathcal{C}(\mathcal{Q})}(M, I_\imath^{\mathcal{Q}}).$$

To do so, we can proceed exactly as in the proof of Proposition 1.8.9 by defining an analogous set $X \subset \mathrm{ind}(\mathcal{Q})$ and a fully faithful functor $T$ from the full additive subcategory of $\mathcal{C}(\mathcal{Q})$ generated by $\mathrm{ind}(\mathcal{Q}) \setminus X$ to some $\mathcal{C}(\mathcal{Q}')$ where $\imath$ is a source of $\mathcal{Q}'$. The only part which we cannot immediately adapt is that $\imath$ is a source in $T(M)$ when $M \notin X$ (we cannot apply Lemma 1.8.7), but this follows from the additional hypothesis that $M \preceq_{\mathcal{C}(\mathcal{Q})} \tau_{\mathcal{Q}}(I_\imath^{\mathcal{Q}})$. Indeed, it implies that

$$T(M) \cong T_{\imath_1} \dots T_{\imath_{k-1}}(S_{\imath_k})$$

for some source sequence $(\imath_1, \dots, \imath_k)$ for $\mathcal{Q}'$ where $\imath$ appears at most once. Since $\imath$ is a source of $\mathcal{Q}'$, we can assume $\imath_1 = \imath$, and then it is not hard to verify that $\imath$ is a source in $T(M)$ using Proposition 1.3.6.

If $r = 2$, this finishes the proof. However, if $\mathcal{Q}$ is of type $\mathsf{G}_2$, it remains to consider the case when $M \preceq_{\mathcal{C}(\mathcal{Q})} \tau_{\mathcal{Q}}^2(I_\imath^{\mathcal{Q}})$ but $M \not\preceq_{\mathcal{C}(\mathcal{Q})} \tau_{\mathcal{Q}}(I_\imath^{\mathcal{Q}})$. In this case, $M$ has to be the unique indecomposable object with residue $\jmath$ such that $d_{\bar{\jmath}} = r$ and that satisfies $\tau_{\mathcal{Q}}(I_\imath^{\mathcal{Q}}) \preceq_{\mathcal{C}(\mathcal{Q})} M \preceq_{\mathcal{C}(\mathcal{Q})} \tau_{\mathcal{Q}}^2(I_\imath^{\mathcal{Q}})$. The formula of the statement can then be easily checked directly (see Example 1.9.5). $\checkmark$

### 1.8.3 Application: inverse quantum Cartan matrices

Let $C = (c_{ij})_{i,j \in I}$ denote the Cartan matrix of the complex finite-dimensional simple Lie algebra $\mathfrak{g}$. Using the associated pair $(\Delta, \sigma)$, we have

$$c_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -\lceil d_j/d_i \rceil & \text{if } i \sim j, \\ 0 & \text{otherwise,} \end{cases}$$

for $i, j \in I$. The *quantum Cartan matrix of* $\mathfrak{g}$ is a matrix $C(q) = (C_{ij}(q))_{i,j \in I}$ with entries in $\mathbb{Z}[q^{\pm 1}]$, where $q$ is an indeterminate. It is defined by

$$C_{ij}(q) = \begin{cases} q^{d_i} + q^{-d_i} & \text{if } i = j, \\ \frac{q^{c_{ij}} - q^{-c_{ij}}}{q - q^{-1}} & \text{if } i \neq j, \end{cases}$$

for $i, j \in I$. Such a matrix is invertible if regarded as a matrix with entries in the field $\mathbb{Q}(\!(q)\!)$ of formal Laurent series over $\mathbb{Q}$. Let $\widetilde{C}(q) = (\widetilde{C}_{ij}(q))_{i,j\in I}$ be its inverse and denote by

$$\widetilde{C}_{ij}(q) = \sum_{u\in\mathbb{Z}} \widetilde{c}_{ij}(u) q^u$$

the formal Laurent expansion of the $(i, j)$-entry.

One of the main results in [FO21] is a combinatorial formula for the coefficients $\widetilde{c}_{ij}(u)$ above, generalizing a formula of Hernandez–Leclerc (cf. [HL15]) to the nonsimply laced case.

**Theorem 1.8.14** ([FO21, Theorem 4.8])**.** *For each $i, j \in I$ and $u \in \mathbb{Z}$, we have*

$$\widetilde{c}_{ij}(u) = \begin{cases} (\varpi_\imath, \tau_{\mathcal{Q}}^{(u+\xi_\jmath-\xi_\imath-d_i)/2}(\gamma_\jmath^{\mathcal{Q}})) & \text{if } u \geq 0 \text{ and } u + \xi_\jmath - \xi_\imath - d_i \in 2\mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

*where $\mathcal{Q} = (\Delta, \sigma, \xi)$ is a Q-datum for $\mathfrak{g}$ and $\imath \in i$, $\jmath \in j$.*

In [Fuj22, Corollary 3.6], Fujita reinterprets the formula above in the simply laced case using representations of a Dynkin quiver of the same type. Our next proposition generalizes his result and proof. Before we state it, we introduce some notation.

Following [FO21, Section 3.8], define

$$\widehat{I} = \{(i, p) \in I \times \mathbb{Z} \mid \exists(\imath, p) \in \widehat{\Delta}_0^\sigma, \bar{\imath} = i\}.$$

The map $f : \widehat{\Delta}_0^\sigma \longrightarrow \widehat{I}$ given by projecting the first coordinate from $\Delta_0$ to $I$ is a bijection called the *folding map*. For a Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$, we have a bijection $H_{\mathcal{Q}} : \widehat{I} \longrightarrow \operatorname{ind}(\mathcal{D}(\mathcal{Q}))$ that sends $(i, p) \in \widehat{I}$ to the indecomposable object of $\mathcal{D}(\mathcal{Q})$ with coordinate $f^{-1}(i, p)$.

**Proposition 1.8.15.** *Let $(i, p), (j, s) \in \widehat{I}$ be such that $p - s + d_i \geq 0$. Suppose one of the following conditions holds:*

*(1)* $\max\{d_i, d_j\} = r$*;*

*(2)* $\mathfrak{g}$ *is of type* $\mathsf{B}_n$ *and* $i = j = n$*;*

*(3)* $\mathfrak{g}$ *is of type* $\mathsf{C}_n$ *and* $i + j \leq n$*;*

*(4)* $\mathfrak{g}$ *is of type* $\mathsf{F}_4$ *and* $i = j = 4$*.*

*Then, for any Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$ for $\mathfrak{g}$, we have*

$$\widetilde{c}_{ij}(p - s + d_i) = \left\langle H_{\mathcal{Q}}(j, s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i, p)) \right\rangle_{\mathcal{Q}}.$$

*Moreover, by varying $p$ and $s$, the value $p - s + d_i$ covers all integers $u \in \mathbb{Z}$ such that $\widetilde{c}_{ij}(u) \neq 0$.*

*Proof.* Let $\imath \in i$ and $\jmath \in j$ be the residues of $H_{\mathcal{Q}}(i,p)$ and $H_{\mathcal{Q}}(j,s)$, respectively. Note that

$$(p - s + d_i) + \xi_\jmath - \xi_\imath - d_i = (\xi_\jmath - s) - (\xi_\imath - p) \in 2\mathbb{Z}$$

since $p$ is congruent to $\xi_\imath$ modulo $2d_i$ and $s$ is congruent to $\xi_\jmath$ modulo $2d_j$. By Theorem 1.8.14, we have

$$\widetilde{c}_{ij}(p - s + d_i) = (\varpi_\imath, \tau_{\mathcal{Q}}^{(\xi_\jmath - s)/2 - (\xi_\imath - p)/2}(\gamma_\jmath^{\mathcal{Q}})).$$

By Proposition 1.8.13, this equals

$$\begin{aligned}
\left\langle \tau_{\mathcal{Q}}^{(\xi_\jmath - s)/2 - (\xi_\imath - p)/2}(I_\jmath^{\mathcal{Q}}), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(I_\imath^{\mathcal{Q}}) \right\rangle_{\mathcal{Q}} &= \left\langle \tau_{\mathcal{Q}}^{(\xi_\jmath - s)/2}(I_\jmath^{\mathcal{Q}}), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(\tau_{\mathcal{Q}}^{(\xi_\imath - p)/2}(I_\imath^{\mathcal{Q}})) \right\rangle_{\mathcal{Q}} \\
&= \left\langle H_{\mathcal{Q}}(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i,p)) \right\rangle_{\mathcal{Q}},
\end{aligned}$$

where the last equality comes from Corollary 1.8.6. The last statement also follows from Theorem 1.8.14 once we realize that $(i, p + 2k) \in \widehat{I}$ for all $k \in \mathbb{Z}$ (see [FO21, Section 3.8]). $\checkmark$

**Remark 1.8.16.** By the formula above, the quasi-periodicity property of $\widetilde{c}_{ij}(u)$ described in [FO21, Corollary 4.10(1)] arises from the behavior of the Euler form under shifts. Indeed, since $\Sigma H_{\mathcal{Q}}(j,s) = H_{\mathcal{Q}}(j^*, s + rh^\vee)$ and $\langle \Sigma M, N \rangle_{\mathcal{Q}} = -\langle M, N \rangle_{\mathcal{Q}}$, we deduce that $\widetilde{c}_{ij}(u + rh^\vee) = -\widetilde{c}_{ij^*}(u)$ for $u \geq 0$. We also have the following interpretation for the positivity property described in [FO21, Corollary 4.10(6)]. By applying a tilting functor that takes $H_{\mathcal{Q}}(i,p)$ to an injective object and writing the Euler form as an alternated sum, we see that there can be at most one term in this sum that does not vanish (the case where $d_i = 1$ and $d_j = r > 1$ was done in the proof of Proposition 1.8.13). If we additionally suppose that $p - s + d_i \leq rh^\vee$, then such a term is nonnegative by [FO21, Corollary 4.10(6)]. Thus, in case this term does not vanish, there must exist a tilting functor taking both objects inside the Euler form into some category $\mathcal{C}(\mathcal{Q}')$, otherwise we would need to apply an *even* power of the shift functor to one of the objects for this to be true and we would get a contradiction to the inequality. We conclude that

$$\widetilde{c}_{ij}(p - s + d_i) = \dim_K \operatorname{Hom}_{\mathcal{D}(\mathcal{Q})} \left( H_{\mathcal{Q}}(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i,p)) \right).$$

This equality is an analog of [Fuj22, Proposition 3.8].

Finally, it is also interesting to remark that the direct sum appearing in the formula when $d_i = 1$ and $d_j = r > 1$ reflects the symmetry described in [FO21, Lemma 4.4(2)]: in this case, if $u = p - s + 1$ satisfies $0 \leq u \pm (r-1) \leq rh^\vee$, we have

$$\begin{aligned}
\widetilde{c}_{ij}(u) &= \sum_{k=0}^{r-1} \langle H_{\mathcal{Q}}(j,s), \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i,p)) \rangle_{\mathcal{Q}} = \sum_{k=0}^{r-1} \langle \tau_{\mathcal{Q}}^k(H_{\mathcal{Q}}(i,p)), S_{\mathcal{Q}}(H_{\mathcal{Q}}(j,s)) \rangle_{\mathcal{Q}} \\
&= \sum_{k=0}^{r-1} \langle H_{\mathcal{Q}}(i,p-2k), H_{\mathcal{Q}}(j,s-2r+rh^\vee) \rangle_{\mathcal{Q}} = \sum_{k=0}^{r-1} \widetilde{c}_{ji}(rh^\vee - (u+(r-1)-2k)) \\
&= \sum_{k=0}^{r-1} \widetilde{c}_{ji}(u+(r-1)-2k),
\end{aligned}$$

where we used Theorem 1.8.10 and [FO21, Corollary 4.10(3)] in the second and fifth equalities, respectively.

**Remark 1.8.17.** The proposition above gives a categorical interpretation for all the entries of the inverse of the quantum Cartan matrix in simply laced types and in type $\mathsf{B}_n$. In the remaining types, there are values of $(i,p)$ and $(j,s)$ for which the formula does not give the desired result. This occurs due to the failure of Lemma 1.8.7 when its hypotheses are removed (see Remark 1.8.8). However, it is interesting to point out a link with the conjectural formula [FO21, Conjecture 6.7] for the denominator of the $R$-matrix between Kirillov–Reshetikhin modules over the quantum affine algebra (see also Conjecture 5.9 and Remark 5.10 in [FM23]). In [FM23, Conjecture 6.15], this conjectural formula is extended to all cases by adding an extra term $\Delta_{ij}(q)$, which is zero precisely when one of the conditions in Proposition 1.8.15 holds (see [FM23, Proposition A.1]).

In [Her04], the inverse of the quantum Cartan matrix appears in the definition of a skew-symmetric pairing $\mathscr{N} : \widehat{I} \times \widehat{I} \longrightarrow \mathbb{Z}$, an essential ingredient to construct a quantization of the Grothendieck ring of a quantum affine algebra. It is defined as

$$\mathscr{N}(i,p;j,s) = \widetilde{c}_{ij}(p-s-d_i) - \widetilde{c}_{ij}(p-s+d_i) - \widetilde{c}_{ij}(s-p-d_i) + \widetilde{c}_{ij}(s-p+d_i).$$

We can describe it using the c-derived category of a Q-datum.

**Proposition 1.8.18.** *For distinct $(i,p),(j,s) \in \widehat{I}$ with $s \geq p$, we have*

$$\mathscr{N}(i,p;j,s) = \langle H_{\mathcal{Q}}(i,p), H_{\mathcal{Q}}(j,s) \rangle_{\mathcal{Q}} + \langle H_{\mathcal{Q}}(j,s), H_{\mathcal{Q}}(i,p) \rangle_{\mathcal{Q}}$$

*for any Q-datum $\mathcal{Q} = (\Delta, \sigma, \xi)$ for $\mathfrak{g}$.*

*Proof.* The result follows from Theorem 1.7.6 and [FO21, Proposition 5.21]. $\checkmark$

## 1.9 Explicit descriptions in the case of Q-data

In this section, we present some explicit descriptions of the indecomposable objects of $\mathcal{C}([\boldsymbol{i}])$ when $[\boldsymbol{i}]$ is a commutation class of reduced words $w_0$ coming from a Q-datum. They are important for the proof of Lemma 1.8.7. What follows depends mostly on Sections 1.4 and 1.8.1. In particular, we adopt the same conventions as at the end of Section 1.4.1 when we deal with representations.

**Example 1.9.1.** Let us fix $(\Delta, \sigma) = (\mathsf{A}_{2n-1}, \vee)$ and let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum of type $\mathsf{B}_n$. If we view $M \in \operatorname{ind}(\mathcal{Q})$ as a representation of the double quiver $\overline{Q}$, then we have a vector space of dimension at most one on each vertex. Since $M$ satisfies the preprojective relations, one can easily deduce that $M$ is the restriction of a representation of some orientation $Q'$ of $\mathsf{A}_{2n-1}$ under the quotient map $K\overline{Q} \longrightarrow KQ'$. In this example, we will prove that there are two orientations $Q^{\leq n} = Q^{\leq n}(\mathcal{Q})$ and $Q^{\geq n} = Q^{\geq n}(\mathcal{Q})$ of $\mathsf{A}_{2n-1}$ such that any indecomposable object of $\mathcal{C}(\mathcal{Q})$ comes from a representation of $Q^{\leq n}$ or $Q^{\geq n}$.

The definition of the orientations $Q^{\leq n}$ and $Q^{\geq n}$ depends on whether the height function $\xi$ satisfies $\xi_n = \xi_{n-1} + 1$ or $\xi_n = \xi_{n+1} + 1$, and is given as follows. Let $\imath, \jmath \in \Delta_0$ be such that $\imath \sim \jmath$. If $\xi_n = \xi_{n-1} + 1$, then there is an arrow $\imath \to \jmath$ in $Q^{\leq n}$ if and only if $\xi_\imath > \xi_\jmath$, and $Q^{\geq n}$ can obtained from $Q^{\leq n}$ by reversing the direction of the arrow between the vertices $n-1$ and $n$. On the other hand, if $\xi_n = \xi_{n+1} + 1$, then there is an arrow $\imath \to \jmath$ in $Q^{\geq n}$ if and only if $\xi_\imath > \xi_\jmath$, and $Q^{\leq n}$ can obtained from $Q^{\geq n}$ by reversing the direction of the arrow between the vertices $n$ and $n+1$.

**Proposition 1.9.2.** *Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum of type $\mathsf{B}_n$, and let $Q^{\leq n}$ and $Q^{\geq n}$ be the orientations of $\Delta = \mathsf{A}_{2n-1}$ defined above. Let $M \in \mathcal{C}(\mathcal{Q})$ be an indecomposable object of residue $\imath \in \Delta_0$ and view it as a representation of the double quiver $\overline{Q}$.*

- *If $\imath \leq n$, then $M$ is the restriction of an indecomposable representation of $Q^{\leq n}$.*
- *If $\imath \geq n$, then $M$ is the restriction of an indecomposable representation of $Q^{\geq n}$.*

*Proof.* Without loss of generality, we assume that $\xi_n = \xi_{n-1} + 1$. Let us first show the following statement: if $\imath \in \Delta_0 \setminus \{n\}$ is a source of $\mathcal{Q}$ and if $s_\imath \mathcal{Q}$ satisfies the proposition, then so does $\mathcal{Q}$. By our hypothesis on $\xi$, we have $\imath \neq n-1$. Note that $\imath$ is also a source for the quivers $Q^{\leq n}(\mathcal{Q})$ and $Q^{\geq n}(\mathcal{Q})$, and we have $Q^{\leq n}(s_\imath \mathcal{Q}) = s_\imath Q^{\leq n}(\mathcal{Q})$ and $Q^{\geq n}(s_\imath \mathcal{Q}) = s_\imath Q^{\geq n}(\mathcal{Q})$. Thus, if we apply the spherical twist $T_\imath$ to an indecomposable representation of $Q^{\leq n}(s_\imath \mathcal{Q})$ or $Q^{\geq n}(s_\imath \mathcal{Q})$ not isomorphic to $S_\imath$ (viewed as an object of $\operatorname{pvd}(\Pi)$ in the canonical way), we get a representation of $Q^{\leq n}(\mathcal{Q})$ or $Q^{\geq n}(\mathcal{Q})$ by Remark 1.3.7. Hence, by Proposition 1.4.8, if $s_\imath \mathcal{Q}$ satisfies the proposition above, so does $\mathcal{Q}$, as desired.

By the previous paragraph, after applying a source sequence, we may assume that $n$ is the unique source of $\mathcal{Q}$. Therefore, up to adding a constant, the height function $\xi$ is given as follows:

$$\xi_\imath = \begin{cases} -d^\sigma_\Delta(\imath, n) & \text{if } \imath \le n, \\ -d^\sigma_\Delta(\imath, n) - 2 & \text{if } \imath > n. \end{cases}$$

The quivers $Q^{\le n} = Q^{\le n}(\mathcal{Q})$ and $Q^{\ge n} = Q^{\ge n}(\mathcal{Q})$ are

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l] \arrow[r, dash, dotted] & \underset{n-2}{\circ} & \underset{n-1}{\circ} \arrow[l] & \underset{n}{\circ} \arrow[l] \arrow[r] & \underset{n+1}{\circ} \arrow[r, dash, dotted] & \underset{2n-2}{\circ} \arrow[r] & \underset{2n-1}{\circ} \end{tikzcd}$$

and

$$\begin{tikzcd} \underset{1}{\circ} & \underset{2}{\circ} \arrow[l] \arrow[r, dash, dotted] & \underset{n-2}{\circ} & \underset{n-1}{\circ} \arrow[l] \arrow[r] & \underset{n}{\circ} \arrow[r] & \underset{n+1}{\circ} \arrow[r, dash, dotted] & \underset{2n-2}{\circ} \arrow[r] & \underset{2n-1}{\circ} \end{tikzcd}$$

respectively.

Note that $\mathcal{Q}^\circ = \mathcal{Q}$. Moreover, there is a unique compatible reading for $X^\circ_{\mathcal{Q}}$ and the corresponding sequence of residues is $(n, n-1, \dots, 1)$. Hence, $\tau_{\mathcal{Q}} = T_n T_{n-1} \cdots T_1 \sigma$ and, by [FO21, Proposition 3.31], we have $\boldsymbol{i} \in [\mathcal{Q}]$ for

$$\boldsymbol{i} = (n, n-1, \dots, 1, \sigma(n), \sigma(n-1), \dots, \sigma(1), \dots\dots, \sigma^{h^\vee-1}(n), \sigma^{h^\vee-1}(n-1), \dots, \sigma^{h^\vee-1}(1)).$$

By item (4) in Proposition 1.3.3, we deduce that $\tau_{\mathcal{Q}}(M^{\boldsymbol{i}}_k) \cong M^{\boldsymbol{i}}_{k+n}$ for $1 \le k \le N-n$. Observe that $M^{\boldsymbol{i}}_1, M^{\boldsymbol{i}}_2, \dots, M^{\boldsymbol{i}}_n$ have residue $n, n-1, \dots, 1$, respectively. In addition, by Proposition 1.8.4, if we repeatedly apply $\tau_{\mathcal{Q}}$ to an indecomposable object of $\mathcal{C}(\mathcal{Q})$, the residue of the objects we get alternate between being less than $n$ and being greater than $n$. With these observations, we will have proved the proposition after we verify the following assertions.

(1) The indecomposable objects $M^{\boldsymbol{i}}_1, \dots, M^{\boldsymbol{i}}_n$ are representations of $Q^{\le n}$ and $M^{\boldsymbol{i}}_1$ is also a representation of $Q^{\ge n}$.

(2) Let $\star$ denote either $\le$ or $\ge$. If $M \in \operatorname{ind}(\mathcal{Q})$ is a representation of $Q^{\star n}$ and $\tau_{\mathcal{Q}}(M) \in \mathcal{C}(\mathcal{Q})$, then $\tau_{\mathcal{Q}}(M)$ is a representation of $Q^{\mathrm{op}(\star)n}$, where $\mathrm{op}(\star)$ denotes the inequality sign opposite to $\star$.

For $1 \le \imath \le \jmath \le 2n-1$ and $\star \in \{\le, \ge\}$, denote by $M^\star(\imath, \jmath)$ the indecomposable representation of $Q^{\star n}$ whose support are the vertices between $\imath$ and $\jmath$. We view it as an object of $\operatorname{pvd}(\Pi_Q)$. One can compute using Proposition 1.3.6 that $M^{\boldsymbol{i}}_k \cong M^{\le}(n-k+1, n)$ for $1 \le k \le n$ and $M^{\boldsymbol{i}}_1 \cong S_n \cong M^{\ge}(n, n)$, which proves (1). The same proposition allows us to calculate that

$$\tau_{\mathcal{Q}}(M^{\star}(\imath,\jmath)) \cong \begin{cases} M^{\mathrm{op}(\star)}(\sigma(\jmath),\sigma(\imath)) & \text{if } \jmath < n-1, \\ M^{\mathrm{op}(\star)}(\sigma(\jmath)-1,\sigma(\imath)) & \text{if } \imath < n \text{ and } n-1 \leq \jmath < 2n-1, \\ M^{\mathrm{op}(\star)}(\sigma(\jmath)-1,\sigma(\imath)-1) & \text{if } \imath \geq n \text{ and } n \leq \jmath < 2n-1, \\ M^{\leq}(n+1,\sigma(\imath)) & \text{if } \imath < n, \jmath = 2n-1 \text{ and } \star = \geq, \\ T_n(M^{\leq}(n,\sigma(\imath))) & \text{if } \imath < n, \jmath = 2n-1 \text{ and } \star = \leq, \\ \Sigma^{-1}M^{\leq}(\sigma(\imath),n) & \text{if } \imath \geq n \text{ and } \jmath = 2n-1. \end{cases}$$

In the fifth case above, note that $M^{\leq}(n,\sigma(\imath))$ does not satisfy the condition of Proposition 1.3.6 for the vertex $n$, hence $T_n(M^{\leq}(n,\sigma(\imath)))$ is not concentrated in degree 0. We deduce that if $\tau_{\mathcal{Q}}(M^{\star}(\imath,\jmath))$ is concentrated in degree 0, then it is a representation of $Q^{\mathrm{op}(\star)n}$. This implies (2) and finishes the proof. $\checkmark$

**Example 1.9.3.** Suppose $(\Delta,\sigma) = (\mathsf{D}_{n+1},\vee)$ is of type $\mathsf{C}_n$. Let $Q$ be the following orientation for $\Delta$:

$$\begin{tikzcd}[row sep=small] & & & & n \\ 1 \arrow[r] & 2 \arrow[r, dotted, no head] & n-2 \arrow[r] & n-1 \arrow[ur] \arrow[dr] & \\ & & & & n+1 \end{tikzcd}$$

Let $\mathcal{Q} = (\Delta,\sigma,\xi)$ be the Q-datum defined by $\xi_k = k$ for $1 \leq k \leq n$ and $\xi_{n+1} = n-2$. The indecomposable objects of $\mathcal{C}(\mathcal{Q})$ can be listed in the following five families.

(1) For $1 \leq \imath \leq \jmath \leq n-1$, define

$$\begin{tikzcd}[column sep=small] &&&&&&&&&&& 0 \arrow[dl, bend left] \\ A^{0,0}_{\imath,\jmath} = 0 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[urr, bend left] \arrow[drr, bend left] && \\ &&&&&&&&&&& 0 \arrow[ull, bend left] \end{tikzcd}$$

If we read from left to right, the first nonzero vector space appears at vertex $\imath$ and the last at vertex $\jmath$.

(2) For $1 \leq \imath \leq n$, define

$$\begin{tikzcd}[column sep=small] &&&&&&&&& K \arrow[dl, bend left, "\mathrm{id}"] \\ A^{1,0}_{\imath} = 0 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[urr, bend left, "0"] \arrow[drr, bend left] && \\ &&&&&&&&& 0 \arrow[ull, bend left] \end{tikzcd}$$

If we read from left to right, the first nonzero vector space appears at vertex $\imath$.

(3) For $1 \leq \imath \leq n$, define

$$A_\imath^{0,1} = \begin{tikzcd} & & & & & & & 0 \arrow[dl, bend left] \\ 0 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[ur, bend left] \arrow[dr, bend left, "0"] & \\ & & & & & & & K \arrow[ul, bend left, "\mathrm{id}"] \end{tikzcd}$$

If we read from left to right, the first nonzero vector space appears at vertex $\imath$ (or at vertex $n+1$ if $\imath = n$).

(4) For $1 \leq \imath \leq n-1$, define

$$A_\imath^{1,1} = \begin{tikzcd} & & & & & & & K \arrow[dl, bend left, "\mathrm{id}"] \\ 0 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[ur, bend left, "0"] \arrow[dr, bend left, "\mathrm{id}"] & \\ & & & & & & & K \arrow[ul, bend left, "0"] \end{tikzcd}$$

If we read from left to right, the first nonzero vector space appears at vertex $\imath$.

(5) For $1 \leq \imath < \jmath \leq n-1$, define

$$B_{\imath,\jmath} = \begin{tikzcd} & & & & & & & & & & & K \arrow[dl, bend left, "i_2"] \\ 0 \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & 0 \arrow[l, shift left] \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "0"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "i_1"] & K^2 \arrow[l, shift left, "\pi_2"] \arrow[r, shift left, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & K^2 \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & \cdots \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & K^2 \arrow[l, shift left, "\mathrm{id}"] \arrow[ur, bend left, "0"] \arrow[dr, bend left, "\pi_2"] & \\ & & & & & & & & & & K \arrow[ul, bend left, "i_1"] \end{tikzcd}$$

If we read from left to right, the first nonzero vector space appears at vertex $\imath$ and the first two-dimensional vector space at vertex $\jmath$.

An easy computation shows that $I_\imath^{\mathcal{Q}} \cong A_\imath^{1,0}$ for $1 \leq \imath \leq n$ and $I_{n+1}^{\mathcal{Q}} \cong A_{n-1}^{1,1}$. By Corollary 1.8.6, we can compute any object in $\mathrm{ind}(\mathcal{Q})$ by applying some power of $\tau_{\mathcal{Q}}$ to $I_\imath^{\mathcal{Q}}$ for some $1 \leq \imath \leq n+1$. Since the set of objects displayed above represents all positive roots of $\mathsf{R}^+$, we only need to show that it is closed (up to isomorphism and up to an application of the suspension functor) under the action of $\tau_{\mathcal{Q}}$. Indeed, knowing that $\tau_{\mathcal{Q}} = T_n \cdots T_2 T_1 \sigma$, one can compute:

$$\tau_{\mathcal{Q}}(A_{\imath,\jmath}^{0,0}) \cong \begin{cases} A_{\imath-1,\jmath-1}^{0,0} & \text{if } \imath \neq 1, \\ \Sigma^{-1} A_\jmath^{1,0} & \text{if } \imath = 1, \end{cases} \qquad \tau_{\mathcal{Q}}(A_\imath^{1,0}) \cong \begin{cases} A_{\imath-1}^{1,1} & \text{if } \imath \neq 1, \\ A_n^{0,1} & \text{if } \imath = 1, \end{cases}$$

$$\tau_{\mathcal{Q}}(A_\imath^{0,1}) \cong \begin{cases} A_{\imath-1,n-1}^{0,0} & \text{if } \imath \neq 1, \\ \Sigma^{-1} A_n^{1,0} & \text{if } \imath = 1, \end{cases} \qquad \tau_{\mathcal{Q}}(A_\imath^{1,1}) \cong \begin{cases} B_{\imath-1,n-1} & \text{if } \imath \neq 1, \\ A_{n-1}^{0,1} & \text{if } \imath = 1, \end{cases}$$

$$\tau_{\mathcal{Q}}(B_{\imath,\jmath}) \cong \begin{cases} B_{\imath-1,\jmath-1} & \text{if } \imath \neq 1, \\ A^{0,1}_{\jmath-1} & \text{if } \imath = 1. \end{cases}$$

By Corollary 1.8.5, we also deduce that $P^{\mathcal{Q}}_{\imath} \cong A^{0,0}_{1,\imath}$ for $1 \leq \imath \leq n-1$, $P^{\mathcal{Q}}_{n} \cong B_{12}$, and $P^{\mathcal{Q}}_{n+1} \cong A^{0,1}_{1}$.

**Example 1.9.4.** Suppose $(\Delta, \sigma) = (\mathsf{E}_6, \vee)$ is of type $\mathsf{F}_4$. Let $Q$ be the following orientation for $\Delta$:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[dr] & & \\ & & 3 \arrow[r] & 6 \\ 5 \arrow[r] & 4 \arrow[ur] & & \end{tikzcd}$$

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be the Q-datum defined by $\xi_1 = 7$, $\xi_2 = 9$, $\xi_3 = 8$, $\xi_4 = 7$, $\xi_5 = 9$, and $\xi_6 = 7$. The indecomposable objects of $\mathcal{C}(\mathcal{Q})$ (with their coordinates in $\Gamma_{\mathcal{Q}}$) are depicted in Figure 1.2. The unspecified objects are as follows:

$$M_1 = \begin{tikzcd} & & K \arrow[d, bend left, "i_1"] & & \\ K \arrow[r, bend left, "0"] & K^2 \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, "\mathrm{id}"] & K^2 \arrow[l, bend left, "0"] \arrow[u, bend left, "\pi_2"] \arrow[r, bend right, "\pi_2"'] & K \arrow[l, bend right, "i_1"'] \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}$$

$$M_2 = \begin{tikzcd} & & K \arrow[d, bend left, "i_1"] & & \\ 0 \arrow[r, bend left] & K \arrow[l, bend left] \arrow[r, bend left, "i_2"] & K^2 \arrow[l, bend left, "0"] \arrow[u, bend left, "\pi_2"] \arrow[r, bend right, "\pi_2"'] & K \arrow[l, bend right, "i_1"'] \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}$$

$$M_3 = \begin{tikzcd} & & K \arrow[d, bend left, "0"] & & \\ K \arrow[r, bend left, "0"] & K^2 \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, "\mathrm{id}"] & K^2 \arrow[l, bend left, "0"] \arrow[u, bend left, "\pi_1+\pi_2"] \arrow[r, bend right, "i_1\pi_2"'] & K^2 \arrow[l, bend right, "i_1\pi_2"'] \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

$$M_4 = \begin{tikzcd} & & K \arrow[d, bend left, "0"] & & \\ K \arrow[r, bend left, "0"] & K^2 \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, "\mathrm{id}"] & K^2 \arrow[l, bend left, "0"] \arrow[u, bend left, "\pi_1+\pi_2"] \arrow[r, bend right, "\pi_2"'] & K \arrow[l, bend right, "0"'] \arrow[r, bend right] & 0 \arrow[l, bend right] \end{tikzcd}$$

$$M_5 = \begin{tikzcd} & & K \arrow[d, bend left, "-i_1"] & & \\ K \arrow[r, bend left, "0"] & K^2 \arrow[l, bend left, "\pi_1"] \arrow[r, bend left, "i_2\pi_1+i_3\pi_2"] & K^3 \arrow[l, bend left, "0"] \arrow[u, bend left, "\pi_3"] \arrow[r, bend right, "-i_1\pi_2+(i_1-i_2)\pi_3"'] & K^2 \arrow[l, bend right, "i_1\pi_2"'] \arrow[r, bend right, "0"'] & K \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

$$M_6 = \begin{tikzcd} & & K \arrow[d, shift left, "-i_1"] & & \\ 0 \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "i_2"] & K^2 \arrow[l, shift left, "0"] \arrow[u, shift left, "\pi_2"] \arrow[r, shift right, "(i_1-i_2)\pi_2"'] & K^2 \arrow[l, shift right, "i_1\pi_2"'] \arrow[r, shift right, "0"'] & K \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

$$M_7 = \begin{tikzcd} & & K^2 \arrow[d, shift left, "-i_1\pi_2"] & & \\ K \arrow[r, shift left, "0"] & K^2 \arrow[l, shift left, "\pi_1"] \arrow[r, shift left, "i_2\pi_1+i_3\pi_2"] & K^3 \arrow[l, shift left, "0"] \arrow[u, shift left, "i_1\pi_2+i_2\pi_3"] \arrow[r, shift right, "i_1\pi_3"'] & K^2 \arrow[l, shift right, "-i_1\pi_1+(i_1-i_2)\pi_2"'] \arrow[r, shift right, "0"'] & K \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

$$M_8 = \begin{tikzcd} & & K \arrow[d, shift left, "0"] & & \\ K \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "i_2"] & K^2 \arrow[l, shift left, "0"] \arrow[u, shift left, "\pi_2"] \arrow[r, shift right, "0"'] & K \arrow[l, shift right, "i_1-i_2"'] \arrow[r, shift right, "0"'] & K \arrow[l, shift right, "\mathrm{id}"'] \end{tikzcd}$$

$$M_9 = \begin{tikzcd} & & K \arrow[d, shift left, "0"] & & \\ K \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "i_2"] & K^2 \arrow[l, shift left, "0"] \arrow[u, shift left, "\pi_1-\pi_2"] \arrow[r, shift right, "i_1\pi_2"'] & K^2 \arrow[l, shift right, "i_1\pi_2"'] \arrow[r, shift right, "0"'] & K \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

$$M_{10} = \begin{tikzcd} & & K \arrow[d, shift left, "i_1"] & & \\ 0 \arrow[r, shift left] & K \arrow[l, shift left] \arrow[r, shift left, "i_2"] & K^2 \arrow[l, shift left, "0"] \arrow[u, shift left, "\pi_2"] \arrow[r, shift right, "\pi_2"'] & K \arrow[l, shift right, "i_1"'] \arrow[r, shift right] & 0 \arrow[l, shift right] \end{tikzcd}$$

$$M_{11} = \begin{tikzcd} & & K \arrow[d, shift left, "i_1"] & & \\ K \arrow[r, shift left, "0"] & K \arrow[l, shift left, "\mathrm{id}"] \arrow[r, shift left, "i_2"] & K^2 \arrow[l, shift left, "0"] \arrow[u, shift left, "\pi_2"] \arrow[r, shift right, "\pi_2"'] & K \arrow[l, shift right, "i_1"'] \arrow[r, shift right] & 0 \arrow[l, shift right] \end{tikzcd}$$

**Example 1.9.5.** Suppose $(\Delta, \sigma) = (\mathsf{D}_4, \widetilde{\vee})$ is of type $\mathsf{G}_2$. Let $Q$ be the following orientation for $\Delta$:

$$\begin{tikzcd}[row sep=small] & & 3 \arrow[dl] \\ 1 \arrow[r] & 2 & \\ & & 4 \arrow[ul] \end{tikzcd}$$

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be the Q-datum defined by $\xi_1 = 3$, $\xi_2 = 4$, $\xi_3 = 1$, and $\xi_4 = 5$. The indecomposable objects of $\mathcal{C}(\mathcal{Q})$ are shown below.

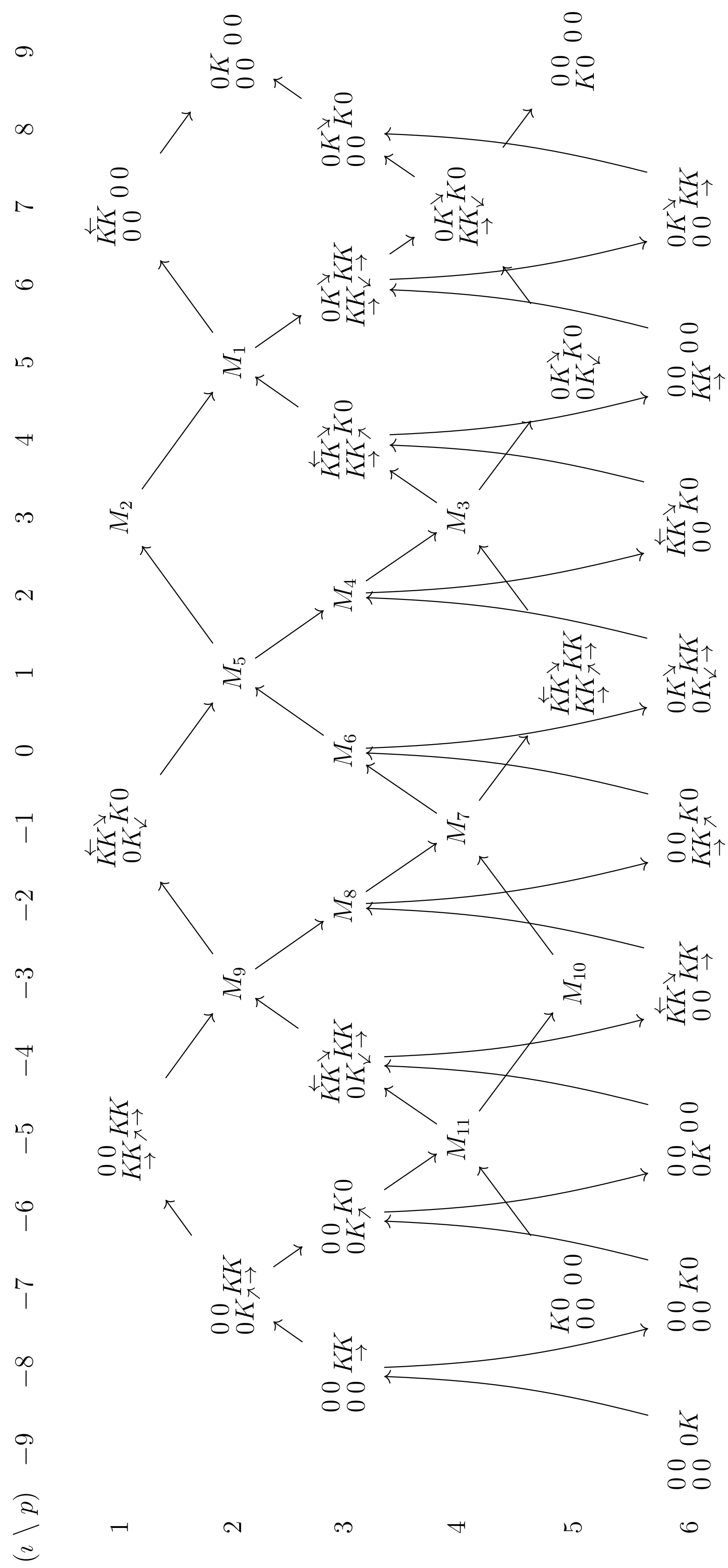
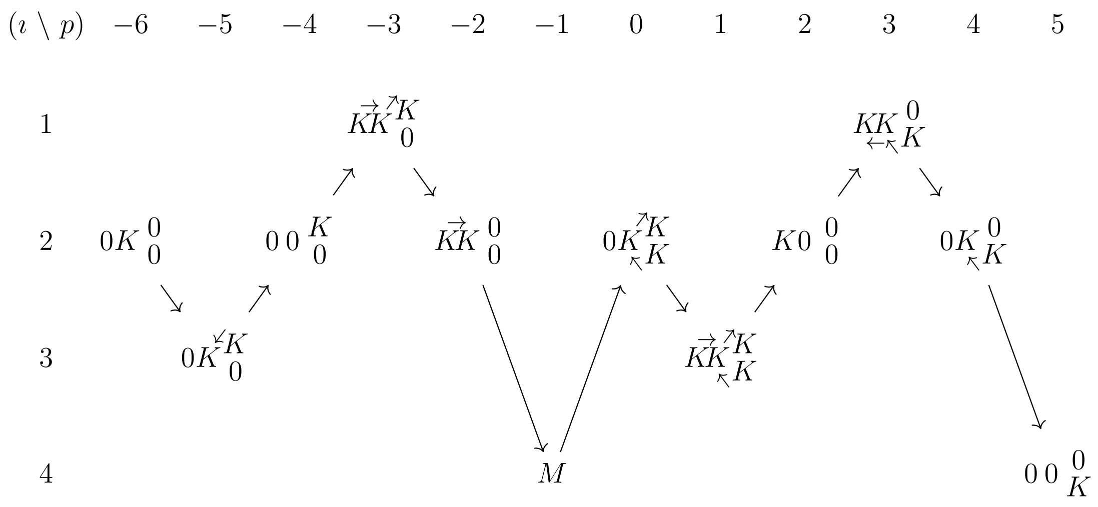



Figure 1.2: Indecomposable objects of $\mathcal{C}(\mathcal{Q})$ for a Q-datum $\mathcal{Q}$ of type $\mathsf{F}_4$ (see Example 1.9.4 for more details).



Here, the object $M$ is the following representation:

$$\begin{tikzcd}
 & & K \arrow[dl, bend left, "i_1"] \\
K \arrow[r, bend left, "-i_1"] & K^2 \arrow[l, bend left, "\pi_2"] \arrow[ur, bend left, "\pi_2"] \arrow[dr, bend left, "0"] & \\
 & & K \arrow[ul, bend left, "i_2"]
\end{tikzcd}$$

# Chapter 2

# Categories of split filtrations and graded quiver varieties

This chapter consists of the article [Can26b], available on arXiv as `arXiv:2601.09509`.

## 2.1 Introduction

The theory of (graded) quiver varieties and its applications to representation theory were developed by Nakajima in a series of papers [Nak94, Nak98, Nak01a, Nak04], motivated notably by earlier work of Ringel [Rin90] and Lusztig [Lus90]. In particular, Nakajima used these varieties to give a geometric realization of certain finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ associated with a finite-dimensional complex simple Lie algebra $\mathfrak{g}$, thereby obtaining important information about their structure and $q$-character theory. The graded version of quiver varieties also plays a prominent role in Nakajima's study of cluster algebras and their monoidal categorifications [Nak11], which was subsequently generalized by Kimura–Qin [KQ14] to graded quiver varieties associated with arbitrary acyclic quivers.

A fruitful approach to the study of graded quiver varieties was pioneered by Hernandez–Leclerc [HL15] and further developed by Leclerc–Plamondon [LP13] and Keller–Scherotzke [KS16]. In the latter two works, affine graded quiver varieties were shown to be isomorphic to representation varieties of a category $\mathcal{S}$, called the *singular Nakajima category*. Moreover, for any Dynkin quiver $Q$ of the same type as $\mathfrak{g}$, Keller and Scherotzke constructed a functor

$$\Phi : \operatorname{mod} \mathcal{S} \longrightarrow \mathcal{D}^b(\operatorname{mod} kQ) \tag{2.1}$$

from the category of finite-dimensional $\mathcal{S}$-modules to the bounded derived category of representations of $Q$ over the field $k = \mathbb{C}$ of complex numbers. They proved that this functor recovers Nakajima's regular stratification on the affine graded quiver variety in the following sense: two $\mathcal{S}$-modules $M$ and $N$ lie in the same stratum if and only if $\Phi(M)$ and $\Phi(N)$ are isomorphic in $\mathcal{D}^b(\operatorname{mod} kQ)$. These results have found several applications, particularly in the theory of quantum groups and their representations (see, for instance, [KS14, Qin16, SS16, LW21, Fuj22]).

Our goal in this chapter is to extend this framework to another class of varieties defined independently by Nakajima [Nak01b] (see also [Nak11] for the graded version), Malkin [Mal03], and Varagnolo–Vasserot [VV03]. We call these varieties the *$n$-fold affine graded tensor product varieties*, where $n \geq 1$ is a positive integer. Through Nakajima's constructions, they provide a geometric realization of the tensor product of a sequence of $n$ standard modules over the quantum affine algebra. These varieties have applications, for example, as valuable tools in the geometric realization of the quantum affine Schur–Weyl duality [Fuj20] and in the study of $R$-matrices and their denominators [Fuj22, FH26].

Our first step is to identify the $n$-fold affine graded tensor product variety with an appropriate representation variety. This is achieved by considering a category of filtrations of $\mathcal{S}$-modules of length $n$ equipped with a splitting, which we call *split filtrations* for short. Let us describe this construction more precisely, and in greater generality, as it may be of independent interest. Fix a base field $k$. Let $\mathcal{A}$ be a small $k$-category and let $\mathcal{B}$ be a (not necessarily full) subcategory of $\mathcal{A}$ containing all objects. The *category of split filtrations of $\mathcal{A}$-modules over $\mathcal{B}$* is the category $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ whose objects are $\mathcal{A}$-modules $M$ equipped with a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

by $\mathcal{A}$-submodules together with $\mathcal{B}$-linear maps $r_i^M : M_i \to M_{i-1}$ that are retractions for the inclusions $M_{i-1} \to M_i$. In particular, when restricting the module structure to $\mathcal{B}$, each inclusion $M_{i-1} \to M_i$ becomes a split monomorphism. We prove that $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ is itself a module category.

**Theorem 2.1.1** (Proposition 2.3.3)**.** *The category* $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ *is equivalent to* $\operatorname{Mod} \mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ *for a certain $k$-category* $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$.

The category $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ is an example of a *triangular matrix category*. This notion generalizes triangular matrix rings, in the spirit of Mitchell's philosophy that "additive categories are rings with several objects" [Mit72]. Triangular matrix categories have appeared in various forms in the literature (see, for example, [Tab08, KL15, LOS23]). For instance, when $n = 2$, one has

$$\mathcal{T}^2_{\mathcal{B}}(\mathcal{A}) = \begin{pmatrix} \mathcal{A} & 0 \\ \ker(\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A}) & \mathcal{A} \end{pmatrix},$$

where the $\mathcal{A}$-$\mathcal{A}$-bimodule on the lower-left corner is the kernel of the morphism induced by the composition of $\mathcal{A}$. A similar description holds for $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ for arbitrary $n$ (see Lemma 2.3.7).

We are primarily interested in the case where $\mathcal{A} = \mathcal{S}$ is the singular Nakajima category and $\mathcal{B} = k\mathcal{S}_0$ is the smallest $k$-subcategory of $\mathcal{A}$ containing all objects. We define the *singular Nakajima category of split filtrations of length* $n$ as the category $\mathcal{S}^{n\text{-filt}} = \mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$. As an immediate consequence of [LP13, Theorem 2.4] and the proof of [Nak01b, Lemma 3.6], the $n$-fold affine graded tensor product varieties are realized as representation varieties of $\mathcal{S}^{n\text{-filt}}$. Although these varieties only depend on the filtration by $\mathcal{S}$-submodules, we emphasize that the splitting over $k\mathcal{S}_0$ is encoded in the group action considered in [Nak01b] and is necessary for Theorem 2.1.1 to hold.

We also provide an alternative description of $\mathcal{S}^{n\text{-filt}}$ in terms of mesh categories, building on Keller–Scherotzke's construction of the category $\mathcal{S}$ (see Proposition 2.4.3). As a consequence, we show in Section 2.4 that many results and arguments in [KS16] extend to our setting. The realization of $\mathcal{S}^{n\text{-filt}}$ as a triangular matrix category plays a crucial role in these generalizations, particularly in the proof of Lemma 2.4.15. Our main result is a generalization of [KS16, Theorem 5.18], describing the stable category $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ of (pseudo-coherent) Gorenstein projective $\mathcal{S}^{n\text{-filt}}$-modules, defined via totally acyclic complete resolutions whose terms are finitely generated projective.

**Theorem 2.1.2** (Theorem 2.4.22)**.** *There is an equivalence of triangulated categories*

$$\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\, k\overleftarrow{\mathsf{A}}_n \otimes kQ),$$

*where $k\overleftarrow{\mathsf{A}}_n$ denotes the path algebra of a linearly oriented Dynkin quiver of type $\mathsf{A}_n$.*

We define the *stratification functor* $\Phi^n$ from the category of finite-dimensional $\mathcal{S}^{n\text{-filt}}$-modules to $\mathcal{D}^b(k\overleftarrow{\mathsf{A}}_n \otimes kQ)$ as the composition

$$\mathrm{mod}\, \mathcal{S}^{n\text{-filt}} \xrightarrow{\Omega} \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}) \xrightarrow{\sim} \mathcal{D}^b(k\overleftarrow{\mathsf{A}}_n \otimes kQ),$$

where $\Omega$ is the syzygy functor, sending an $\mathcal{S}^{n\text{-filt}}$-module $M$ to the kernel of a projective cover $P \to M$. When $n = 1$, this recovers Keller–Scherotzke's functor $\Phi$ in (2.1). We show that $\Phi^n$ is closely related to the stratification of the affine graded quiver variety.

**Theorem 2.1.3** (Corollary 2.4.32)**.** *Let $M$ and $N$ be two points in an $n$-fold affine graded tensor product variety, viewed as $\mathcal{S}^{n\text{-filt}}$-modules. If $\Phi^n(M) \cong \Phi^n(N)$, then for any $0 \leq i < j \leq n$, the subquotients $M_j/M_i$ and $N_j/N_i$ of the filtrations of $M$ and $N$ lie in the same stratum of an affine graded quiver variety.*

As shown in Example 2.4.33, the converse does not hold in general. Nevertheless, it would be interesting to determine when it does, and more generally, to give a geometric description of the partition on the $n$-fold affine graded tensor product variety induced by $\Phi^n$.

### Organization

This chapter is organized as follows. In Section 2.2, we review the background material needed for our results. We recall the definition and basic properties of modules over categories, introduce triangular matrix categories and describe their module categories, and review the construction of mesh categories and Happel's embedding. In Section 2.3, we define and study the category of split filtrations, proving Theorem 2.1.1 and establishing useful properties of the canonical functors introduced in Section 2.3.2. In Section 2.4, we study the category of split filtrations of the singular Nakajima category and explain its connection to graded tensor product varieties. We generalize many results and arguments of [KS16] to our setting. For example, we explicitly describe the projective and injective (co)resolutions of simple $\mathcal{S}^{n\text{-filt}}$-modules, prove that $\mathcal{S}^{n\text{-filt}}$ is weakly Gorenstein, and characterize its stable category of Gorenstein projective (and injective) modules. Finally, we establish compatibility formulas between the stratification functors for different values of $n$, leading to Theorem 2.1.3.

## 2.2 Preliminaries

Throughout this chapter, we fix a field $k$. All our categories are $k$-categories, that is, enriched over the category $\operatorname{Mod} k$ of $k$-vector spaces. Unadorned tensor products are taken over $k$.

### 2.2.1 Modules over categories

Let $\mathcal{A}$ be a small $k$-category. A *right $\mathcal{A}$-module* is a $k$-linear functor $\mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$. We denote by $\operatorname{Mod} \mathcal{A}$ the category of right $\mathcal{A}$-modules, which is an abelian category. Similarly, a *left $\mathcal{A}$-module* is a $k$-linear functor $\mathcal{A} \to \operatorname{Mod} k$. If not stated, a module is by default a *right* module.

**Notation 2.2.1.** Let $M$ be a right $\mathcal{A}$-module and $a : x \to y$ a morphism in $\mathcal{A}$. If $m \in M(y)$, we will denote $M(a)(m) \in M(x)$ simply by $m \cdot a$ or $ma$. A similar notation applies to left modules.

Let $x \in \mathcal{A}$. The *free module* (or the right representable module) is the functor $x^{\wedge} = \mathcal{A}(-, x) : \mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$. The *cofree module* is the functor $x^{\vee} = D(\mathcal{A}(x, ?)) : \mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$, where $D$ denotes the duality over $k$. We have canonical isomorphisms

$$\operatorname{Hom}(x^{\wedge}, M) = M(x) \quad \text{and} \quad \operatorname{Hom}(M, x^{\vee}) = D(M(x))$$

for any $\mathcal{A}$-module $M$, so that free (resp. cofree) modules are projective (resp. injective).

We say that a module $M$ is *finitely generated* (resp. *finitely cogenerated*) if it is isomorphic to a quotient (resp. a submodule) of a finite direct sum of free (resp. cofree) modules.

We will be mainly interested in the case where $\mathcal{A}$ satisfies the following conditions.

**Assumption 2.2.2.** The category $\mathcal{A}$ is

(1) Hom-*finite*, that is, its morphism spaces are finite-dimensional;

(2) *directed*, that is, the endomorphism algebra of each object is $k$, and there is a partial order on the set of objects such that $\mathcal{A}(x, y) \neq 0$ implies $x \leq y$.

We additionally assume that the partial order above is *locally finite*, that is, every bounded interval is finite.

For $x \in \mathcal{A}$, let $S_x$ be the $\mathcal{A}$-module with $S_x(x) = k$ and $S_x(y) = 0$ for $y \neq x$, and the evident action on morphisms. By the directedness assumption, every finite-dimensional simple $\mathcal{A}$-module is of this form, and $S_x$ is the unique simple quotient of $x^\wedge$ and the unique simple submodule of $x^\vee$.

An $\mathcal{A}$-module $M$ is *pointwise finite-dimensional* if $M(x)$ is finite-dimensional for all $x \in \mathcal{A}$. It is *right bounded* (resp. *left bounded*) if there are objects $x_1, \ldots, x_n \in \mathcal{A}$ such that $M(x) \neq 0$ implies $x \leq x_i$ (resp. $x \geq x_i$) for some $1 \leq i \leq n$. For example, finitely generated projective modules are right bounded, while finitely cogenerated injective modules are left bounded. Both are pointwise finite-dimensional by the Hom-finiteness assumption.

If $M$ is pointwise finite-dimensional and right bounded, then $M$ admits a projective cover $P_0 \to M$ where $P_0$ is a (possibly infinite) coproduct of free modules. The multiplicity of $x^\wedge$ in $P_0$ is the dimension of $\mathrm{Hom}(M, S_x)$, which is finite. The kernel of $P_0 \to M$ is again right bounded and pointwise finite-dimensional (by the locally finiteness hypothesis), so that we can iterate this procedure. Consequently, any such module $M$ admits a minimal projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where $P_i$ is a coproduct of free modules and the multiplicity of $x^\wedge$ in $P_i$ is given by the dimension of $\mathrm{Ext}^i(M, S_x)$. We have a dual statement for computing injective coresolutions of pointwise finite-dimensional left bounded modules.

### 2.2.2 Bimodules and tensor products

Let $\mathcal{A}$ and $\mathcal{B}$ be (small) $k$-categories. An $\mathcal{A}$-$\mathcal{B}$-bimodule is a $k$-linear functor $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Mod}\, k$. Here, the tensor product $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A}$ is the category whose objects are pairs of objects in $\mathcal{B}^{\mathrm{op}}$ and $\mathcal{A}$, and whose morphism spaces are the tensor products of those in $\mathcal{B}^{\mathrm{op}}$ and in

$\mathcal{A}$. If $X : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Mod}\, k$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, we may specialize at objects $a \in \mathcal{A}$ and $b \in \mathcal{B}$ to obtain a right $\mathcal{B}$-module $X(-, a)$ or a left $\mathcal{A}$-module $X(b, ?)$. For example, the Hom-functor $\mathcal{A}(-, ?) : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Mod}\, k$ is an $\mathcal{A}$-bimodule whose specializations yield the left and right representable modules. We will also call it the *regular $\mathcal{A}$-bimodule* and denote it by $\mathcal{A}$.

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be $k$-categories. If $X$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule and $Y$ is a $\mathcal{B}$-$\mathcal{C}$-bimodule, the tensor product $X \otimes_{\mathcal{B}} Y$ is the $\mathcal{A}$-$\mathcal{C}$-bimodule defined on objects $a \in \mathcal{A}$ and $c \in \mathcal{C}$ as

$$(X \otimes_{\mathcal{B}} Y)(c, a) = \operatorname{coker} \left( \bigoplus_{b_1, b_2 \in \mathcal{B}} X(b_2, a) \otimes \mathcal{B}(b_1, b_2) \otimes Y(c, b_1) \xrightarrow{\nu} \bigoplus_{b \in \mathcal{B}} X(b, a) \otimes Y(c, b) \right),$$

where $\nu(x \otimes f \otimes y) = xf \otimes y - x \otimes fy$. Since $X \otimes_{\mathcal{B}} Y$ is again a bimodule, this construction may be iterated. It behaves as the tensor product of bimodules over ordinary rings.

### 2.2.3 Triangular matrix categories

Let $\mathcal{A}$ and $\mathcal{B}$ be small $k$-categories. Given a $\mathcal{B}$-$\mathcal{A}$ bimodule $X : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{B} \to \mathrm{Mod}\, k$, we define the *(lower) triangular matrix category*

$$\mathcal{T} = \mathcal{T}(\mathcal{A}, X, \mathcal{B}) = \begin{pmatrix} \mathcal{A} & 0 \\ X & \mathcal{B} \end{pmatrix}$$

to be the $k$-category whose set of objects is the disjoint union of the set of objects of $\mathcal{A}$ and $\mathcal{B}$, and whose morphism spaces are given as follows:

$$\mathcal{T}(x, y) = \begin{cases} \mathcal{A}(x, y), & \text{if } x, y \in \mathcal{A}, \\ \mathcal{B}(x, y), & \text{if } x, y \in \mathcal{B}, \\ X(x, y), & \text{if } x \in \mathcal{A}, y \in \mathcal{B}, \\ 0, & \text{if } x \in \mathcal{B}, y \in \mathcal{A}. \end{cases}$$

The composition law of $\mathcal{T}$ is induced by those of $\mathcal{A}$ and $\mathcal{B}$, and by the bimodule structure of $X$.

**Remark 2.2.3.** If $\mathcal{A}$ and $\mathcal{B}$ contain a single object each, then we may view $\mathcal{A}$ and $\mathcal{B}$ as $k$-algebras. In this case, we can form the triangular matrix algebra associated with $\mathcal{A}$, $\mathcal{B}$, and $X$. It is isomorphic to the matrix ring of the above category $\mathcal{T}$ in the sense of [Mit72, Section 7] and, consequently, it is Morita equivalent to $\mathcal{T}$.

**Remark 2.2.4.** Our definition is taken from [Tab08], where Tabuada defines the (upper) triangular dg category associated with two small dg categories and a dg bimodule. In this setting, a related construction can be found in [KL15]. If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, the notion of a triangular matrix category also appears in [LOS23]. With the appropriate identifications, their category is equivalent to the "additivization" of the category $\mathcal{T}$ above, that is, the category of finitely generated projective $\mathcal{T}$-modules.

As proved in [ARS97] (and in [LOS23] for the version with categories), the category of $\mathcal{T}$-modules is equivalent to the comma category (see [Mac98, Section II.6]) associated with the pair of functors given by $-\otimes_{\mathcal{B}} X : \operatorname{Mod}\mathcal{B} \to \operatorname{Mod}\mathcal{A}$ and the identity functor on $\operatorname{Mod}\mathcal{A}$. Its objects are triples $(M_1, M_2, \varphi)$ where $M_1$ is an $\mathcal{A}$-module, $M_2$ is a $\mathcal{B}$-module, and $\varphi : M_2 \otimes_{\mathcal{B}} X \to M_1$ is a morphism of $\mathcal{A}$-modules. A morphism $(M_1, M_2, \varphi) \to (N_1, N_2, \psi)$ is given by morphisms $f_i : M_i \to N_i$ for $i = 1, 2$ such that the following square commutes:

$$\begin{tikzcd}
M_2 \otimes_{\mathcal{B}} X \arrow[r, "f_2\otimes\mathrm{id}"] \arrow[d, "\varphi"'] & N_2 \otimes_{\mathcal{B}} X \arrow[d, "\psi"] \\
M_1 \arrow[r, "f_1"'] & N_1
\end{tikzcd}$$

The equivalence takes a triple $(M_1, N_1, \varphi)$ to the $\mathcal{T}$-module which is best thought as the "row module" $\begin{pmatrix} M_1 & M_2 \end{pmatrix}$, where $\mathcal{T}$ acts by "matrix multiplication" on the right. From now on, we will identify $\operatorname{Mod}\mathcal{T}$ with this comma category. The category $\operatorname{Mod}\mathcal{T}^{\mathrm{op}}$ of left $\mathcal{T}$-modules can also be identified with a comma category whose objects are triples $(M_1', M_2', \varphi')$ where $M_1'$ is a left $\mathcal{A}$-module, $M_2'$ is a left $\mathcal{B}$-module, and $\varphi' : X \otimes_{\mathcal{A}} M_1' \to M_2'$ is a morphism of left $\mathcal{B}$-modules.

**Remark 2.2.5.** By iterating the definitions above, we may construct triangular matrix categories of arbitrary order. Following [Wan16], they can be explicitly described as follows. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be small $k$-categories and, for all $n \geq i > j \geq 1$, let $X_{ij}$ be an $\mathcal{A}_i$-$\mathcal{A}_j$-bimodule. Suppose we are given bimodule morphisms $\mu_{ikj} : X_{ik} \otimes_{\mathcal{A}_k} X_{kj} \to X_{ij}$ such that the square

$$\begin{tikzcd}
X_{ik} \otimes_{\mathcal{A}_k} X_{kl} \otimes_{\mathcal{A}_l} X_{lj} \arrow[r, "\mathrm{id}\otimes\mu_{klj}"] \arrow[d, "\mu_{ikl}\otimes\mathrm{id}"'] & X_{ik} \otimes_{\mathcal{A}_k} X_{kj} \arrow[d, "\mu_{ikj}"] \\
X_{il} \otimes_{\mathcal{A}_l} X_{lj} \arrow[r, "\mu_{ilj}"'] & X_{ij}
\end{tikzcd} \tag{2.2}$$

commutes for all $n \geq i > k > l > j \geq 1$. This data defines the triangular matrix category

$$\mathcal{T} = \begin{pmatrix} \mathcal{A}_1 & 0 & 0 & \cdots & 0 \\ X_{21} & \mathcal{A}_2 & 0 & \cdots & 0 \\ X_{31} & X_{32} & \mathcal{A}_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & X_{n3} & \cdots & \mathcal{A}_n \end{pmatrix},$$

whose set of objects is the disjoint union of the set of objects of the $\mathcal{A}_i$ for $1 \leq i \leq n$. If $x \in \mathcal{A}_j$ and $y \in \mathcal{A}_i$, then

$$\mathcal{T}(x,y) = \begin{cases} X_{ij}(x,y), & \text{if } i > j, \\ \mathcal{A}_i(x,y), & \text{if } i = j, \\ 0, & \text{if } i < j. \end{cases}$$

The composition law is induced by the bimodule structures and by the maps $\mu_{ikj}$. The square (2.2) ensures that the composition is associative.

A $\mathcal{T}$-module is equivalent to the following data: a tuple $(M_1, \ldots, M_n, \varphi_{ij})$ where $M_i$ is an $\mathcal{A}_i$-module and $\varphi_{ij} : M_i \otimes_{\mathcal{A}_i} X_{ij} \to M_j$ (for $i > j$) is a morphism of $\mathcal{A}_j$-modules such that the square

$$\begin{tikzcd} M_i \otimes_{\mathcal{A}_i} X_{ij} \otimes_{\mathcal{A}_j} X_{jk} \arrow[r, "\mathrm{id}\otimes\mu_{ijk}"] \arrow[d, "\varphi_{ij}\otimes\mathrm{id}"'] & M_i \otimes_{\mathcal{A}_i} X_{ik} \arrow[d, "\varphi_{ik}"] \\ M_j \otimes_{\mathcal{A}_j} X_{jk} \arrow[r, "\varphi_{jk}"'] & M_k \end{tikzcd}$$

commutes for all $n \geq i > j > k \geq 1$. In this setting, a morphism $f : (M_1, \ldots, M_n, \varphi_{ij}) \to (N_1, \ldots, N_n, \psi_{ij})$ corresponds to morphisms $f_i : M_i \to N_i$ for all $1 \leq i \leq n$ such that the square

$$\begin{tikzcd} M_i \otimes_{\mathcal{A}_i} X_{ij} \arrow[r, "f_i\otimes\mathrm{id}"] \arrow[d, "\varphi_{ij}"'] & N_i \otimes_{\mathcal{A}_i} X_{ij} \arrow[d, "\psi_{ij}"] \\ M_j \arrow[r, "f_j"'] & N_j \end{tikzcd} \tag{2.3}$$

commutes for all $i > j$.

### 2.2.4 The eight functors

Let $\mathcal{T} = \mathcal{T}(\mathcal{A}, X, \mathcal{B})$ be a triangular matrix category as in the previous section. In [Lad11], it is shown that $\operatorname{Mod}\mathcal{T}$ can be viewed as a gluing of $\operatorname{Mod}\mathcal{A}$ and $\operatorname{Mod}\mathcal{B}$. This is described by a diagram of functors

$$\begin{tikzcd} \operatorname{Mod}\mathcal{B} \arrow[r, "i_*"] \arrow[r, bend right, "i_!"'] & \operatorname{Mod}\mathcal{T} \arrow[l, "i^{-1}"] \arrow[l, bend right, "i^!"'] \arrow[r, "j^{-1}"] \arrow[r, bend right, "j^\natural"'] & \operatorname{Mod}\mathcal{A} \arrow[l, "j_!"] \arrow[l, bend right, "j_*"'] \end{tikzcd}$$

where we have adjunctions $i_! \dashv i^{-1} \dashv i_* \dashv i^!$ and $j^\natural \dashv j_! \dashv j^{-1} \dashv j_*$. These functors are defined as follows. For modules $A \in \operatorname{Mod}\mathcal{A}$, $B \in \operatorname{Mod}\mathcal{B}$ and $(M_1, M_2, \varphi) \in \operatorname{Mod}\mathcal{T}$, we have

$$i_*(B) = (0, B, 0), \quad j_!(A) = (A, 0, 0) \quad \text{and} \quad i_!(B) = (B \otimes_{\mathcal{B}} X, B, \mathrm{id}),$$

and

$$j^{-1}(M_1, M_2, \varphi) = M_1, \quad i^{-1}(M_1, M_2, \varphi) = M_2 \quad \text{and} \quad j^{\natural}(M_1, M_2, \varphi) = \operatorname{coker} \varphi.$$

The action on morphisms is the evident one. For the functors $i^!$ and $j_*$, it is easier to use an alternative description of the modules over $\mathcal{T}$. By the tensor-Hom adjunction, we may equivalently view $\mathcal{T}$-modules as triples $(M_1, M_2, \psi)$ where $M_1$ is an $\mathcal{A}$-module, $M_2$ is a $\mathcal{B}$-module and $\psi : M_2 \to \operatorname{Hom}_{\mathcal{A}}(X, M_1)$ is a morphism of $\mathcal{B}$-modules. With this description, we have

$$i^!(M_1, M_2, \psi) = \ker \psi \quad \text{and} \quad j_*(A) = (A, \operatorname{Hom}_{\mathcal{A}}(X, A), \mathrm{id})$$

for $A \in \operatorname{Mod} \mathcal{A}$ and $(M_1, M_2, \psi) \in \operatorname{Mod} \mathcal{T}$.

### 2.2.5 Mesh categories and Happel's embedding

Let $Q = (Q_0, Q_1, s, t)$ be a *quiver*, that is, a directed graph with set of vertices $Q_0$ and set of arrows $Q_1$, together with maps $s, t : Q_1 \to Q_0$ indicating the source and target of each arrow, respectively. We always suppose that $Q$ comes with a distinguished subset $F \subseteq Q_0$ (possibly empty) of *frozen vertices*, so that $(Q, F)$ is an *ice quiver*. For simplicity, we omit $F$ from the notation and say directly that $Q$ is an ice quiver. We also assume that $Q$ is finite and acyclic, that is, $Q_0$ and $Q_1$ are finite, and there are no oriented cycles.

The *repetition quiver* $\mathbb{Z}Q$ associated with $Q$ is the ice quiver with set of vertices $(\mathbb{Z}Q)_0 = Q_0 \times \mathbb{Z}$ and with the following set of arrows: given $\alpha : i \to j$ in $Q_1$ and $p \in \mathbb{Z}$, there are arrows $(\alpha, p) : (i, p) \to (j, p)$ and $\sigma(\alpha, p) : (j, p-1) \to (i, p)$ in $(\mathbb{Z}Q)_1$. A vertex $(i, p)$ is frozen in $\mathbb{Z}Q$ if $i$ is frozen in $Q$. We define the bijection $\sigma : (\mathbb{Z}Q)_1 \to (\mathbb{Z}Q)_1$ by $\sigma(\beta) = \sigma(\alpha, p)$, if $\beta = (\alpha, p)$, and $\sigma(\beta) = (\alpha, p-1)$, if $\beta = \sigma(\alpha, p)$. We have an automorphism $\tau : \mathbb{Z}Q \to \mathbb{Z}Q$ which sends a vertex $(i, p)$ to $(i, p-1)$ and an arrow $\beta$ to $\sigma^2(\beta)$.

The *path category* $\mathcal{P}_k(\mathbb{Z}Q)$ is the $k$-category whose objects are the vertices of $\mathbb{Z}Q$ and whose space of morphisms from a vertex $x$ to a vertex $y$ is the $k$-vector space with basis given by the paths from $x$ to $y$ in $\mathbb{Z}Q$. Composition is induced by the concatenation of paths, and the identity morphisms are the stationary paths of length zero. We now define the *mesh category* $k(\mathbb{Z}Q)$ (after [Rie80, Gab80, KS16]) as the quotient of $\mathcal{P}_k(\mathbb{Z}Q)$ by the $k$-ideal generated by the morphisms

$$r_x = \sum_{\substack{\beta \in (\mathbb{Z}Q)_1 \\ \beta : y \to x}} \beta \circ \sigma(\beta) : \tau(x) \longrightarrow x \tag{2.4}$$

for all *non-frozen* vertices $x \in (\mathbb{Z}Q)_0$. Since we assume $Q$ is finite and acyclic, notice that

$\mathcal{P}_k(\mathbb{Z}Q)$ (and hence all its quotients) satisfies Assumption 2.2.2.

For the next result, suppose $Q$ has no frozen vertices. Let $kQ$ be the path algebra of $Q$ and consider the bounded derived category $\mathcal{D}^b(\operatorname{mod} kQ)$ of the category $\operatorname{mod} kQ$ of finite-dimensional right $kQ$-modules. It is a Krull-Schmidt triangulated category. In particular, it can be completely described by its full subcategory $\operatorname{ind}(\mathcal{D}^b(\operatorname{mod} kQ))$ of indecomposable objects. We have the following result due to Happel.

**Theorem 2.2.6** ([Hap88, Proposition I.5.6]). *There is a fully faithful functor*

$$H : k(\mathbb{Z}Q) \longrightarrow \operatorname{ind}(\mathcal{D}^b(\operatorname{mod} kQ)).$$

*sending the vertex $(i, 0)$ to the indecomposable projective $kQ$-module associated with the vertex $i \in Q_0$. It is an equivalence if and only if $Q$ is a* Dynkin quiver, *that is, its underlying graph is a disjoint union of* ADE *Dynkin diagrams.*

Through the functor above, the automorphism $\tau$ corresponds to the Auslander–Reiten translation on $\mathcal{D}^b(\operatorname{mod} kQ)$, which we denote by the same symbol. In the Dynkin case, we can use the equivalence to transport the suspension functor $\Sigma$ of $\mathcal{D}^b(\operatorname{mod} kQ)$ to an automorphism of the mesh category $k(\mathbb{Z}Q)$. In particular, it induces a bijection $\Sigma : (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0$.

## 2.3 The category of split filtrations

In this section, we present the main definition of this chapter: the category of split filtrations. We show that such a category is equivalent to the module category over a certain triangular matrix category. We then use this description and Section 2.2.4 to study natural functors relating split filtrations of different lengths.

For what follows, we fix a field $k$ and an integer $n \geq 1$. Let $\mathcal{A}$ be a small $k$-category and $\mathcal{B}$ a (not necessarily full) $k$-subcategory of $\mathcal{A}$. We always assume that $\mathcal{B}$ contains all objects of $\mathcal{A}$.

### 2.3.1 Definition and main characterization

With the setup above, we give the following definition.

**Definition 2.3.1.** We define $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ as the category of filtrations of length $n$ in $\operatorname{Mod}\mathcal{A}$ together with a splitting over $\operatorname{Mod}\mathcal{B}$. More precisely, an object of $\operatorname{Filt}^n_{\mathcal{B}}(\mathcal{A})$ is an $\mathcal{A}$-module $M$ together with a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

by $\mathcal{A}$-submodules and $\mathcal{B}$-linear maps $r_i^M : M_i \to M_{i-1}$ which are retractions for the inclusions $M_{i-1} \to M_i$ $(1 \leq i \leq n)$. A morphism in $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ is a morphism $f : M \to N$ of $\mathcal{A}$-modules such that $f(M_i) \subseteq N_i$ and $r_i^N \circ f_i = f_{i-1} \circ r_i^M$ as maps of $\mathcal{B}$-modules, for all $1 \leq i \leq n$, where $f_i : M_i \to N_i$ denotes the restriction of $f$. The composition law is given by the composition of morphisms of $\mathcal{A}$-modules.

**Remark 2.3.2.** The category above is equivalent to the category $\mathrm{CoFilt}^n_{\mathcal{B}}(\mathcal{A})$ of "cofiltrations" of length $n$ in $\mathrm{Mod}\,\mathcal{A}$ together with a splitting over $\mathrm{Mod}\,\mathcal{B}$, that is, the analogous category where each object is a sequence of $n-1$ epimorphisms

$$\begin{tikzcd} N_1 \arrow[r, two heads, "f_1"'] & N_2 \arrow[l, dashed, bend right, "s_1"'] \arrow[r, two heads, "f_2"'] & \cdots \arrow[l, dashed, bend right, "s_2"'] \arrow[r, two heads, "f_{n-1}"'] & N_n \arrow[l, dashed, bend right, "s_{n-1}"'] \end{tikzcd}$$

equipped with $\mathcal{B}$-linear sections $s_i$ and with morphisms respecting this structure. The equivalence sends an object $M \in \mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ to the sequence

$$M = M/M_0 \twoheadrightarrow M/M_1 \twoheadrightarrow \cdots \twoheadrightarrow M/M_{n-1}.$$

Each epimorphism has a canonical $\mathcal{B}$-linear section induced by the splitting of $M$.

The goal of this subsection is to prove the following result.

**Proposition 2.3.3.** *The category* $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ *is equivalent to* $\mathrm{Mod}\,\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ *for a certain* $k$*-category* $\mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ *(see Lemma 2.3.7 below for the definition).*

**Remark 2.3.4.** This result is not true if we remove the assumption that $\mathcal{B}$ contains all objects of $\mathcal{A}$. For example, for $n = 2$ and $\mathcal{B}$ the empty subcategory of $\mathcal{A}$, the category $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ is equivalent to the category of short exact sequences of $\mathcal{A}$-modules, which is not even abelian if $\mathcal{A}$ is not empty (see [Mac63, XII.6, p. 375]).

**Lemma 2.3.5.** *The category* $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ *is* $k$*-linear, cocomplete and abelian.*

*Proof.* The only part that is not immediate is the existence of cokernels. Let $f : M \to N$ be a morphism in $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$. If $f_i : M_i \to N_i$ $(1 \leq i \leq n)$ denote the restrictions of $f$, then the filtration of $N$ induces maps $\mu_i : \operatorname{coker} f_{i-1} \to \operatorname{coker} f_i$, while the retractions $r_i^N$ induce $\mathcal{B}$-linear retractions of the $\mu_i$. Since $\mathcal{A}$ and $\mathcal{B}$ have the same set of objects, the existence of such a $\mathcal{B}$-linear retraction is enough to guarantee that $\mu_i$ is a monomorphism of $\mathcal{A}$-modules. Therefore, $\operatorname{coker} f$ comes equipped with a canonical filtration with splitting over $\mathcal{B}$, thus giving an object of $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ with the desired universal property. √

Thus, one way to prove Proposition 2.3.3 is to find a set of compact projective objects in $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ that generates the whole category. It turns out that such a collection of objects is easier to describe in the equivalent category $\mathrm{CoFilt}^n_{\mathcal{B}}(\mathcal{A})$ (see Remark 2.3.2), so we will work with it until the end of this subsection.

Let $\mu : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$ be the morphism of $\mathcal{A}$-$\mathcal{A}$-bimodules induced by the composition of the category $\mathcal{A}$. For $1 \leq k \leq n$, consider the cofiltration of $\mathcal{A}$-bimodules

$$\mathcal{A}^{\otimes_{\mathcal{B}} k} \xrightarrow{\mathrm{id}\otimes\mu} \mathcal{A}^{\otimes_{\mathcal{B}} k-1} \xrightarrow{\mathrm{id}\otimes\mu} \cdots \xrightarrow{\mathrm{id}\otimes\mu} \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \xrightarrow{\mu} \mathcal{A}.$$

It has length $k$. Given an object $x \in \mathcal{A}$, we can specialize the bimodules above to get a cofiltration $\mathcal{P}_x^k$ of right $\mathcal{A}$-modules

$$x^\wedge \otimes_{\mathcal{B}} \mathcal{A}^{\otimes_{\mathcal{B}} k-1} \xrightarrow{\mathrm{id}\otimes\mu} x^\wedge \otimes_{\mathcal{B}} \mathcal{A}^{\otimes_{\mathcal{B}} k-2} \xrightarrow{\mathrm{id}\otimes\mu} \cdots \xrightarrow{\mathrm{id}\otimes\mu} x^\wedge \otimes_{\mathcal{B}} \mathcal{A} \xrightarrow{\mu} x^\wedge,$$

where $x^\wedge = \mathcal{A}(-, x)$ is the representable $\mathcal{A}$-module associated with $x$. Each one of the maps above has a canonical $\mathcal{B}$-linear section which sends a pure tensor $a_1 \otimes \cdots \otimes a_l$ to $a_1 \otimes \cdots \otimes a_l \otimes \mathrm{id}$. Extending it to a cofiltration of length $n$ by putting zeros at the end, we view $\mathcal{P}_x^k$ as an object of $\mathrm{CoFilt}_{\mathcal{B}}^n(\mathcal{A})$.

**Lemma 2.3.6.** *For $1 \leq k \leq n$ and $x \in \mathcal{A}$, the object $\mathcal{P}_x^k$ is compact and projective in $\mathrm{CoFilt}_{\mathcal{B}}^n(\mathcal{A})$. Moreover, the direct sum of these objects as we vary $k$ and $x$ is a generator for $\mathrm{CoFilt}_{\mathcal{B}}^n(\mathcal{A})$.*

*Proof.* We claim that the functor $\mathrm{Hom}(\mathcal{P}_x^k, -)$ is isomorphic to the functor that sends a cofiltration

$$\begin{tikzcd} \mathcal{N} = & N_1 \arrow[r, two heads, "f_1"'] & N_2 \arrow[l, dashed, bend right, "s_1"'] \arrow[r, two heads, "f_2"'] & \cdots \arrow[l, dashed, bend right, "s_2"'] \arrow[r, two heads, "f_{n-1}"'] & N_n \arrow[l, dashed, bend right, "s_{n-1}"'] \end{tikzcd}$$

to the evaluation of the module $\ker f_k$ at the object $x$ (for $k = n$, we set $f_n : N_n \to N_{n+1} = 0$ as the zero morphism). This claim implies the lemma. Indeed, the latter functor commutes with arbitrary direct sums, so $\mathcal{P}_x^k$ is compact. The snake lemma shows that it is exact, so $\mathcal{P}_x^k$ is projective. Moreover, $\mathcal{N}$ is zero if and only if $\ker f_k = 0$ for all $1 \leq k \leq n$, implying that the direct sum of the $\mathcal{P}_x^k$ is a generator.

Let us prove our claim. Part of the data of a morphism $\mathcal{P}_x^k \to \mathcal{N}$ is a commutative diagram

$$\begin{tikzcd} x^\wedge \arrow[r] \arrow[d, "\eta"] & 0 \arrow[d] \\ N_k \arrow[r, "f_k"] & N_{k+1}. \end{tikzcd}$$

By the Yoneda lemma, the morphism on the left corresponds to an element $\eta \in N_k(x)$, and the diagram commutes if and only if $\eta$ lies in the kernel of $f_k$. Reciprocally, let us show that a commutative diagram as the one above uniquely lifts to a morphism $\mathcal{P}_x^k \to \mathcal{N}$. Uniqueness follows from the fact that the image of the $\mathcal{B}$-linear section

$$x^\wedge \otimes_{\mathcal{B}} \mathcal{A}^{\otimes_{\mathcal{B}} l-1} \to x^\wedge \otimes_{\mathcal{B}} \mathcal{A}^{\otimes_{\mathcal{B}} l}$$

in the definition of $\mathcal{P}_x^k$ generates $x^\wedge \otimes_\mathcal{B} \mathcal{A}^{\otimes_\mathcal{B} l}$ as an $\mathcal{A}$-module. Hence, since a morphism $\mathcal{P}_x^k \to \mathcal{N}$ has to be compatible with the sections, it is determined by its component $x^\wedge \to N_k$. This last observation forces us to define the desired morphism as follows: its component $x^\wedge \otimes_\mathcal{B} \mathcal{A} \to N_{k-1}$ sends a pure tensor $a_1 \otimes a_2$ to

$$s_{k-1}(\eta \cdot a_1) \cdot a_2,$$

its component $x^\wedge \otimes_\mathcal{B} \mathcal{A}^{\otimes_\mathcal{B} 2} \to N_{k-2}$ sends a pure tensor $a_1 \otimes a_2 \otimes a_3$ to

$$s_{k-2}(s_{k-1}(\eta \cdot a_1) \cdot a_2) \cdot a_3,$$

and so on. One checks that this is indeed well-defined and yields a morphism from $\mathcal{P}_x^k$ to $\mathcal{N}$. $\surd$

We define a category $\mathcal{T} = \mathcal{T}_\mathcal{B}^n(\mathcal{A})$ as follows. Its set of objects is the disjoint union of $n$ copies of the set of objects of $\mathcal{A}$. For $x \in \mathcal{A}$, we denote the corresponding object in the $i$-th copy by $x^i$. For $x, y \in \mathcal{A}$ and $1 \leq i, j \leq n$, we put

$$\mathcal{T}(x^i, y^j) = \mathrm{Hom}_{\mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})}(\mathcal{P}_x^i, \mathcal{P}_y^j).$$

The composition law comes from that of $\mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})$. By a classical result of Freyd (see [Mit72, Theorem 3.1]), Lemma 2.3.6 implies that $\mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})$ is equivalent to $\mathrm{Mod}\,\mathcal{T}$. The equivalence sends $\mathcal{N} \in \mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})$ to the $\mathcal{T}$-module $F_\mathcal{N}$ defined on objects by

$$F_\mathcal{N}(x^i) = \mathrm{Hom}_{\mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})}(\mathcal{P}_x^i, \mathcal{N}),$$

with the evident action on morphisms.

Since $\mathrm{CoFilt}_\mathcal{B}^n(\mathcal{A})$ is equivalent to $\mathrm{Filt}_\mathcal{B}^n(\mathcal{A})$, this completes the proof of Proposition 2.3.3. However, our definition of the category $\mathcal{T}$ is a bit abstract. We will now provide a more concrete description of it.

For $k \geq 2$, define the $\mathcal{A}$-bimodule $X_k$ to be the kernel of the morphism

$$\mathrm{id}^{\otimes k-2} \otimes \mu : \mathcal{A}^{\otimes_\mathcal{B} k} \longrightarrow \mathcal{A}^{\otimes_\mathcal{B} k-1}$$

given by the composition of the last two factors. We define $X_1$ to be the regular bimodule $\mathcal{A}$. For $k, l \geq 1$, we remark that we have the morphism of bimodules

$$\mu_{kl} : X_k \otimes_\mathcal{A} X_l \longrightarrow X_{k+l-1}$$

that multiplies the last factor of a pure tensor in $X_k$ with the first factor of a pure tensor in $X_l$. In particular, $\mu_{11} = \mu$ is the map induced by the composition of $\mathcal{A}$.

**Lemma 2.3.7.** *The category $\mathcal{T} = \mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$ is isomorphic to the triangular matrix category*

$$\begin{pmatrix} \mathcal{A} & 0 & 0 & \cdots & 0 \\ X_2 & \mathcal{A} & 0 & \cdots & 0 \\ X_3 & X_2 & \mathcal{A} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_n & X_{n-1} & X_{n-2} & \cdots & \mathcal{A} \end{pmatrix}.$$

*More precisely, $\mathcal{T}(x^j, y^i)$ vanishes for $i < j$ and we have an isomorphism*

$$\mathcal{T}(x^j, y^i) \cong X_{i-j+1}(x, y)$$

*for $i \geq j$ in such a way that, under this identification, the composition law in $\mathcal{T}$ is given by the maps $\mu_{kl} : X_k \otimes_{\mathcal{A}} X_l \longrightarrow X_{k+l-1}$.*

*Proof.* The description of $\mathcal{T}(x^j, y^i)$ follows immediately from the characterization of the functor $\mathrm{Hom}(\mathcal{P}^j_x, -)$ given in the proof of Lemma 2.3.6. The verification of the last assertion is left to the reader. $\surd$

### 2.3.2 The canonical functors

Let us describe some canonical functors relating the categories $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ for various $n$. For $m \geq n \geq 1$, we have the functors $\mathrm{sub}^{m,n}, \mathrm{quot}^{m,n} : \mathrm{Filt}^m_{\mathcal{B}}(\mathcal{A}) \to \mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ defined on an object $M$ by

$$\mathrm{sub}^{m,n}(M) = M_n \quad \text{and} \quad \mathrm{quot}^{m,n}(M) = M/M_{m-n}.$$

Both modules above inherit the filtration and the $\mathcal{B}$-linear splitting from $M$. In the other direction, we have the functors $i^{n,m}_{\mathrm{sub}}, i^{n,m}_{\mathrm{quot}} : \mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A}) \to \mathrm{Filt}^m_{\mathcal{B}}(\mathcal{A})$ which take an object $M$ and trivially extend its filtration to length $m$ by repeating $M$ at the end or 0 at the beginning, respectively. Finally, we denote by $\mathrm{tot}^n : \mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A}) \to \mathrm{Mod}\,\mathcal{A}$ the functor which forgets the filtration and the splitting. These functors are all exact.

Most of these functors can be realized in the setting of Section 2.2.4. For this purpose, via Proposition 2.3.3, we identify $\mathrm{Filt}^n_{\mathcal{B}}(\mathcal{A})$ with the category of modules over $\mathcal{T}^n = \mathcal{T}^n_{\mathcal{B}}(\mathcal{A})$. Fix $m, n \geq 1$. The characterization in Lemma 2.3.7 allows us to view $\mathcal{T}^{m+n}$ as the triangular matrix category

$$\begin{pmatrix} \mathcal{T}^m & 0 \\ X_{mn} & \mathcal{T}^n \end{pmatrix},$$

where the rectangular block

$$X_{mn} = \begin{pmatrix} X_{m+1} & X_m & \cdots & X_2 \\ X_{m+2} & X_{m+1} & \cdots & X_3 \\ \vdots & \vdots & & \vdots \\ X_{m+n} & X_{m+n-1} & \cdots & X_{n+1} \end{pmatrix}$$

can be naturally viewed as a $\mathcal{T}^n$-$\mathcal{T}^m$-bimodule using the maps $\mu_{kl}$ from Section 2.3.1. This decomposition gives rise to the eight functors in Section 2.2.4.

**Lemma 2.3.8.** *If we decompose $\mathcal{T}^{m+n}$ as the triangular matrix category above and consider the corresponding diagram as in Section 2.2.4, then the functors $i_*$, $i^{-1}$, $j_!$ and $j^{-1}$ are isomorphic to $i_{\mathrm{quot}}^{n,m+n}$, $\mathrm{quot}^{m+n,n}$, $i_{\mathrm{sub}}^{m,m+n}$ and $\mathrm{sub}^{m+n,m}$, respectively.*

*Proof.* By composing the equivalence $\mathrm{Filt}_{\mathcal{B}}^{m+n}(\mathcal{A}) \to \mathrm{CoFilt}_{\mathcal{B}}^{m+n}(\mathcal{A})$ from Remark 2.3.2 with the equivalence $\mathrm{CoFilt}_{\mathcal{B}}^{m+n}(\mathcal{A}) \to \mathrm{Mod}\,\mathcal{T}_{\mathcal{B}}^{m+n}(\mathcal{A})$ from the proof of Proposition 2.3.3, we see that a split filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m+n} = M$$

in $\mathrm{Filt}_{\mathcal{B}}^{m+n}(\mathcal{A})$ corresponds to the $\mathcal{T}_{\mathcal{B}}^{m+n}(\mathcal{A})$-module

$$(M_1, M_2/M_1, \ldots, M_{m+n}/M_{m+n-1}, \varphi_{ij}).$$

Here, the splitting over $\mathcal{B}$ induces the direct sum decomposition $M \cong M_1 \oplus M_2/M_1 \oplus \cdots \oplus M_{m+n}/M_{m+n-1}$ (as $\mathcal{B}$-modules) and the strucutral maps $\varphi_{ij}$ encode the $\mathcal{A}$-module structure on $M$ in terms of these summands. The lemma then follows without much difficulty from the definition of the functors above. $\surd$

The main result of this subsection is the following.

**Proposition 2.3.9.** *Suppose that the restriction of the regular $\mathcal{A}$-$\mathcal{A}$-bimodule to $\mathrm{Mod}\,\mathcal{B}$ is projective on both sides. Then the functors $\mathrm{sub}^{m+n,m}$ and $\mathrm{quot}^{m+n,n}$ preserve projective and injective objects for all $m, n \geq 1$.*

**Remark 2.3.10.** If $\mathcal{B}$ is the discrete $k$-subcategory of $\mathcal{A}$ generated by all identity morphisms, then any $\mathcal{A}$-$\mathcal{A}$-bimodule is projective on both sides as a $\mathcal{B}$-module, so that we can apply the result above.

To prove the proposition, let us focus first on the case of the functor $\mathrm{sub}^{m+n,m}$. By the identity

$$\mathrm{sub}^{j,i} \circ \mathrm{sub}^{k,j} = \mathrm{sub}^{k,i}$$

valid for $1 \leq i \leq j \leq k$, we can assume that $n = 1$. It is enough to show that both adjoints of $\mathrm{sub}^{m+1,m}$ are exact. By Lemma 2.3.8, the left adjoint is $i_{\mathrm{sub}}^{m,m+1}$, which is indeed exact. By the same lemma, the right adjoint is the functor $j_* : \mathrm{Mod}\,\mathcal{T}^m \to \mathrm{Mod}\,\mathcal{T}^{m+1}$. It is exact if and only if the functor $\mathrm{Hom}_{\mathcal{T}^m}(X_{m1}, -) : \mathrm{Mod}\,\mathcal{T}^m \to \mathrm{Mod}\,\mathcal{A}$ is exact, that is, if and only if $X_{m1}$ is projective on the right. Before we show that this is the case, we need some preparation.

Since $\mathcal{A}$ is projective on both sides as a $\mathcal{B}$-module, we have that $\mathcal{A}^{\otimes_{\mathcal{B}} k}$ is projective on both sides as an $\mathcal{A}$-module for all $k \geq 1$. Consequently, for $k \geq 2$, the exact sequence

$$0 \longrightarrow X_k \longrightarrow \mathcal{A}^{\otimes_{\mathcal{B}} k} \xrightarrow{\mathrm{id}\otimes\mu} \mathcal{A}^{\otimes_{\mathcal{B}} k-1} \longrightarrow 0$$

defining $X_k$ splits and $X_k$ is also projective on both sides. Thus, if we apply $- \otimes_{\mathcal{A}} X_l$ ($l \geq 2$) to the exact sequence above, we get the exact sequence

$$0 \longrightarrow X_k \otimes_{\mathcal{A}} X_l \longrightarrow \mathcal{A}^{\otimes_{\mathcal{B}} k-1} \otimes_{\mathcal{B}} X_l \xrightarrow{\eta_{kl}} \mathcal{A}^{\otimes_{\mathcal{B}} k-2} \otimes_{\mathcal{B}} X_l \longrightarrow 0,$$

where $\eta_{kl}$ multiplies the last factor of a pure tensor in $\mathcal{A}^{\otimes_{\mathcal{B}} k-1}$ with the first factor of a pure tensor in $X_l$. On the other hand, if we tensor the first exact sequence $t$ times on the left by $\mathcal{A}$ over $\mathcal{B}$, we get an exact sequence where the last map is still of the form $\mathrm{id} \otimes \mu$, so we deduce that $\mathcal{A}^{\otimes_{\mathcal{B}} t} \otimes_{\mathcal{B}} X_k \cong X_{k+t}$. Therefore, we may identify the second exact sequence above with the sequence

$$0 \longrightarrow X_k \otimes_{\mathcal{A}} X_l \xrightarrow{\mu_{kl}} X_{k+l-1} \xrightarrow{\eta_{kl}} X_{k+l-2} \longrightarrow 0. \tag{2.5}$$

We can now continue the proof.

**Lemma 2.3.11.** *With the hypothesis of Proposition 2.3.9, the bimodule $X_{m1}$ is projective on the right for all $m \geq 1$.*

*Proof.* For $1 \leq k \leq m$, the $1 \times m$ “row vector”

$$Y_k = \begin{pmatrix} X_{k+1} & \cdots & X_3 & X_2 & 0 & \cdots & 0 \end{pmatrix}$$

is naturally an $\mathcal{A}$-$\mathcal{T}^m$-bimodule via matrix multiplication. Notice that $Y_m = X_{m1}$. We will prove by induction on $k$ that $Y_k$ is projective on the right. For $k = 1$ and an object $x \in \mathcal{A}$, the right $\mathcal{T}^m$-module $Y_1(-,x)$ is given by $i_{\mathrm{sub}}^{1,m}(X_2(-,x))$. By Lemma 2.3.8, the right adjoint of $i_{\mathrm{sub}}^{1,m}$ is $\mathrm{sub}^{m,1}$, which is exact, hence $i_{\mathrm{sub}}^{1,m}$ preserves projective objects. By the discussion above, $X_2(-,x)$ is a projective $\mathcal{A}$-module, thus $Y_1(-,x)$ is a projective $\mathcal{T}^n$-module.

Suppose $k \geq 2$. Consider the $1 \times m$ "row vector"

$$Z_k = \begin{pmatrix} X_2 \otimes_{\mathcal{A}} X_k & \cdots & X_2 \otimes_{\mathcal{A}} X_2 & X_2 \otimes_{\mathcal{A}} \mathcal{A} & 0 & \cdots & 0 \end{pmatrix}.$$

It is an $\mathcal{A}$-$\mathcal{T}^n$-bimodule where the structural maps of the $\mathcal{T}^n$-module structure are of the form $\mathrm{id}_{X_2} \otimes \mu_{ij}$. We claim that it is projective on the right. Let $i_! : \operatorname{Mod} \mathcal{A} \to \operatorname{Mod} \mathcal{T}^k$ be the functor of Section 2.2.4 arising from the decomposition

$$\mathcal{T}^k = \begin{pmatrix} \mathcal{T}^{k-1} & 0 \\ X_{k-1,1} & \mathcal{A} \end{pmatrix}.$$

Notice that $Z_k(-,x) = i^{k,n}_{\mathrm{sub}}(i_!(X_2(-,x)))$ for all $x \in \mathcal{A}$. Since the right adjoint of $i_!$ is $\mathrm{quot}^{1,k}$, which is exact, this functor preserves projectives. But $i^{k,n}_{\mathrm{sub}}$ also preserves projectives and $X_2(-,x)$ is a projective $\mathcal{A}$-module, whence we deduce our claim.

We will prove the existence of an exact sequence

$$0 \longrightarrow Z_k \longrightarrow Y_k \longrightarrow Y_{k-1} \longrightarrow 0.$$

Taking into account the induction hypothesis and the previous paragraph, this will imply that $Y_k$ is projective, concluding the proof. The exact sequence (2.5) induces exact sequences between the components of the bimodules above, so we only need to check that the maps $\mu_{2,i}$ and $\eta_{2,i}$ are compatible with the $\mathcal{T}^n$-module structures. In other words, we have to check the commutativity of the diagrams (2.3) in Remark 2.2.5. This follows from the diagrams

$$\begin{tikzcd}
X_2 \otimes_{\mathcal{A}} X_j \arrow[r, equal] \arrow[d, equal] & X_2 \otimes_{\mathcal{A}} X_j \arrow[r] \arrow[d, "\mu_{2,j}"'] & 0 \arrow[d] \\
X_2 \otimes_{\mathcal{A}} X_j \arrow[r, "\mu_{2,j}"'] & X_{j+1} \arrow[r, "\eta_{2,j}"'] & X_j
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_2 \otimes_{\mathcal{A}} X_i \otimes_{\mathcal{A}} X_j \arrow[r, "\mu_{2,i}\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu_{ij}"'] & X_{i+1} \otimes_{\mathcal{A}} X_j \arrow[r, "\eta_{2,i}\otimes\mathrm{id}"] \arrow[d, "\mu_{i+1,j}"'] & X_i \otimes_{\mathcal{A}} X_j \arrow[d, "\mu_{ij}"] \\
X_2 \otimes_{\mathcal{A}} X_{i+j-1} \arrow[r, "\mu_{2,i+j-1}"'] & X_{i+j} \arrow[r, "\eta_{2,i+j-1}"'] & X_{i+j-1}
\end{tikzcd}$$

which are commutative for $i, j \geq 2$ as one can easily check. $\surd$

This completes the proof of Proposition 2.3.9 for the functor $\mathrm{sub}^{m+n,m}$. For the functor $\mathrm{quot}^{m+n,n}$, a similar argument works, and the proof is reduced to showing that $X_{1n}$ is projective on the left for all $n \geq 1$. This can be proved as in the proof of Lemma 2.3.11.

**Remark 2.3.12.** Using the adjunction $\mathrm{quot}^{m+n,n} \dashv i^{n,m+n}_{\mathrm{quot}}$, one can see that $\mathrm{quot}^{m+n,n}$ sends the free module $(x^i)^\wedge$ in $\operatorname{Mod} \mathcal{T}^{m+n}$ (for $x \in \mathcal{A}$ and $1 \leq i \leq m+n$) to the free module $(x^{i-m})^\wedge$ in $\operatorname{Mod}\mathcal{T}^n$ (if $i > m$) or to zero (if $i \leq m$). In general, it is not clear if we have an analogous property for cofree modules. Nevertheless, if we assume the hypothesis of Proposition 2.3.9 and that the endomorphism algebra of every object in $\mathcal{A}$ is local, then $\mathrm{quot}^{m+n,n}$ sends a cofree module to a product of cofree modules. Indeed, we may suppose $m = 1$. In this case, the left adjoint of $\mathrm{quot}^{n+1,n}$ can be described using the bimodule $X_{1n}$, which is projective on the left. By Kaplansky's theorem [Kap58], any specialization of $X_{1n}$ to a left module is hence a direct sum of free modules. Knowing this, it is not hard to deduce our claim. Dually, under the same hypotheses on $\mathcal{A}$, $\mathrm{sub}^{m+n,m}$ sends a cofree module to another cofree module or to zero, and a free module to a coproduct of free modules.

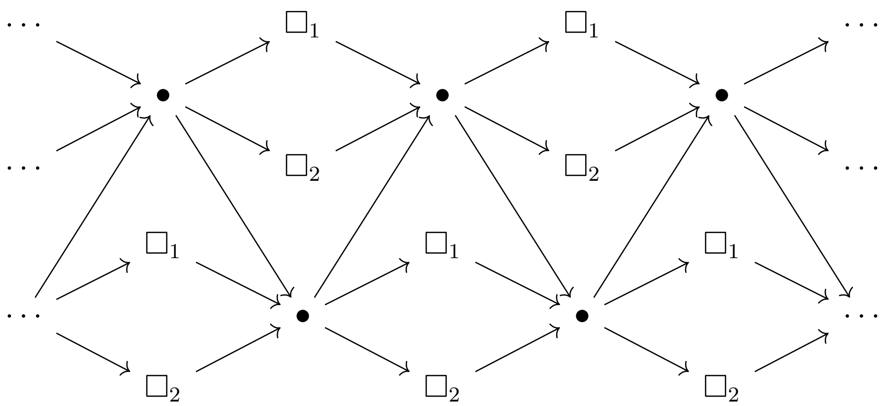
## 2.4 Split filtrations for the singular Nakajima category

We study the category of split filtrations over the singular Nakajima category $\mathcal{S}$, as defined by Keller–Scherotzke in [KS16]. Its properties are similar to those of $\mathcal{S}$, and we adapt some arguments in [KS16] to our setting without much difficulty. The initial results hold for an arbitrary finite acyclic quiver, but we specialize to the case of a Dynkin quiver starting from Section 2.4.5. Most of the results hold for an arbitrary base field $k$, but we will suppose that $k = \mathbb{C}$ whenever we use the varieties defined in Section 2.4.2.

### 2.4.1 Alternative definition using mesh categories

Let $Q$ be a finite acyclic quiver (without frozen vertices) and $n \geq 1$ a positive integer. The *$n$-framed quiver* $Q^{n\text{-fr}}$ is obtained from $Q$ by adding, for each vertex $i \in Q_0$, $n$ new vertices $i_1, i_2, \ldots, i_n$, and $n$ new arrows $i \to i_r$ $(1 \leq r \leq n)$. The new vertices are considered to be frozen.

Let $\mathbb{Z}Q^{n\text{-fr}}$ be the repetition quiver of $Q^{n\text{-fr}}$. For example, if $Q = 1 \to 2$ and $n = 2$, then $\mathbb{Z}Q^{2\text{-fr}}$ is the infinite quiver



where we represent by $\square_r$ the frozen vertices of the form $(i_r, p)$ for $i \in Q_0$. For a non-frozen vertex $x = (i, p)$, we put $\sigma_r(x) = (i_r, p-1)$, so that $\sigma_1(x), \sigma_2(x), \ldots, \sigma_n(x)$ are the frozen vertices immediately preceding $x$. Similarly, we denote by $\sigma_1^{-1}(x), \sigma_2^{-1}(x), \ldots, \sigma_n^{-1}(x)$ the frozen vertices immediately succeding $x$. When $n = 1$, we write $\sigma_1 = \sigma$ and $\sigma_1^{-1} = \sigma^{-1}$.

**Definition 2.4.1.** Let $Q$ be a finite acyclic quiver and $n \geq 1$ a positive integer.

- The *regular Nakajima category of split filtrations of length* $n$ is the quotient $\mathcal{R}^{n\text{-filt}}$ of the mesh category $k(\mathbb{Z}Q^{n\text{-fr}})$ by the smallest $k$-ideal containing all morphisms from a frozen vertex of the form $(i_r, p)$ to a frozen vertex of the form $(j_s, q)$, where $i, j \in Q_0$ and $r > s$.

- The *singular Nakajima category of split filtrations of length* $n$ is the full subcategory $\mathcal{S}^{n\text{-filt}}$ of $\mathcal{R}^{n\text{-filt}}$ whose objects are the frozen vertices.

For $n = 1$, we denote $\mathcal{R}^{1\text{-filt}}$ and $\mathcal{S}^{1\text{-filt}}$ by $\mathcal{R}$ and $\mathcal{S}$, respectively.

**Remark 2.4.2.** When $n = 1$, the given set of relations on $k(\mathbb{Z}Q^{1\text{-fr}})$ is empty, so that $\mathcal{R} = k(\mathbb{Z}Q^{1\text{-fr}})$. Hence, the categories $\mathcal{R}$ and $\mathcal{S}$ are precisely the *regular Nakajima category* and the *singular Nakajima category* defined in [KS16].

**Proposition 2.4.3.** *The category $\mathcal{S}^{n\text{-filt}}$ is isomorphic to the category $\mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$ of Section 2.3.1, where $k\mathcal{S}_0$ denotes the discrete $k$-subcategory of $\mathcal{S}$ generated by all identity morphisms.*

*Proof.* By construction, both $\mathcal{S}^{n\text{-filt}}$ and $\mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$ are spectroids in the sense of [GR92], that is, they are Hom-finite $k$-categories where different objects are non-isomorphic and all endomorphism algebras are local. Thus, by [GR92, Sections 2.5, 3.3], it is enough to prove that $\operatorname{Mod}\mathcal{S}^{n\text{-filt}}$ is equivalent to $\operatorname{Mod}\mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$, which in turn is equivalent to $\operatorname{Filt}^n_{k\mathcal{S}_0}(\mathcal{S})$ by Proposition 2.3.3.

It is not hard to check that a $k(\mathbb{Z}Q^{n\text{-fr}})$-module has the same data as an $\mathcal{R}$-module $M : \mathcal{R}^{\mathrm{op}} \to \operatorname{Mod} k$ together with a vector space decomposition $M(y) = M_1(y) \oplus \cdots \oplus M_n(y)$ for every frozen vertex $y$ in $\mathbb{Z}Q^{1\text{-fr}}$. If we replace $k(\mathbb{Z}Q^{n\text{-fr}})$ by its subcategory of frozen vertices and $\mathcal{R}$ by $\mathcal{S}$, the analogous statement also holds. If we impose the additional relations given in Definition 2.4.1, we deduce that an $\mathcal{S}^{n\text{-filt}}$-module corresponds to an $\mathcal{S}$-module $M : \mathcal{S}^{\mathrm{op}} \to \operatorname{Mod} k$ together with a vector space decomposition $M(y) = M_1(y) \oplus \cdots \oplus M_n(y)$ for every frozen vertex $y$ and such that the first $l$ terms of this decomposition determines a submodule of $M$ for all $1 \leq l \leq n$. In other words, the data of an $\mathcal{S}^{n\text{-filt}}$-module is equivalent to that of a filtration of length $n$ in $\operatorname{Mod}\mathcal{S}$ with a splitting over $k\mathcal{S}_0$. One checks that this correspondence is functorial and gives the desired equivalence. $\surd$

### 2.4.2 The link to graded quiver varieties

For this subsection, we assume $k = \mathbb{C}$ is the field of complex numbers. The categories $\mathcal{R}$ and $\mathcal{S}$ are related to graded quiver varieties as follows. Denote by $\mathcal{R}_0$ and $\mathcal{S}_0$ their set of objects. Let $w : \mathcal{S}_0 \to \mathbb{N}$ be a *dimension vector*, that is, a function with finite support. We define $\mathfrak{M}_0^\bullet(w)$ as the affine variety of $\mathcal{S}$-modules $M$ such that $M(u) = k^{w(u)}$ for $u \in \mathcal{S}_0$. It is the Zariski closed subset of the affine space

$$\prod_{u_1,u_2\in\mathcal{S}_0} \mathrm{Hom}_k(\mathcal{S}(u_1,u_2), \mathrm{Hom}_k(k^{w(u_2)}, k^{w(u_1)}))$$

cut out by the equations encoding the composition of $\mathcal{S}$. By the proof in [LP13, Theorem 2.4] (see also [KS16, Section 2.3]), it can be identified with the *affine graded quiver variety* defined by Nakajima [Nak11] (when $Q$ is bipartite) and by Qin [Qin14] (see also [KQ14]).

The category $\mathcal{R}$ is used to define the *(nonsingular) graded quiver variety* $\mathfrak{M}^\bullet(v, w)$ of [Nak11, Qin14, KQ14] associated with dimension vectors $v : \mathcal{R}_0 \setminus \mathcal{S}_0 \to \mathbb{N}$ and $w : \mathcal{S}_0 \to \mathbb{N}$. To define it, we need an auxiliary construction. We say an $\mathcal{R}$-module is *stable* if it does not contain non-zero submodules concentrated in non-frozen vertices. Consider the affine variety $\widetilde{\mathfrak{M}}^\bullet(v, w)$ of stable $\mathcal{R}$-modules $M$ such that $M(x) = k^{v(x)}$ and $M(\sigma(x)) = k^{w(\sigma(x))}$ for $x \in (\mathbb{Z}Q)_0$. The group $G_v = \prod_{x\in(\mathbb{Z}Q)_0} \mathrm{GL}(k^{v(x)})$ acts on $\widetilde{\mathfrak{M}}^\bullet(v, w)$ by base change on non-frozen vertices. We then define $\mathfrak{M}^\bullet(v, w)$ as the quotient $\widetilde{\mathfrak{M}}^\bullet(v, w)/G_v$. It can be canonically equipped with the structure of a smooth quasi-projective variety. The restriction along the inclusion $\mathcal{S} \to \mathcal{R}$ induces a proper map $\pi : \mathfrak{M}^\bullet(v, w) \longrightarrow \mathfrak{M}_0^\bullet(w)$. Notice that both $\mathfrak{M}^\bullet(v, w)$ and $\mathfrak{M}_0^\bullet(w)$ carry an action of the group $G_w = \prod_{x\in(\mathbb{Z}Q)_0} \mathrm{GL}(k^{w(\sigma(x))})$ by base change on frozen vertices, and the map $\pi$ is $G_w$-equivariant.

In the non-graded setting, Nakajima defines in [Nak01b] his tensor product varieties. A version for graded quiver varieties associated with bipartite quivers can be found in [Nak11, Section 3.5]. We define them for arbitrary quivers by mimicking his construction as follows. Let $w_1, \ldots, w_n : \mathcal{S}_0 \to \mathbb{N}$ be dimension vectors and write $w$ for their sum. Choose a direct sum decomposition $k^{w(u)} = k^{w_1(u)} \oplus \cdots \oplus k^{w_n(u)}$ for each $u \in \mathcal{S}_0$. The $n$*-fold affine graded tensor product variety* $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ is the closed subvariety of $\mathfrak{M}_0^\bullet(w)$ of all modules $M$ such that, for all $1 \leq l \leq n$, the subspaces $k^{w_1(u)} \oplus \cdots \oplus k^{w_l(u)} \subseteq k^{w(u)} = M(u)$ are invariant under the $\mathcal{S}$-module action. This subvariety is invariant under the action of the subgroup $G_{w_1} \times \cdots \times G_{w_n}$ of $G_w$ determined by the direct sum decomposition.

**Remark 2.4.4.** We use a convention opposite to the one in [Nak01b], where the dimension vectors are reversed. For example, in their definition of $\mathfrak{T}_0^\bullet(w_1, w_2)$, the subspaces $k^{w_2(u)}$ are the ones invariant under the module action (see the proof of Lemma 3.6 in loc. cit.).

Our definition immediately identifies the points of $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ with objects in

$\mathrm{Filt}^n_{k\mathcal{S}_0}(\mathcal{S})$. In fact, by Proposition 2.4.3 and its proof, $\mathfrak{T}^\bullet_0(w_1,\dots,w_n)$ is isomorphic to the variety of $\mathcal{S}^{n\text{-filt}}$-modules $M$ such that $M(\sigma_r(x)) = k^{w_r(\sigma(x))}$ for all $x \in (\mathbb{Z}Q)_0$ and $1 \leq r \leq n$. The action by the group $G_{w_1}\times\cdots\times G_{w_n}$ corresponds to base change on the vertices. For a dimension vector $v : \mathcal{R}_0\backslash\mathcal{S}_0 \to \mathbb{N}$, one also has the subvariety $\mathfrak{T}^\bullet(v; w_1,\dots,w_n)$ of $\mathfrak{M}^\bullet(v,w)$ defined by $\pi^{-1}(\mathfrak{T}^\bullet_0(w_1,\dots,w_n))$. It is not hard to show that it has a similar description as $\mathfrak{M}^\bullet(v,w)$, but where $\mathcal{R}$-modules are replaced by $\mathcal{R}^{n\text{-filt}}$-modules.

**Remark 2.4.5.** Nakajima also considers the subvariety $\widetilde{\mathfrak{T}}^\bullet_0(w_1,\dots,w_n)$ of $\mathfrak{T}^\bullet_0(w_1,\dots,w_n)$ consisting of points corresponding to split filtrations of $\mathcal{S}$-modules whose successive quotients are semisimple. Strictly speaking, this is the variety he uses to geometrically realize tensor products of standard modules over the quantum affine algebra. However, since standard modules themselves are tensor products of fundamental modules (see [Nak01b, Corollary 6.13]), when working with this geometric realization we may restrict to the case where each dimension vector $w_i$ is supported at a single vertex, where it has value 1. In this situation, $\widetilde{\mathfrak{T}}^\bullet_0(w_1,\dots,w_n)$ coincides with $\mathfrak{T}^\bullet_0(w_1,\dots,w_n)$.

### 2.4.3 Projective and injective (co)resolutions

In this section, we present projective resolutions and injective (co)resolutions for simple modules over $\mathcal{R}^{n\text{-filt}}$ and $\mathcal{S}^{n\text{-filt}}$.

**Lemma 2.4.6.** *For each non-frozen vertex $x$ of $\mathbb{Z}Q^{n\text{-fr}}$, we have minimal (co)resolutions of simple $\mathcal{R}^{n\text{-filt}}$-modules:*

*(a)* $0 \longrightarrow \tau(x)^\wedge \longrightarrow \sigma_n(x)^\wedge \longrightarrow S_{\sigma_n(x)} \longrightarrow 0,$

*(b)* $0 \longrightarrow S_{\sigma_1(x)} \longrightarrow \sigma_1(x)^\vee \longrightarrow x^\vee \longrightarrow 0,$

*(c)* $0 \longrightarrow \tau(x)^\wedge \longrightarrow \bigoplus_{y\to x} y^\wedge \longrightarrow x^\wedge \longrightarrow S_x \longrightarrow 0,$

*(d)* $0 \longrightarrow S_x \longrightarrow x^\vee \longrightarrow \bigoplus_{x\to y} y^\vee \longrightarrow \tau^{-1}(x)^\vee \longrightarrow 0,$

*where the direct sums vary over all arrows in $\mathbb{Z}Q^{n\text{-fr}}$ ending or starting at $x$, respectively.*

*Proof.* For (a), the only non-trivial verification is that the map $\tau(x)^\wedge \to \sigma_n(x)^\wedge$ is a monomorphism. This follows since there are no relations in the definition of $\mathbb{Z}Q^{n\text{-fr}}$ and $\mathcal{R}^{n\text{-filt}}$ ending with the arrow $\tau(x) \to \sigma_n(x)$. We also deduce that the first map in (c) is a monomorphism. To verify the exactness of (c) at the direct sum, observe that, if a non-trivial linear combination of morphisms becomes zero after composing with the arrows $y \to x$, then this composition must be a consequence of the mesh relation $r_x$ from (2.4). Therefore, such a linear combination is in the image of the first map in (c). The proofs for (b) and (d) are similar. $\surd$

**Corollary 2.4.7.** *Let $x \in (\mathbb{Z}Q^{n\text{-fr}})_0$ be a non-frozen vertex. Then $\mathbf{R}\mathrm{Hom}(S_x, M)$ is acyclic if $M$ is a coproduct of free $\mathcal{R}^{n\text{-filt}}$-modules associated with frozen vertices. Dually, $\mathbf{R}\mathrm{Hom}(N, S_x)$ is acyclic if $N$ is a product of cofree $\mathcal{R}^{n\text{-filt}}$-modules associated with frozen vertices.*

*Proof.* Using the resolutions (c) and (d) of Lemma 2.4.6, one can compute that the dual of $\mathbf{R}\mathrm{Hom}(S_x, M)$ is isomorphic to a shift of $\mathbf{R}\mathrm{Hom}(M, S_{\tau(x)})$ in the derived category of vector spaces. But $M$ is projective, hence this last complex is isomorphic to a shift of $\mathrm{Hom}(M, S_{\tau(x)})$. This space is zero by the hypothesis on $M$ as $\tau(x)$ is not frozen. The other case is analogous. $\surd$

**Lemma 2.4.8.** *For each non-frozen vertex $x$ of $\mathbb{Z}Q^{n\text{-fr}}$ and $1 \leq l \leq n$, we have a minimal resolution of the simple $\mathcal{R}^{n\text{-filt}}$-module $S_{\sigma_l^{-1}(x)}$ of the form:*

$$\cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow x^{\wedge} \longrightarrow \sigma_l^{-1}(x)^{\wedge} \longrightarrow S_{\sigma_l^{-1}(x)} \longrightarrow 0,$$

*where $P_t$ is a coproduct of free modules associated with* frozen *vertices. Dually, we have a minimal coresolution of the simple $\mathcal{R}^{n\text{-filt}}$-module $S_{\sigma_l(x)}$ of the form:*

$$0 \longrightarrow S_{\sigma_l(x)} \longrightarrow \sigma_l(x)^{\vee} \longrightarrow x^{\vee} \longrightarrow I_2 \longrightarrow I_3 \longrightarrow \cdots,$$

*where $I_t$ is a product of cofree modules associated with* frozen *vertices.*

*Proof.* Since Assumption 2.2.2 holds for $\mathcal{R}^{n\text{-filt}}$, this follows from a straightforward computation of $\mathrm{Ext}^p(S_{\sigma_l^{-1}(x)}, S_y)$ for $p \geq 1$ and a non-frozen vertex $y$ using the injective coresolution of $S_y$ from Lemma 2.4.6. The second assertion follows dually. $\surd$

**Remark 2.4.9.** The resolution from Lemma 2.4.8 is usually infinite. In particular, $\mathcal{R}^{n\text{-filt}}$ does not have finite global dimension in general for $n \geq 2$.

For an integer $1 \leq l \leq n$ and a non-frozen vertex $x$ in $\mathbb{Z}Q^{n\text{-fr}}$ (that is, a vertex of $\mathbb{Z}Q$), consider the $\mathcal{R}^{n\text{-filt}}$-modules

$$\begin{aligned} P^{\leq l}(x) &= \bigoplus_{y \in (\mathbb{Z}Q)_0} \mathcal{D}_Q(H(y), H(x)) \otimes (\sigma_1(y)^{\wedge} \oplus \cdots \oplus \sigma_l(y)^{\wedge}), \\ I^{\geq l}(x) &= \prod_{y \in (\mathbb{Z}Q)_0} D\mathcal{D}_Q(H(x), H(y)) \otimes (\sigma_l^{-1}(y)^{\vee} \oplus \cdots \oplus \sigma_n^{-1}(y)^{\vee}), \end{aligned}$$

where $\mathcal{D}_Q = \mathcal{D}^b(\mathrm{mod}\, kQ)$ and $H$ is Happel's embedding. The first one is projective, while the second one is injective. Since the restriction of $\sigma_i(y)^{\wedge}$ (resp. $\sigma_i(y)^{\vee}$) to $\mathrm{Mod}\,\mathcal{S}^{n\text{-filt}}$ is the free (resp. cofree) $\mathcal{S}^{n\text{-filt}}$-module associated with $\sigma_i(y)$, the restrictions of $P^{\leq l}(x)$ and $I^{\geq l}(x)$ are again projective and injective, respectively. By abuse of notation, we also denote these restrictions by $P^{\leq l}(x)$ and $I^{\geq l}(x)$.

**Theorem 2.4.10.** *Suppose $Q$ is connected. Let $x$ be a non-frozen vertex of $\mathbb{Z}Q^{n\text{-fr}}$ and $1 \leq l \leq n$.*

*(a) If $Q$ is a Dynkin quiver, the simple $\mathcal{S}^{n\text{-filt}}$-module $S_{\sigma_l^{-1}(x)}$ admits a minimal projective resolution of the form*

$$\cdots \to P^{\leq l}(\Sigma^{-2}x) \to P^{\leq l}(\Sigma^{-1}x) \to P^{\leq l}(x) \to \sigma_l^{-1}(x)^\wedge \to S_{\sigma_l^{-1}(x)} \to 0$$

*and the simple $\mathcal{S}^{n\text{-filt}}$-module $S_{\sigma_l(x)}$ admits a minimal injective coresolution of the form*

$$0 \to S_{\sigma_l(x)} \to \sigma_l(x)^\vee \to I^{\geq l}(x) \to I^{\geq l}(\Sigma x) \to I^{\geq l}(\Sigma^2 x) \to \cdots .$$

*(b) If $Q$ is not a Dynkin quiver, the simple $\mathcal{S}^{n\text{-filt}}$-module $S_{\sigma_l^{-1}(x)}$ admits a minimal projective resolution of the form*

$$0 \to P^{\leq l}(x) \to \sigma_l^{-1}(x)^\wedge \to S_{\sigma_l^{-1}(x)} \to 0$$

*and the simple $\mathcal{S}^{n\text{-filt}}$-module $S_{\sigma_l(x)}$ admits a minimal injective coresolution of the form*

$$0 \to S_{\sigma_l(x)} \to \sigma_l(x)^\vee \to I^{\geq l}(x) \to 0.$$

The proof follows the same arguments as in [KS16, Section 3]. First, we introduce auxiliary $\mathcal{R}^{n\text{-filt}}$-modules. The quotient of $\mathcal{R}^{n\text{-filt}}$ by the $k$-ideal generated by the identity morphisms of the frozen vertices is isomorphic to the mesh category $k(\mathbb{Z}Q)$. Therefore, by composition with the quotient functor, we may view the embedding $H$ of Theorem 2.2.6 as a functor

$$H : \mathcal{R}^{n\text{-filt}} \longrightarrow \operatorname{ind}(\mathcal{D}_Q)$$

vanishing on frozen vertices. We define the $\mathcal{R}^{n\text{-filt}}$-modules

$$x_{\mathcal{D}}^\wedge = \mathcal{D}_Q(H(?), H(x)) \quad \text{and} \quad x_{\mathcal{D}}^\vee = D\mathcal{D}_Q(H(x), H(?))$$

for a non-frozen vertex $x \in (\mathbb{Z}Q^{n\text{-fr}})_0$.

**Lemma 2.4.11.** *Suppose $Q$ is connected. Let $x$ be a non-frozen vertex of $\mathbb{Z}Q^{n\text{-fr}}$.*

*(a) If $Q$ is a Dynkin quiver, we have a minimal resolution of $\mathcal{R}^{n\text{-filt}}$-modules*

$$0 \longrightarrow (\Sigma^{-1}x)^\wedge \longrightarrow P^{\leq n}(x) \longrightarrow x^\wedge \longrightarrow x_{\mathcal{D}}^\wedge \longrightarrow 0$$

*and a minimal coresolution*

$$0 \longrightarrow x_{\mathcal{D}}^\vee \longrightarrow x^\vee \longrightarrow I^{\geq 1}(x) \longrightarrow (\Sigma x)^\vee \longrightarrow 0.$$

*(b) If $Q$ is not a Dynkin quiver, we have a minimal resolution of $\mathcal{R}^{n\text{-filt}}$-modules*

$$0 \longrightarrow P^{\leq n}(x) \longrightarrow x^{\wedge} \longrightarrow x^{\wedge}_{\mathcal{D}} \longrightarrow 0$$

*and a minimal coresolution*

$$0 \longrightarrow x^{\vee}_{\mathcal{D}} \longrightarrow x^{\vee} \longrightarrow I^{\geq 1}(x) \longrightarrow 0.$$

*Proof.* We sketch a proof for the resolutions of $x^{\wedge}_{\mathcal{D}}$. Since $\mathcal{R}^{n\text{-filt}}$ satisfies Assumption 2.2.2, the proof boils down to the computation of $\operatorname{Ext}^n(x^{\wedge}_{\mathcal{D}}, S_u)$ for $n \geq 0$ and $u \in (\mathbb{Z}Q^{n\text{-fr}})_0$. If $u = \sigma_l(y)$ is frozen, we may compute this extension group using the injective resolution of $S_{\sigma_l(y)}$ described in Lemma 2.4.8. Since $x^{\wedge}_{\mathcal{D}}$ vanishes on frozen vertices, one gets that $\operatorname{Ext}^n(x^{\wedge}_{\mathcal{D}}, S_{\sigma_l(y)}) = 0$ for $n \neq 1$ and

$$\operatorname{Ext}^1(x^{\wedge}_{\mathcal{D}}, S_{\sigma_l(y)}) \cong \operatorname{Hom}(x^{\wedge}_{\mathcal{D}}, y^{\vee}) = D\mathcal{D}_Q(H(y), H(x)).$$

This gives the term $P^{\leq n}(x)$ in the resolutions above. The computation of $\operatorname{Ext}^n(x^{\wedge}_{\mathcal{D}}, S_u)$ for non-frozen vertices $u$ is identical to the one in the proof of [KS16, Theorem 3.7]. $\checkmark$

We can now sketch a proof of Theorem 2.4.10.

*Proof of Theorem 2.4.10.* Identify $\mathcal{S}^{n\text{-filt}}$ with $\mathcal{T}^n_{k\mathcal{S}_0}(\mathcal{S})$ so that we can use the canonical functors from Section 2.3.2. If $1 \leq m \leq n$, notice that $i^{m,n}_{\mathrm{sub}}(\sigma^{\wedge}_l) = \sigma^{\wedge}_l$ and $i^{m,n}_{\mathrm{quot}}(\sigma^{\vee}_l) = \sigma^{\vee}_{l+n-m}$. These isomorphisms follow from the adjunctions coming from Lemma 2.3.8 and Section 2.2.4. Consequently, we can recursively transport the (co)resolutions of the simple $\mathcal{S}^{m\text{-filt}}$-modules to get almost all (co)resolutions of the simple $\mathcal{S}^{n\text{-filt}}$-modules. The only ones missing are the projective resolution of $S_{\sigma_n^{-1}(x)}$ and the injective coresultion of $S_{\sigma_1(x)}$. One can compute them as in [KS16]. We will do it explicitly in the Dynkin case.

Suppose $Q$ is Dynkin. Denote by $\operatorname{res} : \operatorname{Mod}\mathcal{R}^{n\text{-filt}} \to \operatorname{Mod}\mathcal{S}^{n\text{-filt}}$ the restriction functor. Since $H$ vanishes on frozen vertices, we have $\operatorname{res}(x^{\wedge}_{\mathcal{D}}) = 0$ and the resolution in Lemma 2.4.11 induces an exact sequence

$$0 \longrightarrow \operatorname{res}((\Sigma^{-1}x)^{\wedge}) \longrightarrow P^{\leq n}(x) \longrightarrow \operatorname{res}(x^{\wedge}) \longrightarrow 0$$

of $\mathcal{S}^{n\text{-filt}}$-modules, and more generally, an exact sequence

$$0 \longrightarrow \operatorname{res}((\Sigma^{-(p+1)}x)^{\wedge}) \longrightarrow P^{\leq n}(\Sigma^{-p}x) \longrightarrow \operatorname{res}((\Sigma^{-p}x)^{\wedge}) \longrightarrow 0.$$

Splicing these sequences with the restriction of the resolution (a) in Lemma 2.4.6, we get the resolution

$$\cdots \to P^{\leq n}(\Sigma^{-2}x) \to P^{\leq n}(\Sigma^{-1}x) \to P^{\leq n}(x) \to \sigma_n^{-1}(x)^\wedge \to S_{\sigma_n^{-1}(x)} \to 0$$

of $S_{\sigma_n^{-1}(x)}$, as desired. We can similarly find the injective coresolution of $S_{\sigma_1(x)}$. $\surd$

**Corollary 2.4.12.** *Let $x, y \in (\mathbb{Z}Q^{n\text{-fr}})_0$ be non-frozen vertices. For each $p \geq 1$, we have an isomorphism*

$$\operatorname{Ext}^p_{\mathcal{S}^{n\text{-filt}}}(S_{\sigma_i(x)}, S_{\sigma_j(y)}) \cong \begin{cases} 0 & \text{if } i < j, \\ \mathcal{D}_Q(H(x), \Sigma^p H(y)) & \text{if } i \geq j. \end{cases}$$

*Proof.* We may assume $Q$ is connected. Suppose first that $Q$ is Dynkin. We use the injective coresolution of $S_{\sigma_j(y)}$ from Theorem 2.4.10. Since this resolution is minimal, we get $\operatorname{Ext}^p(S_{\sigma_i(x)}, S_{\sigma_j(y)}) \cong \operatorname{Hom}(S_{\sigma_i(x)}, I^{\geq j}(\Sigma^{p-1}y))$. Since

$$I^{\geq j}(\Sigma^{p-1}y) = \prod_{z \in (\mathbb{Z}Q)_0} D\mathcal{D}_Q(\Sigma^{p-1}H(y), H(z)) \otimes (\sigma_j^{-1}(z)^\vee \oplus \cdots \oplus \sigma_n^{-1}(z)^\vee),$$

this morphism space vanishes if $i < j$. If $i \geq j$, it becomes $D\mathcal{D}_Q(\Sigma^{p-1}H(y), H(z))$ for $z$ satisfying $\sigma_i^{-1}(z) = \sigma_i(x)$, that is, for $z = \tau(x)$. Finally, by Serre duality in $\mathcal{D}_Q$, we have

$$D\mathcal{D}_Q(\Sigma^{p-1}H(y), H(\tau(x))) \cong D\mathcal{D}_Q(\Sigma^p H(y), \Sigma\tau H(x)) \cong \mathcal{D}_Q(H(x), \Sigma^p H(y)).$$

If $Q$ is not Dynkin, the same proof works for $p = 1$, and both sides vanish for $p \geq 2$. $\surd$

### 2.4.4 Kan extensions

The restriction functor res : $\operatorname{Mod} \mathcal{R}^{n\text{-filt}} \to \operatorname{Mod} \mathcal{S}^{n\text{-filt}}$ admits a left and a right adjoint. They are given by the left and right Kan extension functors $K_L$ and $K_R$ along the inclusion $\mathcal{S}^{n\text{-filt}} \to \mathcal{R}^{n\text{-filt}}$ (see [Mac98, Section X.3]). Both of them are fully faithful by [Mac98, Corollary X.3.3], which implies that res is a localization of abelian categories (see [Gab62, Proposition 5, p. 374]).

**Lemma 2.4.13.** *An $\mathcal{R}^{n\text{-filt}}$-module $M$ belongs to the image of $K_L$, respectively $K_R$, if and only if, for every $N \in \ker(\text{res})$, we have*

$$\operatorname{Hom}(M, N) = 0 \quad \textit{and} \quad \operatorname{Ext}^1(M, N) = 0,$$

*respectively,*

$$\operatorname{Hom}(N, M) = 0 \quad \textit{and} \quad \operatorname{Ext}^1(N, M) = 0.$$

*If $Q$ is a Dynkin quiver, we may suppose $N = S_x$ for a non-frozen vertex $x \in \mathbb{Z}Q^{n\text{-fr}}$.*

*Proof.* The first statement is [Gab62, Lemme 1, p. 370] and [Gab62, Corollaire, p. 371]. The second can be proved as in [KS16, Lemma 5.3]. $\surd$

Denote by $\eta^R$ and $\varepsilon^R$ the unit and counit of the adjunction $\mathrm{res} \dashv K_R$. Similarly define $\eta^L$ and $\varepsilon^L$ for $K_L \dashv \mathrm{res}$. By [KS14, Lemma 2.3], the following square is commutative:

$$\begin{tikzcd} K_L \circ \mathrm{res} \circ K_R \arrow[r, "K_L\varepsilon^R"] \arrow[d, "\varepsilon^L K_R"'] & K_L \arrow[d, "\eta^R K_L"] \\ K_R \arrow[r, "K_R\eta^L"] & K_R \circ \mathrm{res} \circ K_L \end{tikzcd}$$

Since $K_L$ and $K_R$ are fully faithful, $\eta^L$ and $\varepsilon^R$ are isomorphisms. In particular, the horizontal maps in the diagram above are isomorphisms and, if we invert them, we obtain a canonical morphism $\mathrm{can} : K_L \to K_R$.

**Lemma 2.4.14.** *Suppose $Q$ is a Dynkin quiver. The canonical morphism $K_L(M) \to K_R(M)$ is an isomorphism if $M$ is a coproduct of free $\mathcal{S}^{n\text{-filt}}$-modules or a product of cofree $\mathcal{S}^{n\text{-filt}}$-modules.*

*Proof.* We follow [KS16, Lemma 5.4 (a)]. Suppose that $M$ is a coproduct of free modules. We need to show that $\eta^R_{K_L(M)}$ is an isomorphism. Since $K_R$ is fully faithful, this is equivalent to showing that $K_L(M)$ is in the image of $K_R$. By Lemma 2.4.13, it suffices to show that $\mathbf{R}\mathrm{Hom}(S_x, K_L(M)) = 0$ for any non-frozen vertex $x \in (\mathbb{Z}Q^{n\text{-fr}})_0$. This follows from Corollary 2.4.7. Indeed, knowing that $K_L$ is left adjoint to res, one shows that $K_L(M)$ is a coproduct of free modules associated with frozen vertices. The proof when $M$ is a product of cofree modules is dual. $\surd$

**Lemma 2.4.15.** *Suppose $Q$ is a Dynkin quiver. Let $x$ be a non-frozen vertex of $\mathbb{Z}Q^{n\text{-fr}}$.*

*(a) Let*

$$0 \longrightarrow (\Sigma^{-1}x)^{\wedge}_{\mathcal{R}^{m\text{-filt}}} \longrightarrow P^{\leq m}_{\mathcal{R}^{m\text{-filt}}}(x) \longrightarrow x^{\wedge}_{\mathcal{R}^{m\text{-filt}}} \longrightarrow 0$$

*be the resolution of $x^{\wedge}_{\mathcal{D}}$ in $\mathrm{Mod}\,\mathcal{R}^{m\text{-filt}}$ for some $m \leq n$ from Lemma 2.4.11. Then its image under $\mathrm{Hom}_{\mathcal{S}^{n\text{-filt}}}(i^{m,n}_{\mathrm{sub}} \circ \mathrm{res}(-), P)$ is acyclic for any finitely generated projective $\mathcal{S}^{n\text{-filt}}$-module $P$.*

*(b) Let*

$$0 \longrightarrow x^{\vee}_{\mathcal{R}^{m\text{-filt}}} \longrightarrow I^{\geq 1}_{\mathcal{R}^{m\text{-filt}}}(x) \longrightarrow (\Sigma x)^{\vee}_{\mathcal{R}^{m\text{-filt}}} \longrightarrow 0$$

*be the coresolution of $x^{\vee}_{\mathcal{D}}$ in $\mathrm{Mod}\,\mathcal{R}^{m\text{-filt}}$ for some $m \leq n$ from Lemma 2.4.11. Then its image under $\mathrm{Hom}_{\mathcal{S}^{n\text{-filt}}}(I, i^{m,n}_{\mathrm{quot}} \circ \mathrm{res}(-))$ is acyclic for any finitely cogenerated injective $\mathcal{S}^{n\text{-filt}}$-module $I$.*

*Proof.* For (a), the adjunctions $\mathrm{res} \dashv K_R$ and $i_{\mathrm{sub}}^{m,n} \dashv \mathrm{sub}^{n,m}$ imply that the functors

$$\mathrm{Hom}_{\mathcal{S}^{n\text{-filt}}}(i_{\mathrm{sub}}^{m,n} \circ \mathrm{res}(-), P) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{R}^{m\text{-filt}}}(-, K_R \circ \mathrm{sub}^{n,m}(P))$$

are isomorphic. By Proposition 2.3.9 and Remark 2.3.12, $\mathrm{sub}^{n,m}(P)$ is a coproduct of free $\mathcal{S}^{m\text{-filt}}$-modules. Thus, $K_R(\mathrm{sub}^{n,m}(P))$ is isomorphic to $M = K_L(\mathrm{sub}^{n,m}(P))$ by Lemma 2.4.14. Due to the adjunction $K_L \dashv \mathrm{res}$, notice that $M$ is a coproduct of free $\mathcal{R}^{m\text{-filt}}$-modules associated with frozen vertices. The statement we want to prove is equivalent to the vanishing of $\mathbf{R}\mathrm{Hom}_{\mathcal{R}^{m\text{-filt}}}(x_{\mathcal{D}}^{\wedge}, M)$ in the derived category of vector spaces. Since $x_{\mathcal{D}}^{\wedge}$ is finite-dimensional and concentrated on non-frozen vertices, it suffices to show that $\mathbf{R}\mathrm{Hom}(S_y, M)$ vanishes for any non-frozen vertex $y$. This follows from Corollary 2.4.7. The proof of (b) is analogous. $\surd$

The image of the canonical map $\mathrm{can} : K_L \to K_R$ will be denoted by $K_{LR}$. Notice that $K_{LR}(M)$ is finite-dimensional for any finite-dimensional $\mathcal{S}^{n\text{-filt}}$-module $M$. Indeed, since $K_L$ preserves finitely generated projectives and is right exact, $K_L(M)$ is pointwise finite-dimensional and right bounded. Dually, $K_R(M)$ is pointwise finite-dimensional and left bounded. But $K_{LR}(M)$ is a quotient of $K_L(M)$ and a submodule of $K_R(M)$, whence our claim.

We also define $KK$ and $CK$ as the kernel and the cokernel of can, respectively. One checks that $\mathrm{res}(\mathrm{can})$ is an isomorphism, hence $KK(M)$ and $CK(M)$ are in the kernel of res for any $M \in \mathrm{Mod}\,\mathcal{S}^{n\text{-filt}}$. Consequently, they can be viewed as modules over the quotient $\mathcal{R}^{n\text{-filt}}/\ker(\mathrm{res})$, which is isomorphic to the mesh category $k(\mathbb{Z}Q)$.

**Proposition 2.4.16.** *If $M$ is a finite-dimensional $\mathcal{S}^{n\text{-filt}}$-module, then $KK(M)$ is a finitely generated projective $k(\mathbb{Z}Q)$-module and $CK(M)$ is a finitely cogenerated injective $k(\mathbb{Z}Q)$-module.*

*Proof.* When $n = 1$, this is [KS16, Theorem 4.8]. Their proof works perfectly for $n \geq 1$ after a single adaptation, which we explain now. In the proof of [KS16, Lemma 4.7], one needs to show that $\mathrm{Ext}^1(K_{LR}(M), N)$ is finite-dimensional if $N \in \ker(\mathrm{res})$ is finite-dimensional. In loc. cit., it is proved that $\mathrm{Ext}^2(K_{LR}(M), N)$ vanishes and their argument works for general $n$. Hence, since $K_{LR}(M)$ and $N$ are both finite-dimensional, we may assume $N$ is simple. Since $\mathrm{res}(N) = 0$, we have $N \cong S_x$ for a non-frozen vertex $x \in (\mathbb{Z}Q)_0$. But $\mathrm{Ext}^1(K_{LR}(M), S_x)$ can be computed with the injective coresolution of $S_x$ given in Lemma 2.4.6 and its finite-dimensionality is evident. $\surd$

For the next lemma, we say that an $\mathcal{R}^{n\text{-filt}}$-module $U$ is *stable* (resp. *costable*) if it does not admit any non-zero submodule (resp. quotient) concentrated in non-frozen vertices. From the adjunctions with the restriction functor, it follows, for example, that $K_R(M)$ is stable and $K_L(M)$ is costable for any $\mathcal{S}^{n\text{-filt}}$-module $M$. Since $K_{LR}(M)$ is a submodule of $K_R(M)$ and a quotient of $K_L(M)$, $K_{LR}(M)$ is both stable and costable.

**Lemma 2.4.17.** *If $U$ is a finite-dimensional $\mathcal{R}^{n\text{-filt}}$-module which is both stable and costable, then*

$$\dim \operatorname{Ext}^1(S_x, U) = \dim \operatorname{Ext}^1(U, S_{\tau(x)}) = (w_1 + \cdots + w_n)(\sigma(x)) + (C_q v)(x)$$

*for any $x \in (\mathbb{Z}Q)_0$, where*

$$(C_q v)(x) = v(x) + v(\tau(x)) - \sum_{\substack{y \to x \\ y \in (\mathbb{Z}Q)_0}} v(y)$$

*and $v, w_1, \ldots, w_n$ are the dimension vectors of $U$ (as in Section 2.4.2).*

*Proof.* The case $n = 1$ is [KS16, Lemma 4.13], and the same proof works for $n \geq 2$. $\surd$

Let $M$ be a finite-dimensional $\mathcal{S}^{n\text{-filt}}$-module. By Proposition 2.4.16, there exist objects $\varphi_n(M), \psi_n(M) \in \mathcal{D}^b(\operatorname{mod} kQ)$ such that

$$CK(M) \cong D\mathrm{Hom}(\varphi_n(M), H(?)) \quad \text{and} \quad KK(M) \cong \mathrm{Hom}(H(?), \tau(\psi_n(M))),$$

where we view $KK(M)$ and $CK(M)$ as $k(\mathbb{Z}Q)$-modules.

**Proposition 2.4.18.** *We have*

$$\varphi_1(\mathrm{tot}^n(M)) \cong \varphi_n(M) \quad \textit{and} \quad \psi_1(\mathrm{tot}^n(M)) \cong \psi_n(M)$$

*for every finite-dimensional $\mathcal{S}^{n\text{-filt}}$-module $M$.*

*Proof.* We give a proof for the first isomorphism. The other can be proved similarly. Note that $\varphi_n(M)$ is a direct sum of objects of the form $H(x)$ with $x \in (\mathbb{Z}Q)_0$. The multiplicity of $H(x)$ in this direct sum is given by the dimension of $\mathrm{Hom}(S_x, CK(M))$. From the exact sequence

$$0 \longrightarrow K_{LR}(M) \longrightarrow K_R(M) \longrightarrow CK(M) \longrightarrow 0$$

and the fact that $\mathrm{Hom}(S_x, K_R(M)) = \mathrm{Ext}^1(S_x, K_R(M)) = 0$ by Lemma 2.4.13, such multiplicity coincides with the dimension of $\mathrm{Ext}^1(S_x, K_{LR}(M))$, which is $(w_1 + \cdots + w_n)(\sigma_n(x)) - (C_q v)(x)$ by Lemma 2.4.17, where $(v, w_1, \ldots, w_n)$ is the dimension vector of $K_{LR}(M)$. Similarly, we prove that $\varphi_1(\mathrm{tot}^n(M))$ is a direct sum of objects of the form $H(x)$, where each term appears with multiplicity $w'(\sigma(x)) - (C_q v')(x)$, where $(v', w')$ denotes the dimension vector of the $\mathcal{R}$-module $K_{LR}(\mathrm{tot}^n(M))$. Therefore, the lemma will be proved once we show that $v = v'$ and $w' = w_1 + \cdots + w_n$.

For an $\mathcal{R}^{n\text{-filt}}$-module $U$, denote by $\mathrm{tot}^n(U)$ the $\mathcal{R}$-module given on objects by

$$\mathrm{tot}^n(U)(x) = U(x) \quad \text{and} \quad \mathrm{tot}^n(U)(\sigma(x)) = U(\sigma_1(x)) \oplus \cdots \oplus U(\sigma_n(x))$$

for $x \in (\mathbb{Z}Q)_0$, and with the evident action on morphisms. It can be promoted to a functor $\mathrm{tot}^n : \mathrm{Mod}\,\mathcal{R}^{n\text{-filt}} \to \mathrm{Mod}\,\mathcal{R}$ compatible with the functor $\mathrm{tot}^n : \mathrm{Mod}\,\mathcal{S}^{n\text{-filt}} \to \mathrm{Mod}\,\mathcal{S}$ and the restriction functors. To conclude the proof, it suffices to show that $\mathrm{tot}^n(K_{LR}(M)) \cong K_{LR}(\mathrm{tot}^n(M))$. Observe that the restrictions of these modules to $\mathrm{Mod}\,\mathcal{S}$ are both isomorphic to $\mathrm{tot}^n(M)$. Denote by $w : \mathcal{S}_0 \to \mathbb{N}$ the dimension vector of $\mathrm{tot}^n(M)$. Viewing $\mathrm{tot}^n(K_{LR}(M))$ and $K_{LR}(\mathrm{tot}^n(M))$ as points of an appropriate graded quiver variety, we deduce that they have the same image under the map

$$\pi : \coprod \mathfrak{M}^\bullet(v, w) \longrightarrow \mathfrak{M}_0^\bullet(w),$$

where the disjoint union varies over all dimension vectors $v : \mathcal{R}_0 \setminus \mathcal{S}_0 \to \mathbb{N}$. Now, notice that $\mathrm{tot}^n(K_{LR}(M))$ and $K_{LR}(\mathrm{tot}^n(M))$ are both stable and costable as $\mathrm{tot}^n$ clearly preserves these properties. Since $\pi$ is injective over the subset of points that are both stable and costable (see [KS16, Section 2.6] and the references therein), we obtain the desired isomorphism. $\surd$

### 2.4.5 Gorenstein projective and injective modules

From now on, we assume that $Q$ is a Dynkin quiver. We will study the Gorenstein projective/injective $\mathcal{S}^{n\text{-filt}}$-modules. To do so, it is helpful to prove that $\mathcal{S}^{n\text{-filt}}$ is *weakly Gorenstein of dimension* 1 (in the sense of [KS16]), that is, we have

$$\mathrm{Ext}^p_{\mathcal{S}^{n\text{-filt}}}(M, P) = 0 = \mathrm{Ext}^p_{\mathcal{S}^{n\text{-filt}}}(I, M)$$

whenever $p \geq 2$, $P$ is a finitely generated projective module, $I$ is a finitely cogenerated injective module and $M$ is a finite-dimensional module. As in [KS16, Lemma 5.7], we have the following slightly stronger property.

**Lemma 2.4.19.** *For any finitely cogenerated injective $\mathcal{S}^{n\text{-filt}}$-module $I$ and any pointwise finite-dimensional right bounded $\mathcal{S}^{n\text{-filt}}$-module $M$, we have $\mathrm{Ext}^p(I, M) = 0$ for $p \geq 2$. Dually, for any finitely generated projective $\mathcal{S}^{n\text{-filt}}$-module $P$ and any pointwise finite-dimensional left bounded $\mathcal{S}^{n\text{-filt}}$-module $N$, we have $\mathrm{Ext}^p(N, P) = 0$ for $p \geq 2$.*

*Proof.* We give a proof for the first statement. By the hypothesis on $M$, we can write $M$ as the inverse limit of a system of finite-dimensional modules. By the argument in [KS16, Lemma 5.7], this reduces the proof to the case where $M$ is finite-dimensional, as long as we additionally show that $\mathrm{Ext}^p(I, M)$ is finite-dimensional for $p = 0, 1$. Since $M$

is then finite length, we may further suppose that $M = S_{\sigma_l(x)}$ for some $x \in (\mathbb{Z}Q)_0$ and $1 \leq l \leq n$. To compute $\operatorname{Ext}^p(I, S_{\sigma_l(x)})$, we use the injective coresolution of $S_{\sigma_l(x)}$ from Theorem 2.4.10. It is obtained by splicing the exact sequence of $\mathcal{R}^{(n-l+1)\text{-filt}}$-modules

$$0 \longrightarrow S_{\sigma_1(x)} \longrightarrow \sigma_1(x)^\vee \longrightarrow x^\vee \longrightarrow 0$$

with the sequences

$$0 \longrightarrow (\Sigma^{r-1}x)^\vee \longrightarrow I^{\geq 1}_{\mathcal{R}^{n-l+1\text{-filt}}}(\Sigma^{r-1}x) \longrightarrow (\Sigma^r x)^\vee \longrightarrow 0$$

for $r \geq 1$, and then applying the functors res and $i^{n-l+1,n}_{\text{quot}}$. We conclude that $\operatorname{Ext}^p(I, S_{\sigma_l(x)})$ vanishes for $p \geq 2$ by Lemma 2.4.15. We also deduce that $\operatorname{Ext}^p(I, S_{\sigma_l(x)})$ is finite-dimensional for $p = 0, 1$ since $I$ and the modules in the coresolution of $S_{\sigma_l(x)}$ are finitely cogenerated. $\surd$

Following [EJ95], we say an $\mathcal{S}^{n\text{-filt}}$-module $M$ is *Gorenstein projective* if there is an exact complex

$$P = \quad \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

of finitely generated projective modules such that $M \cong \operatorname{coker} d_1$ and $\operatorname{Hom}(P, P')$ is still exact for any finitely generated projective module $P'$. Dually, an $\mathcal{S}^{n\text{-filt}}$-module $N$ is *Gorenstein injective* if there is an exact complex

$$I = \quad \cdots \longrightarrow I^{-1} \xrightarrow{d^{-1}} I^0 \xrightarrow{d^0} I^1 \longrightarrow \cdots$$

of finitely cogenerated injective modules such that $N \cong \ker d^0$ and $\operatorname{Hom}(I', I)$ is still exact for any finitely cogenerated injective module $I'$. Notice that in these definitions we are only considering finitely generated or cogenerated modules.

Let $\operatorname{gpr}(\mathcal{S}^{n\text{-filt}})$ and $\operatorname{gin}(\mathcal{S}^{n\text{-filt}})$ be the full subcategories of $\operatorname{Mod}\mathcal{S}^{n\text{-filt}}$ consisting of Gorenstein projective and Gorenstein injective modules. The argument in [AR91, Proposition 5.1] shows that they are closed under extensions and direct summands. One can then easily check that they are Frobenius exact categories whose projective-injective objects are the finitely generated projective modules and the finitely cogenerated injective modules, respectively. We denote their stable categories (i.e., their quotients by the class of projective-injective objects) by $\underline{\operatorname{gpr}}(\mathcal{S}^{n\text{-filt}})$ and $\underline{\operatorname{gin}}(\mathcal{S}^{n\text{-filt}})$, which are canonically triangulated categories (see [Hap88]).

For an $\mathcal{S}^{n\text{-filt}}$-module $M$, define $\Omega M$ as the kernel of a fixed epimorphism $P_M \to M$ with $P_M$ projective. Dually, define $\Sigma M$ as the cokernel of a fixed monomorphism $M \to I_M$ with $I_M$ injective. If $M$ is finitely generated (resp. cogenerated), we suppose the same holds for $P_M$ (resp. $I_M$). For the simple modules, we take the canonical epimorphism

$\sigma_l^{-1}(x)^\wedge \to S_{\sigma_l^{-1}(x)}$ and the canonical monomorphism $S_{\sigma_l(x)} \to \sigma_l(x)^\vee$, so that

$$\Omega S_{\sigma_l^{-1}(x)} = i_{\text{sub}}^{l,n} \circ \operatorname{res}(x^\wedge_{\mathcal{R}^{l\text{-filt}}}) \quad \text{and} \quad \Sigma S_{\sigma_l(x)} = i_{\text{quot}}^{n-l+1,n} \circ \operatorname{res}(x^\vee_{\mathcal{R}^{n-l+1\text{-filt}}})$$

by Theorem 2.4.10. These constructions are not functorial on the level of $\operatorname{Mod} \mathcal{S}^{n\text{-filt}}$, but they give rise to well-defined functors

$$\Omega : \underline{\operatorname{gpr}}(\mathcal{S}^{n\text{-filt}}) \longrightarrow \underline{\operatorname{gpr}}(\mathcal{S}^{n\text{-filt}}) \quad \text{and} \quad \Sigma : \underline{\operatorname{gin}}(\mathcal{S}^{n\text{-filt}}) \longrightarrow \underline{\operatorname{gin}}(\mathcal{S}^{n\text{-filt}}),$$

which are in fact equivalences. The second is the suspension functor of the canonical triangulated structure on $\underline{\operatorname{gin}}(\mathcal{S}^{n\text{-filt}})$ (whence the notation), while the first is a quasi-inverse for the suspension functor of $\underline{\operatorname{gpr}}(\mathcal{S}^{n\text{-filt}})$.

**Lemma 2.4.20.** *If $M$ is a finite-dimensional $\mathcal{S}^{n\text{-filt}}$-module, then $\Omega M$ is Gorenstein projective and $\Sigma M$ is Gorenstein injective.*

*Proof.* Since $\operatorname{gpr}(\mathcal{S}^{n\text{-filt}})$ and $\operatorname{gin}(\mathcal{S}^{n\text{-filt}})$ are extension-closed in $\operatorname{Mod} \mathcal{S}^{n\text{-filt}}$, we may suppose $M = S_{\sigma_l^{-1}(x)}$ for some $x \in (\mathbb{Z}Q)_0$ and $1 \leq l \leq n$. For $p \in \mathbb{Z}$, consider the exact sequence of $\mathcal{S}^{l\text{-filt}}$-modules

$$0 \longrightarrow \operatorname{res}(\Sigma^{p-1}x)^\wedge_{\mathcal{R}^{l\text{-filt}}} \longrightarrow P^{\leq l}_{\mathcal{S}^{l\text{-filt}}}(\Sigma^p x) \longrightarrow \operatorname{res}(\Sigma^p x)^\wedge_{\mathcal{R}^{l\text{-filt}}} \longrightarrow 0$$

obtained by restricting to $\operatorname{Mod} \mathcal{S}^{l\text{-filt}}$ the resolution of $(\Sigma^p x)^\wedge_{\mathcal{D}}$ in $\operatorname{Mod} \mathcal{R}^{l\text{-filt}}$ from Lemma 2.4.11. Applying $i_{\text{sub}}^{l,n}$ and splicing these sequences, we get an acyclic complex $P$ of finitely generated projective $\mathcal{S}^{n\text{-filt}}$-modules. By Lemma 2.4.15, $\operatorname{Hom}(P, P')$ is still acyclic. We conclude that $\Omega S_{\sigma_l^{-1}(x)} = i_{\text{sub}}^{l,n} \circ \operatorname{res}(x^\wedge_{\mathcal{R}^{l\text{-filt}}})$ is Gorenstein projective. Dually, one proves that $\Sigma S_{\sigma_l^{-1}(x)}$ is Gorenstein injective. $\surd$

**Lemma 2.4.21.** *If $M$ and $N$ are finite-dimensional $\mathcal{S}^{n\text{-filt}}$-modules, then the map*

$$\operatorname{Ext}^1(M, N) \longrightarrow \operatorname{Ext}^1(\Omega M, \Omega N)$$

*induced by $\Omega$ is surjective. The dual result, where we replace $\Omega$ by $\Sigma$, also holds.*

*Proof.* We give a proof for the first statement. By the horseshoe lemma and the snake lemma, any extension of $M$ by $N$ gives rise to an extension of $\Omega M$ by $\Omega N$. This yields the map above. We can alternatively describe it using the derived category $\mathcal{D}(\mathcal{S}^{n\text{-filt}})$ of $\mathcal{S}^{n\text{-filt}}$-modules. View an extension $\xi \in \operatorname{Ext}^1(M, N)$ as a morphism $M \to N[1]$ in $\mathcal{D}(\mathcal{S}^{n\text{-filt}})$, where $[1]$ denotes the shift functor of $\mathcal{D}(\mathcal{S}^{n\text{-filt}})$. We also have canonical morphisms $\eta_M : M \to \Omega M[1]$ and $\eta_N : N \to \Omega N[1]$ induced by the exact sequences defining $\Omega M$ and $\Omega N$. We claim that the extension $\Omega\xi$ corresponds to the unique morphism $\Omega M \to \Omega N[1]$ such that the following square commutes:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "\xi"'] & \Omega M[1] \arrow[d, "-\Omega\xi[1]"] \\ N[1] \arrow[r, "\eta_N[1]"'] & \Omega N[2] \end{tikzcd}$$

The commutativity of the square and the reason for the negative sign in front of $\Omega\xi[1]$ follow from the $3\times 3$ lemma for triangulated categories (see [BBD82, Proposition 1.1.11]). Applying $\mathrm{Hom}(-,\Omega N)$ to the exact sequence defining $M$, we obtain that composition with $\eta_M$ induces an isomorphism $\mathrm{Ext}^1(\Omega M,\Omega N)\to\mathrm{Ext}^2(M,\Omega N)$, whence the uniqueness in our claim. Therefore, to prove the lemma, it suffices to show that the map $\mathrm{Ext}^1(M,N)\to\mathrm{Ext}^2(M,\Omega N)$ given by composition with $\eta_N[1]$ is surjective. Since $\mathrm{Ext}^2(M,-)$ vanishes on finitely generated projective modules by Lemma 2.4.19, this follows from the long exact sequence on extension groups obtained by applying $\mathrm{Hom}(M,-)$ to the exact sequence defining $\Omega N$. $\surd$

### 2.4.6 The main equivalences

Let $\overleftarrow{\mathsf{A}}_n$ be a linearly oriented Dynkin diagram of type $\mathsf{A}_n$. The vertices are numbered from 1 to $n$ in such a way that 1 is the unique sink and $n$ is the unique source. We set $B_n = k\overleftarrow{\mathsf{A}}_n\otimes kQ$ and write $\mathcal{D}_{B_n}$ for the bounded derived category $\mathcal{D}^b(\operatorname{mod} B_n)$ of finite-dimensional $B_n$-modules. In this section, we will adapt the proof of [KS16, Theorem 5.18] to prove the following result.

**Theorem 2.4.22.** *If $Q$ is a Dynkin quiver, then there are triangle equivalences*

$$F_n:\mathcal{D}_{B_n}\xrightarrow{\sim}\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})\quad\textit{and}\quad G_n:\mathcal{D}_{B_n}\xrightarrow{\sim}\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}).$$

As in the previous section, suppose $Q$ is Dynkin. For each $1\le i\le n$, let $e_i\in k\overleftarrow{\mathsf{A}}$ be the stationary path corresponding to the vertex $i$. The endomorphism algebra of the projective $B_n$-module $(e_i\otimes 1)B_n$ is isomorphic to $kQ$. Thus, the derived tensor product with $(e_i\otimes 1)B_n$ induces a fully faithful triangle functor $\mu_i:\mathcal{D}_Q\to\mathcal{D}_{B_n}$. Since $(e_j\otimes 1)B_n(e_i\otimes 1)$ vanishes for $i<j$ and is otherwise isomorphic to $kQ$ as a left $kQ$-module, we have a natural isomorphism

$$\mathcal{D}_{B_n}(\mu_i(M),\mu_j(N)) = \begin{cases} 0 & \text{if } i<j,\\ \mathcal{D}_Q(M,N) & \text{if } i\ge j,\end{cases}$$

for $M,N\in\mathcal{D}_Q$. We will construct $F_n$ and $G_n$ in such a way that $F_n(\mu_i(H(x)))=\Sigma S_{\sigma_i(x)}$ and $G_n(\mu_i(H(x)))=\Omega S_{\sigma_i(x)}$ for $x\in(\mathbb{Z}Q)_0$. In particular, we need the following lemma.

**Lemma 2.4.23.** *For any two frozen vertices $\sigma_i(x)$ and $\sigma_j(y)$ in $\mathbb{Z}Q^{n\text{-fr}}$ and any $p \in \mathbb{Z}$, there are isomorphisms between $\mathcal{D}_{B_n}(\mu_i(H(x)), \Sigma^p \mu_j(H(y)))$ and the morphism spaces*

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma^p \Sigma S_{\sigma_j(y)}) \quad \textit{and} \quad \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})(\Omega S_{\sigma_i(x)}, \Sigma^p \Omega S_{\sigma_j(y)}).$$

*Proof.* We give a proof for the first isomorphism. Initially, let us suppose $p \geq 2$. Then

$$\begin{aligned} \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma^p \Sigma S_{\sigma_j(y)}) &\cong \mathrm{Ext}^p_{\mathcal{S}^{n\text{-filt}}}(\Sigma S_{\sigma_i(x)}, \Sigma S_{\sigma_j(y)}) \\ &\cong \mathrm{Ext}^{p+1}_{\mathcal{S}^{n\text{-filt}}}(\Sigma S_{\sigma_i(x)}, S_{\sigma_j(y)}), \end{aligned}$$

where the second isomorphism follows from dimension shifting. Since $\mathrm{Ext}^q(-, S_{\sigma_j(y)})$ with $q \geq 2$ vanishes over finitely cogenerated injective modules by Lemma 2.4.19, we can also apply dimension shifting in the first coordinate. Hence, the last space above is isomorphic to $\mathrm{Ext}^p(S_{\sigma_i(x)}, S_{\sigma_j(y)})$, which in turn is isomorphic to $\mathcal{D}_{B_n}(\mu_i(H(x)), \Sigma^p \mu_j(H(y)))$ by Corollary 2.4.12.

Now, suppose $p$ is arbitrary. The short exact sequences obtained by applying $i^{n-j+1}_{\mathrm{quot}}$ores to the coresolution of $(\Sigma^q x)^\vee_{\mathcal{D}}$ in $\mathrm{Mod}\,\mathcal{R}^{(n-j+1)\text{-filt}}$ from Lemma 2.4.11 tell us that

$$\Sigma^q \Sigma S_{\sigma_j(y)} \cong \Sigma S_{\sigma_j(\Sigma^q y)}$$

in $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ for any $q \in \mathbb{Z}$. Consequently,

$$\begin{aligned} \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma^p \Sigma S_{\sigma_j(y)}) &\cong \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma^2 \Sigma S_{\sigma_j(\Sigma^{p-2} y)}) \\ &\cong \mathcal{D}_{B_n}(\mu_i(H(x)), \Sigma^2 \mu_j(H(\Sigma^{p-2} y))) \\ &\cong \mathcal{D}_{B_n}(\mu_i(H(x)), \Sigma^p \mu_j(H(y))), \end{aligned}$$

where the second isomorphism follows from the previous paragraph. $\surd$

Consider the object

$$T_n = \bigoplus_{v \in Q_0} \Sigma S_{\sigma_1((v,0))} \oplus \cdots \oplus \Sigma S_{\sigma_n((v,0))} \in \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}}).$$

Since $\mu_l(H((v,0)))$ gives all indecomposable projective $B_n$-modules as we vary $l$ and $v$, Lemma 2.4.23 implies that the endomorphism algebra of $T_n$ is isomorphic to $B_n$. We also deduce that

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(T_n, \Sigma^p T_n) = 0$$

for any non-zero integer $p$. Since $B_n$ has finite global dimension and $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ is an algebraic triangulated category, there exists a fully faithful triangle functor $F_n : \mathcal{D}_{B_n} \to \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ sending the regular $B_n$-module to $T_n$ (see [Kel06, Theorem 3.8]). A similar argument produces a fully faithful triangle functor $G_n : \mathcal{D}_{B_n} \to \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ sending $\mu_l(H((v,0)))$ to $\Omega S_{\sigma_l((v,0))}$ for all $1 \leq l \leq n$ and $v \in Q_0$.

**Lemma 2.4.24.** *For any $1 \leq l \leq n$ and any non-frozen vertex $x$ in $\mathbb{Z}Q^{n\text{-fr}}$, the objects $\Sigma S_{\sigma_l(x)}$ and $\Omega S_{\sigma_l(x)}$ are in the essential image of $F_n$ and $G_n$, respectively.*

*Proof.* By applying the restriction functor and the functor $i^{n-l+1,n}_{\mathrm{quot}}$ to the injective coresolution of $S_x$ in $\operatorname{Mod}\mathcal{R}^{(n-l+1)\text{-filt}}$ from Lemma 2.4.6, we get a short exact sequence of $\mathcal{S}^{n\text{-filt}}$-modules

$$0 \longrightarrow \Sigma S_{\sigma_l(x)} \longrightarrow \bigoplus_{l\leq i\leq n} \sigma_i^{-1}(x)^{\vee} \oplus \bigoplus_{x\to y} \Sigma S_{\sigma_l(y)} \longrightarrow \Sigma S_{\sigma_l(\tau^{-1}(x))} \longrightarrow 0$$

where $y$ varies over all sucessors of $x$ in $\mathbb{Z}Q$. If we pass to the quotient $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$, the cofree modules above vanish, and we get a distinguished triangle that resembles the Auslander–Reiten triangle in $\mathcal{D}_Q$ associated with the mesh starting at the vertex $x$. Since $\Sigma S_{\sigma_l((v,0))}$ belongs to the image of $F_n$ for all $v \in Q_0$ by construction, we can use these triangles to show that $F_n(\mu_l(H(x))) \cong \Sigma S_{\sigma_l(x)}$ for all $x \in (\mathbb{Z}Q)_0$ via a knitting argument. We can similarly prove the claim for the functor $G_n$. $\surd$

To conclude that $F_n$ and $G_n$ are equivalences, it remains to prove the following result.

**Lemma 2.4.25.** *The functors $F_n$ and $G_n$ are essentially surjective.*

*Proof.* We give a proof for the functor $F_n$. For $1 \leq l \leq n$, let $F_n^l : \mathcal{D}_Q \to \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ be the composition $F_n \circ \mu_l$. We will prove that $F_n^l$ has a right adjoint $R_n^l$ or, equivalently, that the functor

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(F_n^l(-), M) : \mathcal{D}_Q \longrightarrow \operatorname{Mod} k$$

is representable for any $M \in \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$. Since the functor above is cohomological, [BK89, Theorem 2.11] reduces our claim to showing that it is a functor of finite type, i.e., that it takes values on finite-dimensional vector spaces and vanishes for almost all indecomposable objects in $\mathcal{D}_Q$. Let $x \in (\mathbb{Z}Q)_0$. By Lemma 2.4.24, we have $F_n^l(H(x)) \cong \Sigma S_{\sigma_l(x)}$, which is isomorphic to $\Sigma^{-1}\Sigma S_{\sigma_l(\Sigma x)}$ by the argument in the proof of Lemma 2.4.23. Hence,

$$\begin{aligned}\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(F_n^l(H(x)), M) &\cong \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma^{-1}\Sigma S_{\sigma_l(\Sigma x)}, M)\\ &\cong \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_l(\Sigma x)}, \Sigma M)\\ &\cong \operatorname{Ext}^1_{\mathcal{S}^{n\text{-filt}}}(\Sigma S_{\sigma_l(\Sigma x)}, M).\end{aligned}$$

Using that $\Sigma S_{\sigma_l(\Sigma x)}$ is the cokernel of the map $S_{\sigma_l(\Sigma x)} \to \sigma_l(\Sigma x)^{\vee}$ and that the extension group $\operatorname{Ext}^1(\sigma_l(\Sigma x)^{\vee}, M)$ vanishes (as $M$ is Gorenstein injective), we deduce that the last space above is the cokernel of the map

$$\mathrm{Hom}(\sigma_l(\Sigma x)^\vee, M) \longrightarrow \mathrm{Hom}(S_{\sigma_l(\Sigma x)}, M).$$

But the socle of $M$ is finite-dimensional as $M$ is finitely cogenerated, hence the target of the map above is always finite-dimensional and vanishes for almost all $x$, as desired.

Let $M \in \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ and let us show it is in the essential image of $F_n$. We will define recursively a sequence of objects $M_1, M_2, \ldots, M_{n+1}$ as follows. Put $M_1 = M$. For $1 \leq l \leq n$, define $M_{l+1}$ as the cone of the counit map $F_n^l R_n^l(M_l) \to M_l$ from the adjunction $F_n^l \dashv R_n^l$, so that we have a distinguished triangle

$$F_n^l R_n^l(M_l) \longrightarrow M_l \longrightarrow M_{l+1} \longrightarrow \Sigma F_n^l R_n^l(M)$$

in $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$. We claim that $M_{l+1}$ is right orthogonal to the image of the functors $F_n^1$, $F_n^2, \ldots, F_n^l$, that is, $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(F_n^{l'}(X), M_{l+1})$ vanishes for all $1 \leq l' \leq l$ and $X \in \mathcal{D}_Q$. To see this, observe first that the image of $F_n^l$ is right orthogonal to the images of the functors $F_n^1, \ldots, F_n^{l-1}$. This follows from Lemmas 2.4.23 and 2.4.24. Since the right orthogonality property is closed under cones, by induction, it suffices to verify that $M_{l+1}$ is right orthogonal to the image of $F_n^l$. This is true because $F_n^l$ is fully faithful, hence the map

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(N, F_n^l R_n^l(M_l)) \longrightarrow \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(N, M_l)$$

is an isomorphism for every object $N$ in the image of $F_n^l$.

To conclude the proof, it suffices to show that $M_{n+1} = 0$ in $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$. Indeed, knowing that $F_n$ is a full triangle functor, we can use the triangles above to deduce that all $M_l$, including $M_1 = M$, are in the image of $F_n$. By the previous paragraph, $M_{n+1}$ is right orthogonal to the image of the functors $F_n^l$ for $1 \leq l \leq n$. Let us show that this right orthogonal subcategory is zero. Take $M$ in this subcategory. Since $M$ is finitely cogenerated, it has a finite-dimensional socle. We prove that $M$ is an injective $\mathcal{S}^{n\text{-filt}}$-module by induction on the dimension of its socle. If this dimension is zero, then $M$ is the zero module, and we are done. If not, then it contains a simple submodule of the form $S_{\sigma_l(x)}$. By the hypothesis on $M$, the space

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(F_n^l(H(\Sigma^{-1}x)), M) \cong \mathrm{coker}\left(\mathrm{Hom}(\sigma_l(x)^\vee, M) \longrightarrow \mathrm{Hom}(S_{\sigma_l(x)}, M)\right)$$

is zero. In particular, the inclusion $S_{\sigma_l(x)} \to M$ extends to a morphism $\sigma_l(x)^\vee \to M$, which must be injective as it is injective on the socles. But $\sigma_l(x)^\vee$ is an injective module, so it is a direct summand of $M$. The complement $M/\sigma_l(x)^\vee$ is isomorphic to $M$ in $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ (so it is also in the right orthogonal subcategory) and its socle is strictly smaller than that of $M$. By the induction hypothesis, this complement is injective and so $M$ is also injective.

This finishes the proof. $\surd$

By the proof of Theorem 2.4.22, there are simple $\mathcal{S}^{n\text{-filt}}$-modules $S_1, \ldots, S_r$ such that $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ is the thick triangulated subcategory generated by $\Omega S_1, \ldots, \Omega S_r$. This observation gives the following recursive description of $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$. Denote by $E_1$ the strictly full subcategory of $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ whose objects are isomorphic to a direct sum of shifts of the objects $\Omega S_1, \ldots, \Omega S_r$. For $l \geq 2$, define $E_l$ recursively as the strictly full subcategory of objects isomorphic to a direct summand of an object $X$ that fits into a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma A \tag{2.6}$$

where $A \in E_1$ and $B \in E_{l-1}$. Then $\underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ is the union of the $E_l$ for $l \geq 1$. Dually, $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ is the thick triangulated subcategory generated by $\Sigma S_1, \ldots, \Sigma S_r$ and can be similarly described. We obtain two important consequences.

**Corollary 2.4.26.** *For $m \geq n \geq 1$, the functors $i_{\mathrm{sub}}^{n,m}$ and $\mathrm{quot}^{m,n}$ preserve Gorenstein projective modules, while $i_{\mathrm{quot}}^{n,m}$ and $\mathrm{sub}^{m,n}$ preserve Gorenstein injective modules.*

*Proof.* We prove the first part. For $m, n \geq 1$, let $F : \mathrm{Mod}\,\mathcal{S}^{n\text{-filt}} \to \mathrm{Mod}\,\mathcal{S}^{m\text{-filt}}$ be an exact functor which preserves finitely generated projective modules and finite-dimensional modules. For example, these properties are satisfied if $F = i_{\mathrm{sub}}^{n,m}$ or $F = \mathrm{quot}^{n,m}$. We show that $F$ preserves Gorenstein projective modules.

Let $M \in \mathrm{gpr}(\mathcal{S}^{n\text{-filt}})$. Viewed as an object of the stable category, it belongs to $E_l$ for some $l \geq 1$. We prove that $F(M)$ is Gorenstein projective by induction on $l$. Since $F$ is exact and preserves finitely generated projectives, it sends $\Sigma^p(\Omega S)$ for a simple module $S$ and $p \in \mathbb{Z}$ to the direct sum of $\Sigma^p(\Omega F(S))$ with a finitely generated projective. But $F(S)$ is finite-dimensional, hence this direct sum is Gorenstein projective by Lemma 2.4.20. In particular, if $l = 1$, this implies that $F(M)$ is Gorenstein projective. If $l \geq 2$, then $M$ is a direct summand of an object $X$ that sits in a triangle as in (2.6). Consequently, by adding projective summands if necessary, we find an exact sequence

$$0 \longrightarrow A \longrightarrow M \oplus N \longrightarrow B \longrightarrow 0$$

in $\mathrm{gpr}(\mathcal{S}^{n\text{-filt}})$ where $A \in E_1$ and $B \in E_{l-1}$. By the induction hypothesis, $F(A)$ and $F(B)$ are Gorenstein projective, thus so is $F(M)$ since $F$ is exact and $\mathrm{gpr}(\mathcal{S}^{m\text{-filt}})$ is closed under extensions and direct summands. $\surd$

**Corollary 2.4.27.** *Any Gorenstein projective $\mathcal{S}^{n\text{-filt}}$-module is a direct summand of $\Omega X \oplus P$ for some finite-dimensional module $X$ and some finitely generated projective module $P$. Dually, any Gorenstein injective $\mathcal{S}^{n\text{-filt}}$-module is a direct summand of $\Sigma Y \oplus I$ for some finite-dimensional module $Y$ and some finitely cogenerated injective module $I$.*

*Proof.* We prove the first statement. Let $M \in \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$. It belongs to $E_l$ for some $l \geq 1$. We prove by induction on $l$ that $M$ is a direct summand of $\Omega M'$ (in the stable category) for some finite-dimensional module $M'$. By the same argument as in the proof of Lemma 2.4.23, any shift of $\Omega S$ with $S$ simple is again of the form $\Omega S'$ for a simple module $S'$. Hence, our claim is true for $l = 1$. If $l \geq 2$, then $M$ is a direct summand of an object $X$ that fits into a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow \Sigma A$$

where $A \in E_1$ and $B \in E_{l-1}$. By the induction hypothesis, we may add direct summands and assume that $A \cong \Omega A'$ and $B \cong \Omega B'$ where $A'$ and $B'$ are finite-dimensional. By Lemma 2.4.21, there is an extension $X'$ of $B'$ by $A'$ such that $X \cong \Omega X'$. Since $M$ is a direct summand of $X$, this concludes the proof. $\surd$

**Remark 2.4.28.** The same argument shows that if $M \in \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})$ is an iterated extension of objects of the form $\Omega X$ with $X$ finite-dimensional, then $M$ is also of this form.

### 2.4.7 The stratification functors

We continue to assume that $Q$ is a Dynkin quiver. Let $\operatorname{mod} \mathcal{S}^{n\text{-filt}}$ be the category of finite-dimensional $\mathcal{S}^{n\text{-filt}}$-modules. By Lemma 2.4.20, we have the (co)syzygy functors

$$\Omega : \operatorname{mod} \mathcal{S}^{n\text{-filt}} \longrightarrow \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}}) \quad \text{and} \quad \Sigma : \operatorname{mod} \mathcal{S}^{n\text{-filt}} \longrightarrow \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}}).$$

We define the *stratification functors*

$$\Phi^n, \Psi^n : \operatorname{mod} \mathcal{S}^{n\text{-filt}} \longrightarrow \mathcal{D}_{B_n} = \mathcal{D}^b(\operatorname{mod} k\overleftarrow{\mathsf{A}}_n \otimes kQ)$$

as the compositions $G_n^{-1} \circ \Omega$ and $F_n^{-1} \circ \Sigma$, respectively, where $F_n^{-1}$ and $G_n^{-1}$ are quasi-inverses for the equivalences from Theorem 2.4.22. When $n = 1$, we recover the stratification functor of [KS16]. We remark that $\Phi^n$ and $\Psi^n$ are $\delta$-functors in the sense of [Kel91]. This essentially means that they send short exact sequences to distinguished triangles in a functorial way.

**Remark 2.4.29.** In [KS16], it is implicitly shown that $\Phi^n$ is naturally isomorphic to $\Psi^n$ when $n = 1$. This follows from Proposition 2.13 and Lemma 4.14 (and its dual version) in loc. cit. We do not know if this property holds for $n \geq 2$.

Our next goal is to show some compatibility relations between the functors $\Phi^n$ and $\Psi^n$ and the canonical functors from Section 2.3.2.

For $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, there is a canonical truncation map $k\overleftarrow{A}_n \to k\overleftarrow{A}_m$ that sends the unique path between $i_r$ and $i_s$ to the unique path between $r$ and $s$, and the other paths to zero. Tensoring with $kQ$, we get an algebra morphism $t_n^{i_1,\dots,i_m}: B_n \to B_m$. By abuse of notation, we also denote by $t_n^{i_1,\dots,i_m}$ the derived tensor product functor $-\otimes^L_{B_n} B_m : \mathcal{D}_{B_n} \to \mathcal{D}_{B_m}$, where we view $B_m$ as a left $B_n$-module via this truncation map.

**Theorem 2.4.30.** *For $1 \leq m \leq n$, we have the following isomorphisms of functors:*

$$t_n^{1,2,\dots,m} \circ \Psi^n \cong \Psi^m \circ \mathrm{sub}^{n,m},$$
$$t_n^{n-m+1,\dots,n-1,n} \circ \Phi^n \cong \Phi^m \circ \mathrm{quot}^{n,m}.$$

*Proof.* We prove the first isomorphism. The other one can be proved dually. By Corollary 2.4.26 and Proposition 2.3.9, the functor $\mathrm{sub}^{n,m}$ descends to the stable category of Gorenstein injective modules and the left square in the diagram

$$\begin{tikzcd}
\mathrm{mod}\,\mathcal{S}^{n\text{-filt}} \arrow[r, "\Sigma"] \arrow[d, "\mathrm{sub}^{n,m}"'] & \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}}) \arrow[d, "\mathrm{sub}^{n,m}"] & \mathcal{D}_{B_n} \arrow[l, "F_n"'] \arrow[d, "t_n^{1,2,\dots,m}"] \\
\mathrm{mod}\,\mathcal{S}^{m\text{-filt}} \arrow[r, "\Sigma"'] & \underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}}) & \mathcal{D}_{B_m} \arrow[l, "F_m"]
\end{tikzcd}$$

commutes up to natural isomorphism. We prove the same is true for the square on the right. This will conclude the proof.

Let us first show that both compositions in the right square send the regular $B_n$-module to the object $T_m \in \underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}})$ from the proof of Theorem 2.4.22. On one hand, we have

$$\mathrm{sub}^{n,m}(\Sigma S_{\sigma_i(x)}) \cong \Sigma\,\mathrm{sub}^{n,m}(S_{\sigma_i(x)}) = \begin{cases} \Sigma S_{\sigma_i(x)} & \text{if } 1 \leq i \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that $\mathrm{sub}^{m,n}(F_n(B_n)) = \mathrm{sub}^{m,n}(T_n) \cong T_m$. On the other hand, $t_n^{1,2,\dots,m}(B_n) = B_m$, which is sent to $T_m$ by $F_m$.

We now verify that both actions of $B_n$ on $T_m$ defined by the functors above are compatible. In other words, we check that the diagram

$$\begin{tikzcd}
B_n = \mathcal{D}_{B_n}(B_n, B_n) \arrow[r, "\sim"] \arrow[d, "t_n^{1,2,\dots,m}"'] & \underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(T_n, T_n) \arrow[d, "\mathrm{sub}^{n,m}"] \\
B_m = \mathcal{D}_{B_m}(B_m, B_m) \arrow[r, "\sim"] & \underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}})(T_m, T_m)
\end{tikzcd}$$

is commutative. It suffices to show that, for any $x, y \in (\mathbb{Z}Q)_0$ and $1 \leq i, j \leq m$ with $i \geq j$, the morphism

$$\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma S_{\sigma_j(y)}) \longrightarrow \underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}})(\Sigma S_{\sigma_i(x)}, \Sigma S_{\sigma_j(y)})$$

yielded by $\mathrm{sub}^{n,m}$ coincides with the identity map $\mathcal{D}_Q(H(x), H(y)) \to \mathcal{D}_Q(H(x), H(y))$

under the identifications of Lemma 2.4.23. Since $\mathrm{sub}^{n,m}$ is exact and preserves injective modules, we can use the isomorphisms in the proof of this lemma to identify the map above with the map induced on extension groups:

$$\mathrm{Ext}^2_{\mathcal{S}^{n\text{-filt}}}(S_{\sigma_i(x)}, S_{\sigma_j(\Sigma^{-2}y)}) \longrightarrow \mathrm{Ext}^2_{\mathcal{S}^{m\text{-filt}}}(S_{\sigma_i(x)}, S_{\sigma_j(\Sigma^{-2}y)}).$$

We use the injective coresolution of $S_{\sigma_j(\Sigma^{-2}y)}$ from Theorem 2.4.10 to compute the groups above. Observe that $\mathrm{sub}^{n,m}$ sends the injective module $I^{\geq j}_{\mathcal{S}^{n\text{-filt}}}(z)$ to $I^{\geq j}_{\mathcal{S}^{m\text{-filt}}}(z)$ for any $z \in (\mathbb{Z}Q)_0$. Hence, the map above becomes the map

$$\mathrm{Hom}_{\mathcal{S}^{n\text{-filt}}}(S_{\sigma_i(x)}, I^{\geq j}_{\mathcal{S}^{n\text{-filt}}}(\Sigma^{-1}y)) \longrightarrow \mathrm{Hom}_{\mathcal{S}^{m\text{-filt}}}(S_{\sigma_i(x)}, I^{\geq j}_{\mathcal{S}^{m\text{-filt}}}(\Sigma^{-1}y))$$

induced by $\mathrm{sub}^{n,m}$. Computing the spaces above as in Corollary 2.4.12 and knowing that $\mathrm{sub}^{n,m}$ acts as the identity on the $i$-th component of a split filtration (as $i \leq m$), the map above identifies with the identity on $\mathcal{D}_Q(H(x), H(y))$, as desired.

We are now ready to finish the proof using some properties of algebraic triangulated categories (see [Kel06]). The categories $\underline{\mathrm{gin}}(\mathcal{S}^{n\text{-filt}})$ and $\underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}})$ can each be viewed as a full triangulated subcategory of the derived category of some dg category. In this way, since $\mathrm{sub}^{n,m}$ is an exact functor and preserves the projective-injective objects of $\mathrm{gin}(\mathcal{S}^{n\text{-filt}})$, it can be identified with the derived tensor product functor associated with a bimodule on the level of these derived categories. By construction, $F_n$, $F_m$ and $t_n^{1,2,\dots,m}$ are also of this form. Consequently, if $\mathcal{A}$ is the dg category above whose derived category $\mathcal{D}(\mathcal{A})$ contains $\underline{\mathrm{gin}}(\mathcal{S}^{m\text{-filt}})$, then there are $B_n$-$\mathcal{A}$-bimodules $X$ and $Y$ such that $\mathrm{sub}^{n,m} \circ F_n$ and $F_m \circ t_n^{1,2,\dots,m}$ identify with the functors $- \otimes^L_{B_n} X$ and $- \otimes^L_{B_n} Y$ from $\mathcal{D}_{B_n}$ to $\mathcal{D}(\mathcal{A})$. By the previous paragraphs, the restrictions of $X$ and $Y$ to $\mathcal{A}$ are both isomorphic to $T_m$ and the homotopy actions $B_n \to \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(T_m, T_m)$ induced by the left action of $B_m$ on $X$ and $Y$ agree. Since $T_m$ has no negative self-extensions, this information allows one to construct an isomorphism between $X$ and $Y$ in the derived category of $B_n$-$\mathcal{A}$-bimodules by [Kel00, Theorem 2.1] (see also [HK24, Lemma 2.6]). This yields the desired natural isomorphism from $\mathrm{sub}^{n,m} \circ F_n$ to $F_m \circ t_n^{1,2,\dots,m}$. $\surd$

For $1 \leq i \leq n$, let $\mu_i : \mathcal{D}_Q \to \mathcal{D}_{B_n}$ be the functor from the previous section. We denote its left and right adjoints by $\mu_i^L$ and $\mu_i^R$.

**Theorem 2.4.31.** *For $1 \leq m \leq n$, we have isomorphisms*

$$(\mu_m^R \circ \Phi^n)(M) \cong (\Phi^1 \circ \mathrm{tot}^m \circ \mathrm{sub}^{n,m})(M)$$
$$(\mu_{n-m+1}^L \circ \Psi^n)(M) \cong (\Psi^1 \circ \mathrm{tot}^m \circ \mathrm{quot}^{n,m})(M)$$

*for any $M \in \mathrm{mod}\, \mathcal{S}^{n\text{-filt}}$.*

*Proof.* We only give a proof for the first isomorphism. By [KS16, Proposition 2.13], $\Phi^1$ coincides with the functor $\varphi_1$ from Section 2.4.4. By Proposition 2.4.18, the proof reduces to finding an isomorphism of $k(\mathbb{Z}Q)$-modules

$$CK(\mathrm{sub}^{n,m}(M)) \cong D\mathcal{D}_Q(\mu_m^R \Phi^n(M), H(?))$$

for $M \in \operatorname{mod} \mathcal{S}^{n\text{-filt}}$. We do it by adapting the proof of [KS16, Proposition 2.13].

Let $P \to M$ be an epimorphism where $P$ is a finitely generated projective module. It is still surjective after applying the exact functor $\mathrm{sub}^{n,m}$ and induces the following commutative diagram:

$$\begin{tikzcd} K_L(\mathrm{sub}^{n,m}(P)) \arrow[r] \arrow[d] & K_R(\mathrm{sub}^{n,m}(P)) \arrow[d] \\ K_L(\mathrm{sub}^{n,m}(M)) \arrow[r] & K_R(\mathrm{sub}^{n,m}(M)) \end{tikzcd} \tag{2.7}$$

The map on the left is again surjective as $K_L$ is a left adjoint. By Proposition 2.3.9 and Remark 2.3.12, $\mathrm{sub}^{n,m}(P)$ is a coproduct of free modules, so Lemma 2.4.14 implies that the top map in the diagram is an isomorphism. We conclude that the cokernel of the bottom map, i.e. $CK(\mathrm{sub}^{n,m}(M))$, is isomorphic to the cokernel of the map on the right.

Let us now compute this last cokernel on a non-frozen vertex $x \in \mathbb{Z}Q$. First, we have

$$\begin{aligned} K_R(\mathrm{sub}^{n,m}(M))(x) \cong \mathrm{Hom}(x^\wedge, K_R(\mathrm{sub}^{n,m}(M))) &\cong \mathrm{Hom}(i_{\mathrm{sub}}^{m,n} \circ \mathrm{res}(x^\wedge), M) \\ &= \mathrm{Hom}(\Omega S_{\sigma_m^{-1}(x)}, M) \end{aligned}$$

and similarly for $P$ instead of $M$. Thus, we have the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\sigma_m^{-1}(x)^\wedge, P) \arrow[r] \arrow[d] & \mathrm{Hom}(\Omega S_{\sigma_m^{-1}(x)}, P) \arrow[r] \arrow[d] & \mathrm{Ext}^1(S_{\sigma_m^{-1}(x)}, P) \arrow[r] \arrow[d] & 0 \\ \mathrm{Hom}(\sigma_m^{-1}(x)^\wedge, M) \arrow[r] & \mathrm{Hom}(\Omega S_{\sigma_m^{-1}(x)}, M) \arrow[r] & \mathrm{Ext}^1(S_{\sigma_m^{-1}(x)}, M) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact and where the second vertical map identifies with the map on the right in (2.7) evaluated at $x$. But the first vertical map is surjective as $\sigma_m^{-1}(x)^\wedge$ is projective, hence the cokernel of the second vertical map is isomorphic to that of the third one. By Lemma 2.4.19, $\mathrm{Ext}^2(S_{\sigma_m^{-1}(x)}, P)$ is zero, so the cokernel of the third vertical map above is isomorphic to $\mathrm{Ext}^2(S_{\sigma_m^{-1}}(x), \Omega M)$. We then have the identifications:

$$\mathrm{Ext}^2(S_{\sigma_m^{-1}}(x), \Omega M) \cong \mathrm{Ext}^1(\Omega S_{\sigma_m^{-1}}(x), \Omega M) \cong \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})(\Omega S_{\sigma_m^{-1}}(x), \Sigma\Omega M).$$

Now, if $G_n$ denotes the equivalence from Theorem 2.4.22, we have

$$G_n(\mu_m(\tau^{-1}H(x))) = G_n(\mu_m(H(\tau^{-1}(x)))) = \Omega S_{\sigma_m(\tau^{-1}(x))} = \Omega S_{\sigma_m^{-1}(x)}.$$

Therefore, since $\Phi^n = G_n^{-1} \circ \Omega$, we have

$$\begin{aligned} \underline{\mathrm{gpr}}(\mathcal{S}^{n\text{-filt}})(\Omega S_{\sigma_m^{-1}}(x), \Sigma\Omega M) &\cong \mathcal{D}_{B_n}(\mu_m(\tau^{-1}H(x)), \Sigma\Phi^n(M)) \\ &\cong \mathcal{D}_Q(\tau^{-1}H(x), \Sigma\mu_m^R\Phi^n(M)) \\ &\cong \mathcal{D}_Q(H(x), \Sigma\tau\mu_m^R\Phi^n(M)) \\ &\cong D\mathcal{D}_Q(\mu_m^R\Phi^n(M), H(x)), \end{aligned}$$

where the last isomorphism is Serre duality in $\mathcal{D}_Q$. This concludes the proof. $\surd$

Let $w_1, \ldots, w_n : \mathcal{S}_0 \to \mathbb{N}$ be dimension vectors. We define an equivalence relation on the $n$-fold affine graded tensor product variety $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ as follows. Identifying two points $M, N \in \mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ as finite-dimensional $\mathcal{S}^{n\text{-filt}}$-modules, we say that $M$ and $N$ are equivalent if and only if $\Phi^n(M) \cong \Phi^n(N)$ in $\mathcal{D}_{B_n}$. When $n = 1$, the induced partition on $\mathfrak{M}_0^\bullet(w_1)$ coincides with Nakajima's stratification [Nak01a, Nak11] by [KS16, Theorem 2.7].

In general, this partition contains information about the stratum of the subquotients of the filtration of $M$. To state this more precisely, note that the canonical functors $\mathrm{sub}^{n,m}$ and $\mathrm{quot}^{n,m}$ can be viewed as morphisms of varieties from $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ to $\mathfrak{T}_0^\bullet(w_1, \ldots, w_m)$ and $\mathfrak{T}_0^\bullet(w_{n-m+1}, \ldots, w_n)$, respectively. For $0 \le i < j \le n$, the composition

$$\mathrm{subquot}_{i,j}^n : \mathfrak{T}_0^\bullet(w_1, \ldots, w_n) \xrightarrow{\mathrm{sub}^{n-i,j-i} \circ \mathrm{quot}^{n,n-i}} \mathfrak{T}_0^\bullet(w_{i+1}, \ldots, w_j) \hookrightarrow \mathfrak{M}_0^\bullet(w_{i+1} + \cdots + w_j)$$

corresponds to the functor which sends a split filtration $M \in \mathrm{Filt}_{k\mathcal{S}_0}^n(\mathcal{S})$ to the $\mathcal{S}$-module $M_j/M_i$.

**Corollary 2.4.32.** *Let $w_1, \ldots, w_n : \mathcal{S}_0 \to \mathbb{N}$ be dimension vectors and let $M, N \in \mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$. If $M$ and $N$ are equivalent, then the subquotients $\mathrm{subquot}_{i,j}^n(M)$ and $\mathrm{subquot}_{i,j}^n(N)$ belong to the same stratum of $\mathfrak{M}_0^\bullet(w_{i+1} + \cdots + w_j)$ for all $0 \le i < j \le n$.*

*Proof.* This follows from [KS16, Theorem 2.7] and the fact that the composition

$$\Phi^1 \circ \mathrm{tot}^{j-i} \circ \mathrm{sub}^{n-i,j-i} \circ \mathrm{quot}^{n,n-i}$$

factors through $\Phi^n$ by Theorems 2.4.30 and 2.4.31. $\surd$

The converse does not hold. If it were true, then $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ would contain only finitely many equivalence classes, since Nakajima's stratification always consists of finitely many strata. However, because the algebra $B_n$ is rarely of finite representation type when $n \geq 2$, it is not hard to find dimension vectors for which $\mathfrak{T}_0^\bullet(w_1, \ldots, w_n)$ has infinitely many equivalence classes. We give an example below.

**Example 2.4.33.** Take $n = 2$ and let $Q$ be the following Dynkin quiver of type $\mathsf{D}_4$:

$$\begin{tikzcd}[row sep=small]
 & & 3 \\
Q:\ 1 & 2 \arrow[l] \arrow[ur] \arrow[dr] & \\
 & & 4
\end{tikzcd}$$

The algebra $B_2$ is isomorphic to the path algebra of the following quiver modulo the relations making the three squares commutative:

$$\begin{tikzcd}[row sep=small]
 & & (3,1) & \\
(1,1) & (2,1) \arrow[l] \arrow[ur] \arrow[rr] & & (4,1) \\
 & & (3,2) \arrow[uu] & \\
(1,2) \arrow[uu] & (2,2) \arrow[l] \arrow[uu] \arrow[ur] \arrow[rr] & & (4,2) \arrow[uu]
\end{tikzcd}$$

The subquiver given by the vertices $(2,1)$ and $(i,2)$ for $1 \leq i \leq 4$ is an affine Dynkin quiver of type $\widetilde{\mathsf{D}}_4$. Its path algebra is a quotient of $B_2$, so $B_2$ is of infinite representation type. For example, it has infinitely many non-isomorphic representations of the form

$$\begin{tikzcd}[row sep=small]
 & & 0 \arrow[dl] \arrow[dd] & \\
0 \arrow[r] \arrow[dd] & k \arrow[dd] & & 0 \arrow[ll] \arrow[dd] \\
 & & k \arrow[dl] & \\
k \arrow[r] & k^2 & & k \arrow[ll]
\end{tikzcd}$$

We prove that any such representation is isomorphic to $\Phi^2(X)$ in $\mathcal{D}_{B_2}$ for some finite-dimensional $\mathcal{S}^{2\text{-filt}}$-module $X$. Moreover, the dimension vectors $w_1, w_2 : \mathcal{S}_0 \to \mathbb{N}$ of $X$ can be chosen to be the same for all these representations. As a result, we obtain that the equivalence relation on $\mathfrak{T}_0^\bullet(w_1, w_2)$ induced by $\Phi^2$ has infinitely many equivalence classes.

Since the representations above are iterated extensions of simple modules, Corollary 2.4.27 and Remark 2.4.28 reduce our claim to showing that every simple $B_2$-module is in the essential image of $\Phi^2$. Let $x_i \in (\mathbb{Z}Q)_0$ be such that $H(x_i)$ is the simple $kQ$-module corresponding to the vertex $i \in Q_0$. If $T_{i,j}$ denotes the simple $B_2$-module corresponding to the vertex $(i,j)$ of the quiver above, then $T_{i,2} = \mu_2(H(x_i))$ and we have a distinguished triangle

$$\mu_2(H(x_i)) \longrightarrow \mu_1(H(x_i)) \longrightarrow T_{i,1} \longrightarrow \Sigma\mu_2(H(x_i))$$

in $\mathcal{D}_{B_2}$. Applying the equivalence $G_n$ to the triangle above, we deduce that $G_n(T_{i,1})$ is an extension of $\Omega S_{\sigma_2(\Sigma x_i)}$ by $\Omega S_{\sigma_1(x_i)}$ in $\underline{\mathrm{gpr}}(\mathcal{S}^{2\text{-filt}})$. By Corollary 2.4.27, there is an extension $M_i$ of $S_{\sigma_2(\Sigma x_i)}$ by $S_{\sigma_1(x_i)}$ such that $\Omega M_i \cong G_n(T_{i,1})$. We conclude that $\Phi^2(M_i) \cong T_{i,1}$ and $\Phi^2(S_{\sigma_2(x_i)}) \cong T_{i,2}$, as desired. This proves our claim and shows that the dimension vectors $w_1$ and $w_2$ from the previous paragraph can be taken to be

$$w_1(u) = \begin{cases} 1, & \text{if } u = \sigma(x_2), \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad w_2(u) = \begin{cases} 2, & \text{if } u = \sigma(x_2), \\ 1, & \text{if } u = \sigma(x_1), \sigma(x_3), \sigma(x_4) \text{ or } \sigma(\Sigma x_2), \\ 0, & \text{otherwise}, \end{cases}$$

for $u \in \mathcal{S}_0$.

# Chapter 3

# Additive categorification of the monoidal $\Lambda$-invariant

This chapter consists of the article [CCJ26], available on arXiv as `arXiv:2605.03925` and written in collaboration with Peigen Cao and Geoffrey Janssens.

## 3.1 Introduction

### Background and outline

Cluster algebras are a class of commutative algebras introduced by Fomin–Zelevinsky [FZ02] in the early 2000s. By definition, they possess a distinguished set of generators constructed via an iterative procedure called mutation. A fruitful approach to understanding their intricate combinatorics has been to interpret them using certain categories, whose rich structure provides more conceptual tools for their study. There are two types of categorification of cluster algebras: additive and monoidal. Although they employ very different techniques, recent work by many authors has revealed correspondences between them that go well beyond what was initially expected. These notably include Fujita [Fuj22, Fuj24], Fujita–Oh [FO21], Fujita–Murakami [FM23], Duan–Schiffler [DS26b, DS26a], Contu [Con26], Baur–Fu–Li [BFL24], Cao [Cao25], and Casbi [Cas26]. In this chapter, we contribute to this connection by providing an additive reinterpretation of the $\Lambda$-invariant of Kashiwara–Kim–Oh–Park [KKOP20], an important tool in their study of monoidal categorification [KKOP24a, KKOP25].

The *additive* categorification program was initiated soon after the introduction of cluster algebras, starting from the observation of similarities between the combinatorics of mutations and those of tilting theory for finite-dimensional hereditary algebras (cf. [MRZ03]). After the invention of cluster categories by Caldero–Chapoton–Schiffler [CCS06] and Buan–Marsh–Reineke–Reiten–Todorov [BMRRT06], it was further developed and expanded in numerous articles, such as [CC06, GLS06, CK08, CK06, BIRS09, FK10, DWZ08, DWZ10, Ami09, Pal08, Pla11b, Pla11a, Pre17], culminating in the categorification of skew-symmetric cluster algebras with coefficients by Yilin Wu [Wu23b] and Keller–Wu [KW23b]. In these two papers, to an ice quiver with potential $(Q, F, W)$, one associates a differential graded algebra $\widehat{\boldsymbol{\Gamma}}(Q, F, W)$ called the *completed relative Ginzburg dg algebra*. From its derived category, the authors of [Wu23b, KW23b] construct the *Higgs category* $\mathcal{H}(Q, F, W)$ together with a *cluster character* which, under mild assumptions, categorify the cluster algebra $\mathcal{A}_{Q,F}$ (with non-invertible coefficients) associated with the ice quiver $(Q, F)$. For instance, the cluster character establishes a bijection between the isomorphism classes of reachable rigid indecomposable objects in $\mathcal{H}(Q, F, W)$ and the cluster variables in $\mathcal{A}_{Q,F}$.

In another direction, Hernandez–Leclerc [HL10] introduced monoidal categorifications of cluster algebras, motivated by character identities in the representation theory of quantum affine algebras. A monoidal abelian length category $\mathcal{C}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if it is endowed with an isomorphism of $\mathbb{Z}$-algebras from the Grothendieck ring $K_0(\mathcal{C})$ to $\mathcal{A}$ that identifies the set of cluster monomials of $\mathcal{A}$ with a subset of the set of isomorphism classes of simple objects in $\mathcal{C}$, whose elements are called the *reachable* simple objects. For a complex finite-dimensional simple Lie algebra $\mathfrak{g}_0$, Hernandez–Leclerc [HL10, HL16] defined monoidal subcategories $\mathscr{C}_\ell$ $(\ell \geq 1)$ and $\mathscr{C}^-$ of the category of finite-dimensional modules over the untwisted quantum affine algebra $U_q'(\widehat{\mathfrak{g}}_0)$, and showed that their Grothendieck rings admit natural cluster algebra structures. They conjectured that these categories provide monoidal categorifications of these cluster algebras. After pioneering work by Hernandez–Leclerc and Nakajima [Nak11] (see also Kimura–Qin [KQ14]), a major breakthrough came with the work of Kang–Kashiwara–Kim–Oh [KKKO18] and Qin [Qin17]. Building on the tools developed in [KKKO18], Hernandez–Leclerc's conjectures were completely proved by Kashiwara–Kim–Oh–Park [KKOP20, KKOP24a, KKOP25], who generalized them to a much larger family of monoidal subcategories $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ of the category of representations of an arbitrary quantum affine algebra $U_q'(\mathfrak{g})$.

For a pair $(V, W)$ of finite-dimensional integrable simple modules over $U_q'(\mathfrak{g})$, Kashiwara–Kim–Oh–Park [KKOP20] defined the $\Lambda$-*invariant* $\Lambda(V, W) \in \mathbb{Z}$ and the $\mathfrak{d}$-*invariant* $\mathfrak{d}(V, W) \in \mathbb{Z}_{\geq 0}$, inspired by similar invariants constructed in [KKKO18] for representations of quiver Hecke algebras. These invariants are extracted from the commutation morphism $V \otimes W \to W \otimes V$ arising from the universal $R$-matrix, and they encode important information about the structure of these tensor products. They play a fundamental role in the proofs of [KKOP20, KKOP24a, KKOP25] and allow Kashiwara–Kim–Oh–Park to establish a more refined result: the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ is a $\Lambda$-*monoidal categorification*. This implies, in particular, that the commutative ring $K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0})$ has a compatible

Poisson structure [GSV03] and is a $\Lambda$-*cluster algebra*. A $\Lambda$-cluster algebra is a cluster algebra in which each seed with extended exchange matrix $\widetilde{B}$ is endowed with an additional skew-symmetric integer matrix $\Lambda$, called the *Poisson coefficient matrix*, such that the pair $(\widetilde{B}, \Lambda)$ is compatible in the sense of Berenstein–Zelevinsky [BZ05]. Moreover, the compatible pairs associated with two neighboring seeds are related by a mutation as defined in loc. cit. In the case of the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$, if $\{V_i\}_{i\in K}$ is the family of simple modules corresponding to the cluster variables of a seed, then the associated Poisson coefficient matrix $\Lambda = (\lambda_{ij})_{i,j\in K}$ is given by $\lambda_{ij} = \Lambda(V_i, V_j)$ for $i, j \in K$.

Motivated in part by these results, Peigen Cao [Cao25] (see also [CL20]) introduced the *tropical invariant* and the $F$*-invariant* for pairs of cluster monomials (or, more generally, of *good elements*) in an arbitrary (upper) $\Lambda$-cluster algebra. These invariants can be computed using $F$-polynomials and extended $g$-vectors, and he showed that they recover the $\Lambda$-invariant and the $\mathfrak{d}$-invariant for pairs of reachable simple modules in the $\Lambda$-cluster algebras of Kashiwara–Kim–Oh–Park. Moreover, he proved that the $F$-invariant agrees with the $E$-invariant of decorated representations of quivers with potential in the additive categorification framework of Derksen–Weyman–Zelevinsky [DWZ08, DWZ10] (see also [DF15]), confirming a conjecture by Fujita [Fuj24].

Our goal in this chapter is to complete this picture by providing an additive interpretation of the tropical invariant (and hence of the $\Lambda$-invariant) in the setting of Higgs categories. For a non-degenerate ice quiver with potential $(Q, F, W)$ such that $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is a *proper* dg algebra, we prove that the associated cluster algebra $\mathcal{A}_{Q,F}$ admits a structure of $\Lambda$-cluster algebra defined in terms of *negative extensions* in the Higgs category $\mathcal{H}(Q, F, W)$, and we give a homological formula to compute the corresponding tropical and $F$-invariants. Inspired by the work of Contu [Con26], we upgrade the initial exchange matrix of Kashiwara–Kim–Oh–Park for $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ into an ice quiver with potential when the interval $[a, b]$ is finite, and consider the associated Higgs category. When $U_q'(\mathfrak{g})$ is untwisted of simply-laced type and the PBW-pair $(\mathcal{D}, \underline{w}_0)$ is *adapted*, we show that the corresponding relative Ginzburg dg algebra is proper and that our $\Lambda$-cluster algebra structure coincides with the one defined by Kashiwara–Kim–Oh–Park, thereby providing an additive categorification of their $\Lambda$- and $\mathfrak{d}$-invariants.

We now explain our main results in more detail.

#### A $\Lambda$-cluster algebra structure through additive categorification

Let $(Q, F)$ be a finite ice quiver. We allow arrows between frozen vertices, which may be either unfrozen or frozen. Assume that the sets of vertices are $Q_0 = \{1, 2, \ldots, m\}$ and

$F_0 = \{n+1, \dots, m\}$ for $1 \leq n \leq m$. We consider the $m \times m$ matrix $\widehat{B} = (b_{ij})$ defined by

$$b_{ii} = \begin{cases} 0 & \text{if } 1 \leq i \leq n, \\ 1 & \text{otherwise,} \end{cases}$$

and, for $i \neq j$,

$$b_{ij} = \#\{\text{unfrozen arrows } i \to j \text{ in } Q\} - \#\{\text{arrows } j \to i \text{ in } Q\}.$$

The first $n$ columns form the extended exchange matrix $\widetilde{B}$ of the ice quiver $(Q, F)$. If $\widehat{B}$ is invertible over $\mathbb{Q}$, we define

$$\Lambda = |\det \widehat{B}| \cdot (\widehat{B}^{-T} - \widehat{B}^{-1}).$$

In this case, we show that the pair $(\widetilde{B}, \Lambda)$ is compatible (see Proposition 3.4.1) and call it the *compatible pair associated with* $(Q, F)$. This construction is compatible with mutation of ice quivers (taking into account the arrows between frozen vertices!) as defined by Pressland [Pre20] (see Lemma 3.4.3).

If one equips $(Q, F)$ with a potential $W$, then the matrix $\widehat{B}$ is the matrix of the Euler form of the perfectly valued derived category $\mathrm{pvd}(\widehat{\mathbf{\Gamma}})$ with respect to the basis of simple dg modules, where $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}(Q, F, W)$ is the completed relative Ginzburg dg algebra. When $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper, it follows that $\widehat{B}$ is invertible (see Lemma 3.4.4), and hence the compatible pair associated with $(Q, F)$ is defined. We will interpret its Poisson coefficient matrix using the Higgs category $\mathcal{H} = \mathcal{H}(Q, F, W)$. To this end, we define

$$[M, N]_{\mathcal{H}} = \sum_{p \geq 0} (-1)^p (\dim \mathrm{Ext}_{\mathcal{H}}^{-p}(M, N) - \dim \mathrm{Ext}_{\mathcal{H}}^{-p}(N, M)), \tag{3.1}$$

for $M, N \in \mathcal{H}$, where the negative extensions are computed in the relative cluster category (see Section 3.3.3). Under the assumption that $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper, this sum is finite (see Lemma 3.4.5).

To any vertex $t$ of the labeled $n$-regular tree $\mathbb{T}_n$, we associate an ice quiver with potential $(Q_t, F_t, W_t)$ obtained from $(Q, F, W)$ by iterated mutation along the path in $\mathbb{T}_n$ from the root to $t$. Our first main result is the following.

**Theorem 3.1.1** (Theorem 3.4.6)**.** *Let $(Q, F, W)$ be a non-degenerate ice quiver with potential such that $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper. Let $\mathcal{H} = \mathcal{H}(Q, F, W)$ be the associated Higgs category. Then the compatible pair associated with $(Q_t, F_t)$ is defined for all $t \in \mathbb{T}_n$, and its Poisson coefficient matrix $\Lambda_t = (\lambda_{ij}^t)$ is given by*

$$\lambda_{ij}^t = [M_i^t, M_j^t]_{\mathcal{H}},$$

*where $M_i^t \in \mathcal{H}$ is the reachable rigid indecomposable object corresponding to the $i$-th cluster variable in the seed of $\mathcal{A}_{Q,F}$ associated with $t$. In particular, this endows $\mathcal{A}_{Q,F}$ with a $\Lambda$-cluster algebra structure.*

If $\widehat{\mathbf{\Gamma}}(Q,F,W)$ is moreover concentrated in degree zero, then the Higgs category is a Frobenius exact category by a result of Yilin Wu [Wu23b, Theorem 6.2]. As observed in Remark 3.4.7, it follows that

$$[M,N]_{\mathcal{H}} = \dim \operatorname{Hom}_{\mathcal{H}}(M,N) - \dim \operatorname{Hom}_{\mathcal{H}}(N,M)$$

for $M,N \in \mathcal{H}$. Consequently, the $\Lambda$-cluster algebra structure on $\mathcal{A}_{Q,F}$ arising from Theorem 3.1.1 coincides with that of Geiß–Leclerc–Schröer [GLS13] and Grabowski–Pressland [GP25, Theorem 6.22].

The next result gives a homological interpretation of Cao's tropical and $F$-invariants for pairs of cluster monomials in $\mathcal{A}_{Q,F}$, computed using the $\Lambda$-cluster algebra structure of Theorem 3.1.1.

**Theorem 3.1.2** (Proposition 3.4.8)**.** *Let $(Q,F,W)$ be a non-degenerate ice quiver with potential such that $\widehat{\mathbf{\Gamma}}(Q,F,W)$ is proper. Let $\mathcal{H} = \mathcal{H}(Q,F,W)$ be the associated Higgs category. For reachable rigid objects $M,N \in \mathcal{H}$, we have*

$$\langle CC(M), CC(N)\rangle_{\mathrm{trop}} = \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}}$$

*and*

$$(CC(M) \mid\mid CC(N))_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N),$$

*where $CC$ denotes the canonical cluster character on $\mathcal{H}$.*

As explained in Remark 3.4.9, a result of Plamondon [Pla11b] allows us to express $\dim \operatorname{Ext}^1_{\mathcal{H}}(M,N)$ in the theorem above as an $E$-invariant between the decorated representations corresponding to $M$ and $N$. Hence, we recover, in our setting, the link between the $E$-invariant and the $F$-invariant established in [Cao25, Theorem 6.11].

### Additive categorification of the monoidal $\Lambda$-matrix

Let us recall the main ingredients in the definition of the categories $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ and of their initial monoidal seeds, introduced by Kashiwara–Kim–Oh–Park. Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra and consider the associated quantum affine algebra $U_q'(\mathfrak{g})$. By [KKOP22, Theorem 3.6], one can attach to $U_q'(\mathfrak{g})$ a simply-laced Dynkin diagram $\Delta$ of finite type whose root system controls the block decomposition of the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. For example, if $\mathfrak{g}$ is of type $\mathsf{A}^{(1)}$, $\mathsf{D}^{(1)}$, or $\mathsf{E}^{(1)}$, then $\Delta$ is the corresponding Dynkin diagram of finite type. In [KKOP24b], the notion of a *complete PBW-pair* is introduced. It consists of a pair $(\mathcal{D}, \underline{w}_0)$, where $\mathcal{D}$ is a *complete duality datum* and $\underline{w}_0$ is a reduced expression of the longest element of the Weyl group of $\Delta$. The category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ depends on the choice of an integer interval $[a,b]$, where $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, and on a complete PBW-pair $(\mathcal{D}, \underline{w}_0)$.

When $b \in \mathbb{Z}$, Kashiwara–Kim–Oh–Park describe the ice quiver of an initial seed for the cluster algebra structure on $K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0})$ by adapting earlier constructions from [BFZ05, BIRS09, GLS11]. We denote it by $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$, as it does not depend on $\mathcal{D}$. When the interval $[a,b]$ is finite, we endow this ice quiver with a potential $W^{[a,b]}(\underline{w}_0)$, building on [BIRS11, HL16, Con26]. The definition of the resulting ice quiver with potential is given in Section 3.5.3. This allows us to study the Higgs category [Wu23b] that categorifies the corresponding cluster algebra. We remark that Contu [Con26] has already considered an additive counterpart to the monoidal categorification of Kashiwara–Kim–Oh–Park. However, for his purposes, it was sufficient to categorify the cluster algebra *without* coefficients, whereas retaining coefficients is crucial for our results.

Let $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ be the completed relative Ginzburg dg algebra of the ice quiver with potential $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$. We conjecture that it is always a proper dg algebra.

**Conjecture 3.1.3** (Conjecture 3.8.1)**.** The relative Ginzburg dg algebra $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ is proper.

The next result confirms this conjecture in two particular cases.

**Theorem 3.1.4** (Corollaries 3.8.8 and 3.8.20)**.** *Let $[a,b]$ be a finite integer interval and $\underline{w}_0$ a reduced expression of the longest element $w_0$ of the Weyl group of $\Delta$. The completed relative Ginzburg dg algebra $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ is a proper dg algebra if one of the following conditions holds:*

*(1) $\underline{w}_0$ is a source sequence for some orientation of $\Delta$;*

*(2) $b - a + 1$ is a multiple of the length of $w_0$.*

To prove (1), we first show that $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ can be replaced by the *non-completed* relative Ginzburg dg algebra $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ (see Lemma 3.3.3). By a relative version of a result of Keller [Kel11a] (see Theorem 3.7.6), $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ is a relative Calabi–Yau completion in the sense of Yeung [Yeu22]. This completion is proper if a certain relative inverse dualizing bimodule is nilpotent. We prove that this is indeed the case by decomposing it and reducing the argument to the case of $\Delta = \mathsf{A}_1$. Along the way, we establish a "gluing" result for relative inverse dualizing bimodules which may be of independent interest (see Proposition 3.7.8). To prove (2), we study the effect of commutation and braid moves on $\underline{w}_0$, following ideas of Contu [Con26, Section 6.1]. See also Proposition 3.8.19 for a slight strengthening of the theorem.

We now state the main result of this chapter. We assume that $\mathfrak{g}$ is untwisted of simply-laced type, that is, of type $\mathsf{A}^{(1)}$, $\mathsf{D}^{(1)}$, or $\mathsf{E}^{(1)}$. We further assume that the PBW-pair $(\mathcal{D}, \underline{w}_0)$ is *adapted to an orientation of* $\Delta$, meaning that $\mathcal{D}$ is the complete duality datum associated with a height function on $\Delta$ (see Section 3.5.3) and $\underline{w}_0$ is a source sequence for the corresponding orientation. Let $[a,b]$ be a finite integer interval. By Theorem 3.1.4, $\widehat{\boldsymbol{\Gamma}}^{[a,b]}(\underline{w}_0)$ is proper, so the bracket (3.1) is defined for objects in the associated Higgs category $\mathcal{H}^{[a,b]}(\underline{w}_0)$. Let $CC$ be the canonical cluster character from $\mathcal{H}^{[a,b]}(\underline{w}_0)$ to the corresponding cluster algebra, denoted $\mathcal{A}^{[a,b]}(\underline{w}_0)$, and let $\varphi : K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}) \to \mathcal{A}^{[a,b]}(\underline{w}_0)$ be the isomorphism defining the $\Lambda$-monoidal categorification of Kashiwara–Kim–Oh–Park.

**Theorem 3.1.5** (Theorem 3.9.2). *Suppose $\mathfrak{g}$ is untwisted of simply-laced type. Let $(\mathcal{D}, \underline{w}_0)$ be a complete PBW-pair adapted to an orientation of $\Delta$, and let $a, b \in \mathbb{Z}$ with $a \leq b$. For reachable simple objects $V, W \in \mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$, we have*

$$\Lambda(V,W) = \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}},$$

*where $\mathcal{H} = \mathcal{H}^{[a,b]}(\underline{w}_0)$ and $M, N \in \mathcal{H}$ are the corresponding reachable rigid objects such that $\varphi(V) = CC(M)$ and $\varphi(W) = CC(N)$ in $\mathcal{A}^{[a,b]}(\underline{w}_0)$.*

We prove this theorem by showing that the $\Lambda$-cluster algebra structures on $\mathcal{A}^{[a,b]}(\underline{w}_0)$ arising from Theorem 3.1.1 and from the $\Lambda$-monoidal categorification coincide. To this end, we use a formula of Kashiwara–Kim–Oh–Park expressing the $\Lambda$-invariant in terms of the $\mathfrak{d}$-invariant and the left duality functor $\mathscr{D}^{-1}$ (see Proposition 3.5.1), namely

$$\Lambda(V,W) = \mathfrak{d}(V,W) + \sum_{n\geq 1} (-1)^{n-1} \left[\mathfrak{d}(V, \mathscr{D}^{-n}(W)) - \mathfrak{d}(W, \mathscr{D}^{-n}(V))\right].$$

This expression closely resembles the definition of the bracket $[M,N]_{\mathcal{H}}$ given in (3.1). A comparison of the two formulas shows that it is enough to prove that

$$\mathfrak{d}(V, \mathscr{D}^{-n}(W)) = \dim \operatorname{Ext}^{1-n}_{\mathcal{H}}(M,N)$$

for $n \geq 0$. This is established in Proposition 3.9.3 when $V$ and $W$ belong to the initial monoidal seed, which was previously described by Hernandez–Leclerc in [HL16]. By exploiting an infinite sequence of mutations considered in loc. cit., we show in Proposition 3.6.9 that $\mathscr{D}^{-1}(W)$ can be computed via a maximal green sequence. Up to some technical details, the desired equality then follows from results of Keller [Kel11b, Kel12], together with the additive and monoidal categorifications of the $F$-invariant. We remark that the connection between the duality functor on the monoidal side and the suspension functor on the additive side has been observed before; see, for instance, [KK19, Section 3.6] and

[Qin23, Table 3].

Keeping in mind that $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ is expected to be proper for arbitrary finite intervals $[a,b]$ and reduced expressions $\underline{w}_0$, we expect that the hypotheses on $\mathfrak{g}$ and $(\mathcal{D}, \underline{w}_0)$ in Theorem 3.1.5 are not necessary and can be removed.

### Organization

This chapter is organized as follows. In Section 3.2, we briefly review the theory of $\Lambda$-cluster algebras and introduce the tropical and $F$-invariants from [Cao25]. In Section 3.3, we recall the additive categorification of cluster algebras with coefficients following [Wu23b, KW23b], focusing on the Jacobi-finite case. In Section 3.4, we establish the $\Lambda$-cluster algebra structure arising from proper relative Ginzburg dg algebras and provide an additive interpretation of the tropical invariant. Section 3.5 contains the necessary background on quantum affine algebras and the work of Kashiwara–Kim–Oh–Park. In Section 3.6, we present results on maximal green sequences and explain how they can be used to realize the left duality functor $\mathscr{D}^{-1}$ on certain $U_q'(\mathfrak{g})$-modules. In Section 3.7, we develop the required results on inverse dualizing bimodules and relative derived preprojective algebras, including the proof of the gluing theorem (Proposition 3.7.8). The proof of Theorem 3.1.4 is given in Section 3.8. A substantial part of this section is devoted to Theorem 3.8.7, which provides a key technical result on the coincidence of extensions for relative Ginzburg dg algebras in the adapted case. Finally, in Section 3.9, we prove Theorem 3.1.5 using the results developed in the previous sections.

## 3.2 Review on cluster algebras and Cao's invariants

We give a brief reminder on the theory of $\Lambda$-cluster algebras (also known as cluster algebras with compatible Poisson structures), following mainly [Cao25]. We then recall the definition and main properties of the tropical invariant and the $F$-invariant introduced in loc. cit.

Let $K = K^{\mathrm{un}} \sqcup K^{\mathrm{fr}}$ be a countable index set which decomposes into a non-empty subset $K^{\mathrm{un}}$ of *unfrozen* indices and a subset $K^{\mathrm{fr}}$ of *frozen* indices. Following [BZ05] (and [KKOP20, Section 5]), we say that a pair $(\widetilde{B}, \Lambda)$ is a *compatible pair* if

(1) $\widetilde{B} = (b_{ij})$ is a $K \times K^{\mathrm{un}}$ integer matrix such that, for any $j \in K^{\mathrm{un}}$, there are finitely many $i \in K$ such that $b_{ij} \neq 0$;

(2) $\Lambda = (\lambda_{ij})$ is a $K \times K$ skew-symmetric integer matrix;

(3) we have $\widetilde{B}^T \Lambda = (S \mid \mathbf{0})$, where $S$ is a $K^{\mathrm{un}} \times K^{\mathrm{un}}$ diagonal matrix whose diagonal entries are strictly positive integers and $\mathbf{0}$ is the $K^{\mathrm{un}} \times K^{\mathrm{fr}}$ zero matrix.

We call $\widetilde{B}$ the *extended exchange matrix*, $\Lambda$ the *Poisson coefficient matrix*, and $S$ the *type* of the compatible pair. If $B$ denotes the submatrix of $\widetilde{B}$ indexed by $K^{\mathrm{un}} \times K^{\mathrm{un}}$, we call $B$ the *principal part* of $\widetilde{B}$. By [BZ05, Proposition 3.3], $B$ is a skew-symmetrizable matrix with skew-symmetrizer $S$ and, if $K$ is finite, $\widetilde{B}$ has full rank.

**Remark 3.2.1.** In most of this chapter, we will work with compatible pairs $(\widetilde{B}, \Lambda)$ such that the principal part of $\widetilde{B}$ is skew-symmetric. In this case, the data of $\widetilde{B}$ is essentially the same as that of a pair $(Q, F)$ where $Q = (Q_0, Q_1)$ is a quiver without loops or 2-cycles and $F \subseteq Q_0$ is a subset of *frozen vertices*. We also require that any unfrozen vertex is incident to finitely many arrows. Given such a pair and relabeling the vertices so that $Q_0 = K$ and $F = K^{\mathrm{fr}}$, the corresponding matrix $\widetilde{B} = (b_{ij})$ is defined by

$$b_{ij} = \#\{i \to j \text{ in } Q_1\} - \#\{j \to i \text{ in } Q_1\}$$

for $i \in K$ and $j \in K^{\mathrm{un}}$. Notice that the number of arrows between frozen vertices is irrelevant. However, when $K$ is finite, we will need to upgrade $F$ to a *subquiver* of $Q$ for the additive categorification of cluster algebras with coefficients. The *frozen arrows* will play an important role (see Section 3.3). As we will see in Section 3.4.1, this extra data also encodes a matrix $\Lambda$ and yields a compatible pair $(\widetilde{B}, \Lambda)$.

Let $\mathbb{F}$ be the field of rational functions on $|K|$ variables over $\mathbb{Q}$. We call a triple $(\mathbf{x}, \widetilde{B}, \Lambda)$ a $\boldsymbol{\Lambda}$-*seed* in $\mathbb{F}$ if

(1) $\mathbf{x} = \{x_i\}_{i \in K}$ is a *free generating set* of $\mathbb{F}$, that is, a transcendence basis of $\mathbb{F}$ that generates it;

(2) $(\widetilde{B}, \Lambda)$ is a compatible pair.

We call $\mathbf{x}$ the *cluster* of the $\boldsymbol{\Lambda}$-seed, and its elements the *cluster variables*. We say that the cluster variables $x_i$ for $i \in K^{\mathrm{un}}$ are *unfrozen*, while the remaining ones are *frozen*. A monomial on the cluster variables of a $\boldsymbol{\Lambda}$-seed is called a *cluster monomial*.

Let $(\mathbf{x}, \widetilde{B}, \Lambda)$ be a $\boldsymbol{\Lambda}$-seed in $\mathbb{F}$ and write $\mathbf{x} = \{x_i\}_{i \in K}$, $\widetilde{B} = (b_{ij})$ and $\Lambda = (\lambda_{ij})$. For $v \in K^{\mathrm{un}}$, we define the *mutation* $\mu_v(\mathbf{x}, \widetilde{B}, \Lambda)$ of $(\mathbf{x}, \widetilde{B}, \Lambda)$ *in the direction of* $v$ as a $\boldsymbol{\Lambda}$-seed $(\mathbf{x}', \mu_v(\widetilde{B}), \mu_v(\Lambda))$ where $\mathbf{x}' = \{x'_i\}_{i \in K}$, $\mu_v(\widetilde{B}) = (b'_{ij})$, and $\mu_v(\Lambda) = (\lambda'_{ij})$ are defined by

$$x'_i = \begin{cases} x_i & \text{if } i \neq v, \\ x_v^{-1} \cdot \left(\prod_{j \in K} x_j^{[b_{jv}]_+} + \prod_{j \in K} x_j^{[-b_{jv}]_+}\right) & \text{otherwise,} \end{cases}$$

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = v \text{ or } j = v, \\ b_{ij} + [b_{iv}]_+[b_{vj}]_+ - [-b_{iv}]_+[-b_{vj}]_+ & \text{otherwise,} \end{cases}$$

$$\lambda'_{ij} = \begin{cases} -\lambda_{iv} + \sum_{l \in K} [b_{lv}]_+ \lambda_{il} & \text{if } i \neq v \text{ and } j = v, \\ -\lambda_{vj} + \sum_{l \in K} [b_{lv}]_+ \lambda_{lj} & \text{if } i = v \text{ and } j \neq v, \\ \lambda_{ij} & \text{otherwise.} \end{cases}$$

Here $[a]_+$ denotes $\max\{a, 0\}$ for $a \in \mathbb{Z}$. Note that the sums and products above are all finite by the definition of the extended exchange matrix. We remark that $(\mu_v(\widetilde{B}), \mu_v(\Lambda))$ is indeed a compatible pair and has the same type as $(\widetilde{B}, \Lambda)$ by [BZ05, Proposition 3.4]. Moreover, mutation of $\Lambda$-seeds in a fixed direction is an involution.

Let $\mathbb{T}_{|K^{\mathrm{un}}|}$ denote the $|K^{\mathrm{un}}|$-regular tree. Label its edges by $K^{\mathrm{un}}$ so that edges incident to the same vertex have different labels. A $\Lambda$-*seed pattern*

$$\mathcal{S} = \{(\mathbf{x}_t, \widetilde{B}_t, \Lambda_t) \mid t \in \mathbb{T}_{|K^{\mathrm{un}}|}\}$$

is an assignment of $\Lambda$-seeds to the vertices of $\mathbb{T}_{|K^{\mathrm{un}}|}$ such that $(\mathbf{x}_{t'}, \widetilde{B}_{t'}, \Lambda_{t'}) = \mu_v(\mathbf{x}_t, \widetilde{B}_t, \Lambda_t)$ for any vertices $t$ and $t'$ connected by an edge labeled by $v$. If we fix a root vertex $t_0 \in \mathbb{T}_{|K^{\mathrm{un}}|}$, then $\mathcal{S}$ is determined by the $\Lambda$-seed $(\mathbf{x}, \widetilde{B}, \Lambda)$ associated with $t_0$, which we call the *initial* $\Lambda$-seed. The $\Lambda$-*cluster algebra (with non-invertible coefficients)* $\mathcal{A} = \mathcal{A}_{\widetilde{B}}$ associated with $\mathcal{S}$ is the $\mathbb{Z}$-subalgebra of $\mathbb{F}$ generated by all the cluster variables of all the $\Lambda$-seeds in $\mathcal{S}$. If $\widetilde{B}$ comes from a pair $(Q, F)$ as in Remark 3.2.1, we will denote $\mathcal{A}$ by $\mathcal{A}_{Q,F}$. If $K$ is finite (resp. infinite), we say that $\mathcal{A}$ has *finite* (resp. *infinite*) *rank*.

**Remark 3.2.2.** As the notation suggests, the $\Lambda$-cluster algebra does not depend a priori on the chosen initial matrix $\Lambda$. In fact, as originally done by Fomin–Zelevinsky [FZ02], by disregarding the Poisson coefficient matrices in all the definitions above, we can define a *cluster algebra* for any matrix $\widetilde{B}$ satisfying the finiteness condition above and whose principal part is skew-symmetrizable. However, we remark that this extra data can be used to define a compatible Poisson structure [GSV03] on the $\Lambda$-cluster algebra, and the corresponding quantization is called a *quantum cluster algebra* [BZ05]. Although we do not work with quantum cluster algebras, we still want to keep track of the Poisson coefficient matrices of the $\Lambda$-seeds of $\mathcal{A}$. This is the reason for the definition of $\Lambda$-cluster algebras given above.

For the rest of the section, we assume that $\mathcal{A}$ has finite rank. In this case, we will always take $K = \{1, \ldots, m\}$ and $K^{\mathrm{un}} = \{1, \ldots, n\}$ for fixed integers $m \geq n \geq 1$.

Each $\Lambda$-seed $(\mathbf{x}_t, \widetilde{B}_t, \Lambda_t)$ of $\mathcal{A}$ determines a partial order $\preceq_t$ on $\mathbb{Z}^m$ defined as follows: for $\mathbf{g}, \mathbf{g}' \in \mathbb{Z}^m$, we have $\mathbf{g}' \preceq_t \mathbf{g}$ if there is $\mathbf{v} \in \mathbb{N}^n$ such that $\mathbf{g}' = \mathbf{g} + \widetilde{B}_t \mathbf{v}$. This is indeed a partial order because $\widetilde{B}_t$ has full rank. We write $\mathbf{g}' \prec \mathbf{g}$ if we additionally have $\mathbf{g} \neq \mathbf{g}'$. For $\mathbf{g} = (g_1, \ldots, g_m) \in \mathbb{Z}^m$, we denote $\mathbf{x}_t^{\mathbf{g}} = x_{1;t}^{g_1} \cdots x_{m;t}^{g_m}$, where $\mathbf{x}_t = (x_{1;t}, \ldots, x_{m;t})$. Notice that $\mathbf{g}' = \mathbf{g} + \widetilde{B}_t \mathbf{v}$ is equivalent to $\mathbf{x}_t^{\mathbf{g}'} = \mathbf{x}_t^{\mathbf{g}} \cdot \widehat{\mathbf{y}}_t^{\mathbf{v}}$, where $\widehat{\mathbf{y}}_t^{\mathbf{v}} = \mathbf{x}_t^{\widetilde{B}_t \mathbf{v}}$ is a monomial in the variables $(\widehat{y}_{1;t}, \ldots, \widehat{y}_{n;t})$ defined by $\widehat{y}_{t;j} = \prod_{i=1}^m x_{i;t}^{b_{ij}}$.

**Theorem 3.2.3** ([FZ07, GHKK18]). *For $t \in \mathbb{T}_n$, any cluster monomial $u \in \mathcal{A}$ can be written (uniquely) as*

$$u = \mathbf{x}_t^{\mathbf{g}_u^t} + \sum_{\mathbf{h} \prec \mathbf{g}_\mathbf{u}^\mathbf{t}} b_\mathbf{h} \mathbf{x}_t^\mathbf{h} = \mathbf{x}_t^{\mathbf{g}_u^t} \cdot F_u^t(\widehat{y}_{1;t}, \ldots, \widehat{y}_{n;t}),$$

*where $b_\mathbf{h} \in \mathbb{Z}_{\geq 0}$ and $F_u^t \in \mathbb{Z}[y_1, \ldots, y_n]$ is a polynomial with constant term 1.*

The vector $\mathbf{g}_u^t \in \mathbb{Z}^m$ is called the *(extended) g-vector* of $u$ with respect to $t$, and $F_u^t$ is the *F-polynomial* of $u$ with respect to $t$. We remark that $\mathbf{g}_{uu'}^t = \mathbf{g}_u^t + \mathbf{g}_{u'}^t$ for two cluster monomials $u, u' \in \mathcal{A}$ associated to the same $\Lambda$-seed by [Qin24, Lemma 3.2.6]. In particular, $g$-vectors are determined by the $g$-vectors of the cluster variables. We give a categorical interpretation for them in Section 3.3.4.

For a nonzero polynomial

$$F = \sum_{\mathbf{v} \in \mathbb{N}^n} c_\mathbf{v} \mathbf{y}^\mathbf{v} \in \mathbb{Z}[y_1, \ldots, y_n],$$

its *tropical evaluation* on $\mathbf{r} \in \mathbb{Z}^n$ is defined as

$$F[\mathbf{r}] = \max\{\mathbf{v}^T \mathbf{r} \mid c_\mathbf{v} \neq 0\} \in \mathbb{Z}.$$

If $F$ has a nonzero constant term, the integer above is non-negative. We can finally define the invariants of [Cao25]. For two cluster monomials $u, u' \in \mathcal{A}$, their *tropical invariant* is defined as

$$\langle u, u' \rangle_{\text{trop}} = (\mathbf{g}_u^t)^T \Lambda_t \mathbf{g}_{u'}^t + F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] \in \mathbb{Z} \tag{3.2}$$

for any $t \in \mathbb{T}_n$, where $\widetilde{B}_t^T \Lambda_t = (S \mid \mathbf{0})$. By [Cao25, Theorem 1.3], it does not depend on the choice of $t$. The *F-invariant* is given by

$$(u \mid\mid u')_F = \langle u, u' \rangle_{\text{trop}} + \langle u', u \rangle_{\text{trop}} = F_u^t[(S \mid \mathbf{0})\mathbf{g}_{u'}^t] + F_{u'}^t[(S \mid \mathbf{0})\mathbf{g}_u^t] \in \mathbb{Z}_{\geq 0} \tag{3.3}$$

for any $t \in \mathbb{T}_n$. The last equality follows from the definition of the tropical invariant and the fact that $\Lambda_t$ is skew-symmetric. Consequently, observe that the $F$-invariant does not depend on the Poisson coefficient matrices. In fact, it can also be defined for finite rank cluster algebras with trivial coefficients without a compatible Poisson structure (see [Cao25, Section 4.4]).

**Theorem 3.2.4** ([Cao25, Theorem 1.5]). *Let $u, u' \in \mathcal{A}$ be two cluster monomials. They are cluster monomials associated with the same ($\Lambda$-)seed if and only if $(u \mid\mid u')_F = 0$.*

**Remark 3.2.5.** The original definitions of the tropical and $F$-invariants are not the ones given above, but they coincide by [Cao25, Theorem 1.3]. Moreover, they can be defined for a larger class of elements in the *upper* Λ-cluster algebra called *good elements* in loc. cit. (building on the notion of *compatibly pointed elements* of [Qin24]). These are certain elements of the upper Λ-cluster algebra that have a decomposition similar to that of Theorem 3.2.3.

## 3.3 Additive categorification of cluster algebras with coefficients

We present the necessary background on the additive categorification of cluster algebras with coefficients using relative Ginzburg dg algebras and their Higgs categories, following mainly [Wu23b] and [KW23b]. We will focus on the Jacobi-finite case, even though the theory can be developed more generally.

From now on, $k$ will denote an algebraically closed field. We assume $k = \mathbb{C}$ in Section 3.3.4 and in the results that depend on the cluster character that is introduced there.

### 3.3.1 Ice quivers with potential

In this section, we give the necessary background on ice quivers with potential and their mutations. More details can be found in Sections 2, 3, and 4 of [Pre20], which builds on the foundational work of [DWZ08].

An *ice quiver* is a pair $(Q, F)$ consisting of a quiver $Q$ and a subquiver $F$. In this case, we refer to $F$ as the *frozen* subquiver of $Q$, and its vertices and arrows are called *frozen*.

Let $Q$ be a finite quiver and denote by $\widehat{kQ}$ the completion of its path algebra with respect to the ideal generated by all arrows. Let $HH_0(\widehat{kQ})$ be the *zeroth continuous Hochschild cohomology* of $\widehat{kQ}$, that is, the quotient of $\widehat{kQ}$ by the closure of its subspace generated by all commutators. It has a topological basis given by all cycles in $Q$ up to cyclic permutation. A *potential* $W$ on a quiver $Q$ is an element of $HH_0(\widehat{kQ})$ that is a (possibly infinite) linear combination of cycles of length at least two such that any term involving a loop has degree at least three. An *ice quiver with potential* is a triple $(Q, F, W)$ where $(Q, F)$ is a *finite* ice quiver and $W$ is a potential in $Q$. If $F$ is empty, we call $(Q, W)$ a *quiver with potential*. We say that $W$ is *irredundant* if each term of $W$ contains at least one unfrozen arrow. We define $(Q, F, W)$ to be *reduced* if $W$ is irredundant and none of its terms is a 2-cycle.

For a cyclic path $p = \alpha_1 \cdots \alpha_n$ in $Q$ (with $\alpha_i \in Q_1$) and an arrow $\alpha \in Q_1$, the *cyclic derivative* of $p$ with respect to $\alpha$ is defined by

$$\partial_\alpha(p) = \sum_{\alpha_i = \alpha} \alpha_{i+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1}.$$

We can extend $\partial_\alpha$ by linearity and continuity to a well-defined linear map $\partial_\alpha : HH_0(\widehat{kQ}) \to \widehat{kQ}$. For an ice quiver with potential $(Q, F, W)$, we define the *relative Jacobian algebra* as the quotient of $\widehat{kQ}$ by the closure of the ideal generated by the elements $\partial_\alpha(W)$ for all *unfrozen* arrows $\alpha$. We call $(Q, F, W)$ *Jacobi-finite* if $J(Q, F, W)$ is finite-dimensional.

By [Pre20, Theorem 3.6], for any ice quiver with potential $(Q, F, W)$, there exists a *reduced* ice quiver with potential $(Q_{\mathrm{red}}, F_{\mathrm{red}}, W_{\mathrm{red}})$ such that

$$J(Q, F, W) \cong J(Q_{\mathrm{red}}, F_{\mathrm{red}}, W_{\mathrm{red}}).$$

If $W$ is irredundant, then $(Q_{\mathrm{red}}, F_{\mathrm{red}}, W_{\mathrm{red}})$ is unique up to right equivalence (see Definition 3.7 and Proposition 3.15 in [Pre20]). In this case, $(Q_{\mathrm{red}}, F_{\mathrm{red}}, W_{\mathrm{red}})$ is called the *reduction* of $(Q, F, W)$. It is an important ingredient for the mutation of ice quivers with potential, which we now define. From now on, suppose $W$ is irredundant. Let $v \in Q_0 \setminus F_0$ be an unfrozen vertex that is not incident to any loop or 2-cycle. Define an ice quiver with potential $\widetilde{\mu}_v(Q, F, W)$ by the following procedure:

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen arrow $[\beta\alpha] : u \to w$.

(2) Replace each arrow $\alpha$ incident with $v$ by an arrow $\alpha^*$ going in the opposite direction. This gives a new ice quiver $(\widetilde{\mu}_v(Q), F)$.

(3) Choose a representative for $W$ in $\widehat{kQ}$ such that no term begins at $v$. For each pair of arrows as in (1), replace each occurrence of $\beta\alpha$ by the newly introduced arrow $[\beta\alpha]$, and add the term $[\beta\alpha]\alpha^*\beta^*$. This gives a new potential $\widetilde{\mu}_v(W)$.

(4) Set $\widetilde{\mu}_v(Q, F, W) = (\widetilde{\mu}_v(Q), F, \widetilde{\mu}_v(W))$.

The *mutation* $\mu_v(Q, F, W) = (\mu_v(Q), \mu_v(F), \mu_v(W))$ of $(Q, F, W)$ is defined to be the reduction of $\widetilde{\mu}_v(Q, F, W)$. We say that $(Q, F, W)$ is *non-degenerate* if any iterated sequence of mutations of $(Q, F, W)$ does not produce 2-cycles containing unfrozen arrows in the resulting ice quiver. By [Pre20, Proposition 4.10], this holds for example if $(Q, F, W)$ is reduced and *rigid*, that is, any cycle in $Q$ containing an unfrozen arrow is zero in $J(Q, F, W)$ up to cyclic equivalence (see also Definition 4.8 in loc. cit.).

**Remark 3.3.1.** If $(\overline{Q}, \overline{W})$ denotes the quiver with potential obtained from $(Q, F, W)$ by deleting the frozen vertices, then $(\overline{Q}, \overline{W})$ is non-degenerate and $\mu_v(\overline{Q}, \overline{W}) = (\overline{\mu_v(Q)}, \overline{\mu_v(W)})$ for any unfrozen vertex $v$.

**Remark 3.3.2.** By [Pre20, Proposition 4.6], if $(\mu_v(Q), \mu_v(F))$ does not have 2-cycles containing unfrozen arrows, then it coincides with the *extended Fomin–Zelevinsky mutation* $\mu_v^{\mathrm{FZ}}(Q, F)$ of the ice quiver $(Q, F)$, which is constructed by the following procedure:

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen arrow $[\beta\alpha] : u \to w$.

(2) Replace each arrow $\alpha$ incident with $v$ by an arrow $\alpha^*$ going in the opposite direction.

(3) Remove a maximal collection of 2-cycles involving only unfrozen arrows.

(4) Choose a maximal collection of half-frozen 2-cycles, that is, involving exactly one frozen arrow. Replace each 2-cycle in this collection by a frozen arrow pointing in the same direction as the unfrozen arrow of the 2-cycle.

The resulting quiver is defined up to isomorphism. If we disregard the frozen arrows, then the procedure above gives the classical mutation rule (as defined in Section 3.2 in terms of extended exchange matrices).

### 3.3.2 Relative Ginzburg dg algebras

Let $(Q, F, W)$ be a finite ice quiver with potential. We define $\widetilde{Q}$ to be the graded quiver whose vertices are the same as those of $Q$ and whose arrows are

- a copy of each arrow of $Q$, with degree 0;
- an arrow $\alpha^* : j \to i$ of degree $-1$ for every *unfrozen* arrow $\alpha : i \to j$ of $Q$;
- a loop $t_i : i \to i$ of degree $-2$ for every *unfrozen* vertex $i$ of $Q$.

We define the *completed relative Ginzburg dg algebra* $\widehat{\mathbf{\Gamma}}(Q, F, W)$ as follows. Its underlying graded algebra is the completion of the graded path algebra $k\widetilde{Q}$ with respect to the ideal generated by the arrows. The completion is taken in the category of graded algebras, so that the $n$-th component of $\widehat{\mathbf{\Gamma}}(Q, F, W)$ has as a topological basis the set of all paths in $\widetilde{Q}$ of degree $n$. The differential is the unique continuous linear endomorphism of degree 1 which satisfies $d(uv) = du \cdot v + (-1)^p u \cdot dv$ for all homogeneous elements $u$ of degree $p$ and all $v$, and takes the following values on the arrows of $\widetilde{Q}$:

- $d\alpha = 0$ for every $\alpha \in Q_1$;
- $d\alpha^* = \partial_\alpha(W)$ for every $\alpha \in Q_1 \setminus F_1$;
- $dt_i = e_i \left(\sum_{\alpha \in Q_1} [\alpha, \alpha^*]\right) e_i$ for every $i \in Q_0 \setminus F_0$, where $e_i$ denotes the lazy path of length zero at vertex $i$.

Note that $J(Q, F, W)$ is isomorphic to $H^0(\widehat{\mathbf{\Gamma}}(Q, F, W))$.

When $W$ involves finitely many cycles, we also define the *non-completed relative Ginzburg dg algebra* $\mathbf{\Gamma}(Q, F, W)$ as the dg subalgebra of $\widehat{\mathbf{\Gamma}}(Q, F, W)$ whose underlying algebra is the non-completed path algebra $k\widetilde{Q}$. We have the following result linking the two of them.

**Lemma 3.3.3.** *Let $(Q, F, W)$ be an ice quiver with potential that is Jacobi-finite and such that $W$ involves finitely many cycles. Suppose that we can give a non-negative Adams grading to the arrows of $\widetilde{Q}$ such that the differential $d$ defined above is homogeneous of Adams degree zero and no cycle in $\widetilde{Q}$ has Adams degree zero. Then the canonical morphism*

$$\mathbf{\Gamma}(Q, F, W) \longrightarrow \widehat{\mathbf{\Gamma}}(Q, F, W)$$

*is a quasi-isomorphism.*

*Proof.* We start with the following observation. Since $(Q, F, W)$ is Jacobi-finite and $\mathbf{\Gamma}(Q, F, W)$ is concentrated in non-negative degrees and smooth (by [Yeu22, Corollary 5.4(4)]), we can apply [KY16, Proposition 2.5] to conclude that all the cohomologies of $\mathbf{\Gamma}(Q, F, W)$ are finite-dimensional. Now, both $\widehat{\mathbf{\Gamma}}(Q, F, W)$ and $\mathbf{\Gamma}(Q, F, W)$ can be equipped with an Adams grading on top of the cohomological one using the extra grading on $\widetilde{Q}$. By our hypothesis, there are only finitely many paths on $\widetilde{Q}$ of a given Adams degree, so the component $(k\widetilde{Q})^{r,p}$ of $k\widetilde{Q}$ of cohomological degree $r$ and Adams degree $p$ has finite dimension. This allows us to write the $r$-th cohomological components of $\widehat{\mathbf{\Gamma}}(Q, F, W)$ and $\mathbf{\Gamma}(Q, F, W)$ as

$$\widehat{\mathbf{\Gamma}}(Q, F, W)^r = \prod_{p\geq 0} (k\widetilde{Q})^{r,p} \quad \text{and} \quad \mathbf{\Gamma}(Q, F, W)^r = \bigoplus_{p\geq 0} (k\widetilde{Q})^{r,p}.$$

Since $d$ is homogeneous of Adams degree zero, we may restrict it to a map $d^{r,p}$ from $(k\widetilde{Q})^{r,p}$ to $(k\widetilde{Q})^{r+1,p}$. If we denote the restriction of $d$ to $\mathbf{\Gamma}(Q, F, W)$ as $d_{\mathbf{\Gamma}}$, we deduce that

$$\ker d^r = \prod_{p\geq 0} \ker d^{r,p} \quad \text{and} \quad \ker d^r_{\mathbf{\Gamma}} = \bigoplus_{p\geq 0} \ker d^{r,p}.$$

But $H^r(\mathbf{\Gamma}(Q, F, W))$ is finite-dimensional, so there is $p_r \geq 0$ such that the image of $d_{\mathbf{\Gamma}}$ contains $\ker d^{r,p}$ for all $p \geq p_r$. By the continuity of $d$, we also deduce that the image of $d$ contains $\prod_{p\geq p_r} \ker d^{r,p}$. We conclude that both $H^r(\mathbf{\Gamma}(Q, F, W))$ and $H^r(\widehat{\mathbf{\Gamma}}(Q, F, W))$ are given by the finite direct sum

$$\bigoplus_{0\leq p<p_r} \frac{\ker d^{r,p}}{\imath d^{r,p}}.$$

This gives the desired quasi-isomorphism. $\surd$

**Remark 3.3.4.** Suppose we can give a strictly positive grading to the arrows of $Q$ such that the potential $W$ is homogeneous of degree $n$ and all arrows have degree strictly less than $n$. Then we can extend it to an Adams grading on the arrows of $\widetilde{Q}$ as follows. For $\alpha \in Q_1$, we set the Adams degree of $\alpha^*$ as $n - |\alpha|$, and we set each $t_i$ to be of degree $n$. Each arrow in $\widetilde{Q}$ then has positive Adams degree, and the conditions in the lemma above are satisfied. For example, if all cycles in $W$ have the same length, we can set each arrow of $Q$ to have degree 1 and then the resulting Adams grading satisfies the required conditions.

### 3.3.3 Relative cluster categories and Higgs categories

In this section, we fix a Jacobi-finite ice quiver with potential $(Q, F, W)$ and recall some of the constructions and results of [Wu23b]. We remark that they can be generalized to the Jacobi-infinite setting [KW23b], but we will not need this level of generality.

**Remark 3.3.5.** Our hypothesis that $(Q, F, W)$ is Jacobi-finite ensures that the technical condition [KW23b, Assumption 1] holds. This guarantees that the next results and the results in Section 3.3.4 are true without extra assumptions.

Let $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}(Q, F, W)$. Denote by $\mathcal{D}(\widehat{\mathbf{\Gamma}})$ its derived category and by $\Sigma$ the suspension functor on $\mathcal{D}(\widehat{\mathbf{\Gamma}})$. We denote by $\mathrm{per}(\widehat{\mathbf{\Gamma}})$ the *perfect derived category* of $\widehat{\mathbf{\Gamma}}$, that is, the thick subcategory of $\mathcal{D}(\widehat{\mathbf{\Gamma}})$ generated by $\widehat{\mathbf{\Gamma}}$. We let $\mathrm{pvd}(\widehat{\mathbf{\Gamma}})$ be the *perfectly valued derived category* of $\widehat{\mathbf{\Gamma}}$, that is, the full subcategory of $\mathcal{D}(\widehat{\mathbf{\Gamma}})$ whose objects are the dg $\widehat{\mathbf{\Gamma}}$-modules with finite-dimensional total cohomology. By the proof of [KY11, Theorem 2.19(a)], $\mathrm{pvd}(\widehat{\mathbf{\Gamma}})$ is a thick subcategory of $\mathrm{per}(\widehat{\mathbf{\Gamma}})$ and is generated by all simple $H^0(\widehat{\mathbf{\Gamma}})$-modules. Note that there is one simple $H^0(\widehat{\mathbf{\Gamma}})$-module for each vertex of $Q$.

The *relative cluster category* $\mathcal{C}(Q, F, W)$ is defined as the Verdier quotient of $\mathrm{per}(\widehat{\mathbf{\Gamma}})$ by the thick subcategory generated by all simple $H^0(\widehat{\mathbf{\Gamma}})$-modules associated with *unfrozen* vertices. It is a Hom-finite, idempotent complete, and Krull–Schmidt triangulated category by [KY11, Lemma 2.17] and [Wu23b, Section 4.4]. The *Higgs category* $\mathcal{H}(Q, F, W)$ is its full subcategory whose objects are the $X \in \mathcal{C}(Q, F, W)$ such that

$$\mathrm{Ext}^n_{\mathcal{C}(Q,F,W)}(X, e_i\widehat{\mathbf{\Gamma}}) = \mathrm{Ext}^n_{\mathcal{C}(Q,F,W)}(e_i\widehat{\mathbf{\Gamma}}, X) = 0$$

for all $n > 0$ and $i \in F_0$. We remark that $\widehat{\mathbf{\Gamma}} \in \mathcal{H}(Q, F, W)$. The Higgs category is closed under extensions in $\mathcal{C}(Q, F, W)$ and hence canonically inherits from it the structure of an extriangulated category [NP19] equipped with a bivariant $\delta$-functor (as in [GNP21, Example 4.9(2)]). In particular, we have positive and negative (!) extension groups between objects in $\mathcal{H}(Q, F, W)$, which are computed as extension groups in $\mathcal{C}(Q, F, W)$. Moreover, $\mathcal{H}(Q, F, W)$ is a stably 2-Calabi–Yau Frobenius extriangulated category whose subcategory $\mathcal{P}$ of projective-injective objects is the additive subcategory generated by the objects of the form $e_i\widehat{\mathbf{\Gamma}}$ for a frozen vertex $i \in F_0$. This implies, in particular, that there is a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{H}(Q,F,W)}(M, N) \cong D\mathrm{Ext}^1_{\mathcal{H}(Q,F,W)}(N, M)$$

for $M, N \in \mathcal{H}(Q, F, W)$, where $D$ denotes duality over $k$.

**Proposition 3.3.6** ([Che24, Lemma 3.30])**.** *For $i, j \in Q_0$ and $n \leq 1$, the canonical map*

$$\mathrm{Ext}^n_{\widehat{\mathbf{\Gamma}}}(e_i\widehat{\mathbf{\Gamma}}, e_j\widehat{\mathbf{\Gamma}}) \longrightarrow \mathrm{Ext}^n_{\mathcal{C}(Q,F,W)}(e_i\widehat{\mathbf{\Gamma}}, e_j\widehat{\mathbf{\Gamma}}) = \mathrm{Ext}^n_{\mathcal{H}(Q,F,W)}(e_i\widehat{\mathbf{\Gamma}}, e_j\widehat{\mathbf{\Gamma}})$$

*induced by the quotient functor* $\mathrm{per}(\widehat{\mathbf{\Gamma}}) \to \mathcal{C}(Q, F, W)$ *is an isomorphism.*

Let $(\overline{Q}, \overline{W})$ be the quiver with potential obtained from $(Q, F, W)$ by deleting the frozen vertices. We have a dg quotient morphism

$$p : \widehat{\mathbf{\Gamma}}(Q, F, W) \longrightarrow \widehat{\mathbf{\Gamma}}(\overline{Q}, \overline{W})$$

and, in particular, $(\overline{Q}, \overline{W})$ is again Jacobi-finite by [Wu23b, Proposition 4.5]. The *(absolute) cluster category* $\mathcal{C}(\overline{Q}, \overline{W})$ is defined as above if we regard $(\overline{Q}, \overline{W})$ as an ice quiver with potential with empty frozen part, and it agrees with Amiot's cluster category [Ami09]. The morphism $p$ above induces a triangulated quotient functor

$$p^* : \mathcal{C}(Q, F, W) \longrightarrow \mathcal{C}(\overline{Q}, \overline{W}),$$

which in turn yields an equivalence of triangulated categories

$$\mathcal{H}(Q, F, W)/[\mathcal{P}] \xrightarrow{\sim} \mathcal{C}(\overline{Q}, \overline{W}) \tag{3.4}$$

by [KW23b, Proposition 4.17]. As a consequence, we have the following result.

**Proposition 3.3.7** ([Wu23b, Proposition 5.36])**.** *For $X, Y \in \mathcal{H}(Q, F, W)$ and $n \geq 1$, the canonical map*

$$\mathrm{Ext}^n_{\mathcal{H}(Q,F,W)}(X, Y) \longrightarrow \mathrm{Ext}^n_{\mathcal{C}(\overline{Q},\overline{W})}(p^*(X), p^*(Y))$$

*is an isomorphism.*

We conclude this subsection with the effect of the mutation of ice quivers with potential in the associated categories. Suppose $(Q, F, W)$ is reduced and let $v \in Q_0 \setminus F_0$ be an unfrozen vertex that is not incident to any loop or 2-cycle. We denote $\widehat{\mathbf{\Gamma}}' = \widehat{\mathbf{\Gamma}}(Q', F', W')$, where $\mu_v(Q, F, W) = (Q', F', W')$. We remark that $(Q', F', W')$ is again Jacobi-finite. By [Wu23a, Theorem 1.1(b)], there are two triangle equivalences $\Phi_{v,\pm} : \mathcal{D}(\widehat{\mathbf{\Gamma}}') \to \mathcal{D}(\widehat{\mathbf{\Gamma}})$ that restrict to equivalences $\mathrm{per}(\widehat{\mathbf{\Gamma}}') \to \mathrm{per}(\widehat{\mathbf{\Gamma}})$ and $\mathrm{pvd}(\widehat{\mathbf{\Gamma}}') \to \mathrm{pvd}(\widehat{\mathbf{\Gamma}})$. By [KW23b, Proposition 3.29], they both descend to equivalences between the relative cluster categories and, on this level, they become naturally isomorphic functors. We denote this common equivalence by $\Phi_v : \mathcal{C}(Q, F, W) \to \mathcal{C}(Q', F', W')$. In turn, $\Phi_v$ restricts to an equivalence between the corresponding Higgs categories and preserves the classes of projective-injective objects. Consequently, we also have an induced triangle equivalence $\mathcal{C}(\overline{Q}, \overline{W}) \to \mathcal{C}(\overline{Q'}, \overline{W'})$, which coincides with the one defined in [KY11].

For the next result, recall that a dg algebra is *proper* if its total cohomology is finite-dimensional.

**Lemma 3.3.8.** *The completed relative Ginzburg dg algebra $\widehat{\mathbf{\Gamma}}$ is proper if and only if its mutation $\widehat{\mathbf{\Gamma}}'$ is proper.*

*Proof.* This follows from the discussion above since a completed relative Ginzburg dg algebra is proper if and only if its perfect and perfectly valued derived categories coincide. $\surd$

### 3.3.4 The categorification

For this section, the base field is $k = \mathbb{C}$. Let $(Q, F, W)$ be an ice quiver with potential. We assume that it is Jacobi-finite and non-degenerate. We also require that $Q$ has no loops and no 2-cycles containing unfrozen arrows, and we relabel the vertices so that $Q_0 = \{1, \dots, m\}$ and $F_0 = \{n+1, \dots, m\}$ for $1 \leq n \leq m$. Let $\mathbb{F}$ be the field of rational functions in $m$ variables over $\mathbb{Q}$ and let $\mathcal{A}_{Q,F} \subseteq \mathbb{F}$ be cluster algebra associated with $(Q, F)$ (see Section 3.2).

In [KW23b], building on previous works such as [Pal08], [FK10], and [Pla11b], the authors construct a canonical *cluster character* $CC : \mathrm{Ob}(\mathcal{H}) \longrightarrow \mathbb{F}$ on the Higgs category $\mathcal{H} = \mathcal{H}(Q, F, W)$. This means that

(1) $CC$ is constant on isomorphism classes;

(2) $CC(L \oplus M) = CC(L) \cdot CC(M)$ for $L, M \in \mathcal{H}$;

(3) if $L, M \in \mathcal{H}$ are such that $\mathrm{Ext}^1_{\mathcal{H}}(L, M)$ is one-dimensional (hence $\mathrm{Ext}^1_{\mathcal{H}}(M, L)$ is also one-dimensional), and

$$L \rightarrowtail E \twoheadrightarrow M \dashrightarrow \quad \text{and} \quad M \rightarrowtail E' \twoheadrightarrow L \dashrightarrow$$

are non-split conflations in $\mathcal{H}$, then $CC(L) \cdot CC(M) = CC(E) + CC(E')$.

We say that $M \in \mathcal{H}$ is *rigid* if $\mathrm{Ext}^1_{\mathcal{H}}(M, M) = 0$. It is *reachable* if it is a direct summand of a direct sum of copies of the image of $\widehat{\mathbf{\Gamma}}(Q', F', W')$ under the equivalence

$$\Phi_{v_1} \Phi_{v_2} \cdots \Phi_{v_r} : \mathcal{H}(Q', F', W') \longrightarrow \mathcal{H}$$

induced by a sequence of mutations at unfrozen vertices $v_1, \dots, v_r$. We call $T \in \mathcal{H}$ a *cluster-tilting object* if it is rigid and, if $L \in \mathcal{H}$ satisfies $\mathrm{Ext}^1_{\mathcal{H}}(L, T) = 0$, then $L \in \mathrm{add}\, T$, where $\operatorname{add} T \subseteq \mathcal{H}$ denotes the smallest full subcategory containing $T$ and closed under direct sums and summands. If $T$ is also *basic*, that is, its indecomposable direct summands are pairwise non-isomorphic, then it is isomorphic to the image of $\widehat{\mathbf{\Gamma}}(Q', F', W')$ under an equivalence as above.

**Theorem 3.3.9** ([KW23b, Theorem 5.9]). *The canonical cluster character* $CC : \mathrm{Ob}(\mathcal{H}) \to \mathbb{F}$ *induces a bijection between the sets of isomorphism classes of reachable rigid indecomposable objects in* $\mathcal{H}$ *and cluster variables in* $\mathcal{A}_{Q,F}$*. Under this bijection, basic reachable cluster-tilting objects correspond to clusters.*

Let $T \in \mathcal{H}$ be a reachable cluster-tilting object. Its corresponding cluster is associated with a vertex $t \in \mathbb{T}_n$ in the $n$-regular tree. By considering the sequence of mutations from the initial root to $t$, we can also attach to it an ice quiver with potential $(Q_t, F_t, W_t)$. The split Grothendieck group $K_0(\operatorname{add} T)$ is then isomorphic to the Grothendieck group of $\operatorname{per}(\widehat{\mathbf{\Gamma}}(Q_t, F_t, W_t))$, which we identify with $\mathbb{Z}^m$ by sending $e_i \widehat{\mathbf{\Gamma}}(Q_t, F_t, W_t)$ to the $i$-th vector of the canonical basis.

Now take $M \in \mathcal{H}$. By the construction of the Higgs category in [KW23b], there is a conflation

$$T_1 \rightarrowtail T_0 \twoheadrightarrow M \dashrightarrow$$

where $T_0, T_1 \in \operatorname{add} T$. By [GP25, Proposition 3.2] (see also the references mentioned therein), the element $[T_0] - [T_1] \in K_0(\operatorname{add} T)$ does not depend on the chosen conflation. Via the identification above, we can then attach a vector $\operatorname{ind}_T(M) \in \mathbb{Z}^m$ that depends only on $T$ and $M$, called the *index* of $M$ with respect to $T$. It is a categorical version of the $g$-vector by the following result.

**Theorem 3.3.10** ([GP25, Theorem 4.39]). *Let* $T \in \mathcal{H}$ *be a reachable cluster-tilting object corresponding to* $t \in \mathbb{T}_n$*. If* $M \in \mathcal{H}$ *is a reachable rigid object, then* $\operatorname{ind}_T(M)$ *is the* $g$*-vector* $\mathbf{g}_u^t$ *of the cluster monomial* $u = CC(M) \in \mathcal{A}_{Q,F}$ *with respect to* $t$*.*

## 3.4 An additive Λ-invariant

We start this section by attaching a compatible pair to any finite ice quiver whenever the so-called Euler matrix is invertible. For an ice quiver with potential $(Q, F, W)$, we prove the Euler matrix is invertible whenever $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper, and give a homological interpretation for the corresponding Poisson coefficient matrix using the Higgs category. Our main result is Theorem 3.4.6, which gives a $\Lambda$-cluster algebra structure for $\mathcal{A}_{Q,F}$ and generalizes previous results by [GLS13] and [GP25]. We then interpret the associated tropical invariant categorically. Our formula should be thought of as an "additive" Λ-invariant, since it will be used in Theorem 3.9.2 to recover the monoidal Λ-invariant in certain cases.

### 3.4.1 The compatible pair associated with an ice quiver

Let $(Q, F)$ be a finite ice quiver. We label its vertices so that $Q_0 = \{1, \dots, m\}$ and $F_0 = \{n+1, \dots, m\}$ for $1 \leq n \leq m$. We define the *Euler matrix* of $(Q, F)$ to be the $m \times m$ integer matrix $\widehat{B} = (b_{ij})$ given by

$$b_{ii} = \begin{cases} 0 & \text{if } 1 \leq i \leq n, \\ 1 & \text{otherwise,} \end{cases}$$

and, for $i \neq j$,

$$b_{ij} = \#\{i \to j \text{ in } Q_1 \setminus F_1\} - \#\{j \to i \text{ in } Q_1\}.$$

In the next section, we give an alternative description of this matrix using the Euler form of the perfectly valued derived category of a relative Ginzburg dg algebra.

Note that $b_{ij} = -b_{ji}$ if $i$ or $j$ is unfrozen, but $\widehat{B}$ is not skew-symmetric if $n < m$. Moreover, the submatrix of $\widehat{B}$ defined as

$$\widetilde{B} \coloneqq \text{ first } n \text{ columns of } \widehat{B} \tag{3.5}$$

is the extended exchange matrix associated with $(Q, F_0)$ as in Remark 3.2.1. If $\det \widehat{B} \neq 0$, we define

$$\Lambda \coloneqq |\det \widehat{B}| \cdot (\widehat{B}^{-T} - \widehat{B}^{-1}). \tag{3.6}$$

This is an $m \times m$ skew-symmetric integer matrix.

**Proposition 3.4.1.** *Assume* $\det \widehat{B} \neq 0$. *The pair* $(\widetilde{B}, \Lambda)$ *defined in* (3.5) – (3.6) *is compatible. Its type is* $2|\det \widehat{B}| \cdot I_n$, *where* $I_n$ *is the* $n \times n$ *identity matrix.*

*Proof.* We have $\widetilde{B}^T \widehat{B}^{-T} = (\widehat{B}^{-1}\widetilde{B})^T = (I_n \mid \mathbf{0})$ because $\widetilde{B}$ consists of the first $n$ columns of $\widehat{B}$. Moreover, since $b_{ij} = -b_{ji}$ if $i$ or $j$ is not frozen, the submatrix of $\widehat{B}$ consisting of the first $n$ rows is $-\widetilde{B}^T$, so $\widetilde{B}^T \widehat{B}^{-1} = (-I_n \mid \mathbf{0})$. We conclude that $\widetilde{B}^T \Lambda = 2|\det \widehat{B}| \cdot (I_n \mid \mathbf{0})$, as desired. $\surd$

We refer to $(\widetilde{B}, \Lambda)$ as *the compatible pair associated with the ice quiver* $(Q, F)$, which is defined whenever the Euler matrix is invertible. Let us study its behavior under mutations.

Let $1 \leq v \leq n$ be an unfrozen vertex not incident to loops or 2-cycles. Let $E_v = (e_{ij})$ be the $m \times m$ integer matrix given by

$$e_{ij} \coloneqq \begin{cases} \delta_{ij} & \text{if } j \neq v, \\ [b_{iv}]_+ & \text{if } i \neq v \text{ and } j = v, \\ -1 & \text{if } i = j = v. \end{cases}$$

We remark that $E_v$ squares to the identity and the mutation rule for Poisson coefficient matrices can be written as $\mu_v(\Lambda) = E_v^T \Lambda E_v$. Indeed, $E_v$ is the matrix denoted by $E_-$ in [BZ05].

**Lemma 3.4.2.** *Let $\mu_v^{\mathrm{FZ}}(Q,F)$ be the extended Fomin–Zelevinsky mutation of $(Q,F)$ and denote by $\mu_v(\widehat{B})$ its Euler matrix. Then $\mu_v(\widehat{B}) = E_v \widehat{B} E_v^T$.*

*Proof.* Since $b_{vv} = 0$, the entries of the product above are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = v \text{ or } j = v, \\ b_{ij} + b_{iv}[b_{jv}]_+ + [b_{iv}]_+ b_{vj} & \text{otherwise.} \end{cases}$$

Using that $b_{jv} = -b_{vj}$ and dividing in cases depending on the signs of $b_{iv}$ and $b_{vj}$, one shows that the second line above is equal to

$$b_{ij} + [b_{iv}]_+[b_{vj}]_+ - [-b_{iv}]_+[-b_{vj}]_+.$$

If $i = v$ or $j = v$, then it is clear that $b'_{ij}$ is also the entry $(i,j)$ of $\mu_v(\widehat{B})$, since the step (2) in the extended Fomin–Zelevinsky mutation reverses arrows incident to $v$ (which are all unfrozen) and the other steps do not influence this entry. If $i$ and $j$ are different from $v$, at most one of the terms $[b_{iv}]_+[b_{vj}]_+$ and $[-b_{iv}]_+[-b_{vj}]_+$ in the expression above is nonzero. Since $v$ is not incident to 2-cycles, they count the number of arrows added between $i$ and $j$ at step (2) of the mutation. Therefore, it is clear that $E_v \widehat{B} E_v^T$ is the Euler matrix of the ice quiver $(\widetilde{\mu}_v(Q), F)$. One can easily check that the steps (3) and (4) of the extended Fomin–Zelevinsky mutation do not change the Euler matrix, so we conclude the proof. $\surd$

**Proposition 3.4.3.** *Assume $\det \widehat{B} \neq 0$ and let $(\widetilde{B}, \Lambda)$ be the compatible pair associated with the finite ice quiver $(Q,F)$. If $v \in Q_0 \setminus F_0$ is an unfrozen vertex not incident to loops or 2-cycles, then $(\mu_v(\widetilde{B}), \mu_v(\Lambda))$ is the compatible pair associated with $\mu_v^{\mathrm{FZ}}(Q,F)$.*

*Proof.* By Lemma 3.4.2, we have that $\mu_v(\widehat{B}) = E_v \widehat{B} E_v^T$ is the Euler matrix of $\mu_v^{\mathrm{FZ}}(Q,F)$. In particular, $\mu_v(\widehat{B})$ is invertible if $\widehat{B}$ is invertible, and they have the same determinant. We have

$$\begin{aligned} \mu_v(\Lambda) = E_v^T \Lambda E_v &= |\det \widehat{B}| \cdot (E_v^T \widehat{B}^{-T} E_v - E_v^T \widehat{B}^{-1} E_v) \\ &= |\det \widehat{B}| \cdot ((E_v \widehat{B} E_v^T)^{-T} - (E_v \widehat{B} E_v^T)^{-1}) \\ &= |\det \mu_v(\widehat{B})| \cdot (\mu_v(\widehat{B})^{-T} - \mu_v(\widehat{B})^{-1}), \end{aligned}$$

where we used that $E_v^{-1} = E_v$ in the third equality. This concludes the proof. $\surd$

### 3.4.2 Homological interpretation via the Euler form

Let $(Q, F, W)$ be an ice quiver with potential. As before, we label the vertices as $Q_0 = \{1, \ldots, m\}$ and $F_0 = \{n+1, \ldots, m\}$ for $1 \leq n \leq m$. Denote $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}(Q, F, W)$. For $P \in \operatorname{per}(\widehat{\mathbf{\Gamma}})$ and $M \in \operatorname{pvd}(\widehat{\mathbf{\Gamma}})$, the complex $\mathbf{R}\mathrm{Hom}_{\widehat{\mathbf{\Gamma}}}(P, M)$ has finite-dimensional total cohomology. Therefore, if $K_0(\operatorname{per}(\widehat{\mathbf{\Gamma}}))$ and $K_0(\operatorname{pvd}(\widehat{\mathbf{\Gamma}}))$ denote the Grothendieck groups of the corresponding triangulated categories, we have a pairing

$$\langle -, - \rangle : K_0(\operatorname{per}(\widehat{\mathbf{\Gamma}})) \times K_0(\operatorname{pvd}(\widehat{\mathbf{\Gamma}})) \longrightarrow \mathbb{Z}$$

defined by

$$\langle [P], [M] \rangle \coloneqq \sum_{p \in \mathbb{Z}} (-1)^p \dim H^p(\mathbf{R}\mathrm{Hom}_{\widehat{\mathbf{\Gamma}}}(P, M)) = \sum_{p \in \mathbb{Z}} (-1)^p \dim \mathrm{Ext}^p_{\widehat{\mathbf{\Gamma}}}(P, M).$$

It is called the *Euler form* of $\widehat{\mathbf{\Gamma}}$. For $i \in Q_0$, let $S_i \in \operatorname{pvd}(\widehat{\mathbf{\Gamma}}) \subseteq \operatorname{per}(\widehat{\mathbf{\Gamma}})$ denote the simple $H^0(\widehat{\mathbf{\Gamma}})$-module associated with $i$. By [KY11, Lemma 2.15], the dimension of $\mathrm{Ext}^p_{\widehat{\mathbf{\Gamma}}}(S_i, S_j)$ is the number $a^p_{ij}$ of arrows from $j$ to $i$ of degree $-p+1$ in the quiver $\widetilde{Q}$ in the definition of $\widehat{\mathbf{\Gamma}}$ (or $a^p_{ij} + 1$ if $p = 0$ and $i = j$). Consequently, if $\widehat{B} = (b_{ij})$ is the Euler matrix of $(Q, F)$, then one can easily check that $b_{ij} = \langle [S_i], [S_j] \rangle$.

Suppose $\widehat{\mathbf{\Gamma}}$ is proper. In this case, the inclusion $\operatorname{pvd}(\widehat{\mathbf{\Gamma}}) \subseteq \operatorname{per}(\widehat{\mathbf{\Gamma}})$ becomes an equality, and we can compute the Euler form between a pair of perfect dg $\widehat{\mathbf{\Gamma}}$-modules. This allows us to compute the inverse of the Euler matrix.

**Lemma 3.4.4.** *If $\widehat{\mathbf{\Gamma}}$ is proper, then the associated Euler matrix $\widehat{B}$ is invertible over $\mathbb{Z}$. In this case, the entry $(i, j)$ of $\widehat{B}^{-1}$ is given by $\langle [e_j \widehat{\mathbf{\Gamma}}], [e_i \widehat{\mathbf{\Gamma}}] \rangle$ for any $1 \leq i, j \leq m$.*

*Proof.* Since the objects in $\operatorname{pvd}(\widehat{\mathbf{\Gamma}})$ have bounded cohomology, it is not hard to show that $K_0(\operatorname{pvd}(\widehat{\mathbf{\Gamma}}))$ is isomorphic to the Grothendieck group of the category of finite-dimensional $H^0(\widehat{\mathbf{\Gamma}})$-modules. Since we are in the completed setting, this abelian group is free of rank $m$ and has as a basis the classes $[S_i]$ of the simple $H^0(\widehat{\mathbf{\Gamma}})$-modules. On the other hand, one can use [Pla11b, Lemma 2.14] to show that $K_0(\operatorname{per}(\widehat{\mathbf{\Gamma}}))$ is also free of rank $m$ and has as a basis the classes $[e_i \widehat{\mathbf{\Gamma}}]$ of the indecomposable projective dg $\widehat{\mathbf{\Gamma}}$-modules. Under the hypothesis that $\widehat{\mathbf{\Gamma}}$ is proper, these two Grothendieck groups coincide. Using that $\langle e_i \widehat{\mathbf{\Gamma}}, S_j \rangle = \delta_{ij}$, we see that the transition matrix from the first basis to the second one is $(\langle [e_i \widehat{\mathbf{\Gamma}}], [e_j \widehat{\mathbf{\Gamma}}] \rangle)_{ij}$, while the transition matrix in the other direction is $\widehat{B}^T$. The lemma then follows. $\surd$

Lemma 3.4.4 combined with Proposition 3.4.1 implies that the compatible pair $(\widetilde{B}, \Lambda)$ of $(Q, F)$, constructed in (3.5) – (3.6), exists when $\widehat{\mathbf{\Gamma}}$ is proper. Moreover, if $\Lambda = (\lambda_{ij})$, then

$$\lambda_{ij} = \langle [e_i \widehat{\mathbf{\Gamma}}], [e_j \widehat{\mathbf{\Gamma}}] \rangle - \langle [e_j \widehat{\mathbf{\Gamma}}], [e_i \widehat{\mathbf{\Gamma}}] \rangle$$

for all $1 \leq i, j \leq m$. We will now interpret this formula using the Higgs category $\mathcal{H} = \mathcal{H}(Q, F, W)$. Note that $(Q, F, W)$ is Jacobi-finite if we assume $\widehat{\boldsymbol{\Gamma}}$ proper. We define

$$[M, N]_{\mathcal{H}} := \sum_{p \geq 0} (-1)^p (\dim \operatorname{Ext}_{\mathcal{H}}^{-p}(M, N) - \dim \operatorname{Ext}_{\mathcal{H}}^{-p}(N, M)). \tag{3.7}$$

for $M, N \in \mathcal{H}$. The extension groups above are finite-dimensional since the relative cluster category is Hom-finite.

**Lemma 3.4.5.** *If $\widehat{\boldsymbol{\Gamma}}$ is proper, then the sum $[M, N]_{\mathcal{H}}$ is finite for all $M, N \in \mathcal{H}$. Moreover, we have*

$$[e_i\widehat{\boldsymbol{\Gamma}}, e_j\widehat{\boldsymbol{\Gamma}}]_{\mathcal{H}} = \langle [e_i\widehat{\boldsymbol{\Gamma}}], [e_j\widehat{\boldsymbol{\Gamma}}] \rangle - \langle [e_j\widehat{\boldsymbol{\Gamma}}], [e_i\widehat{\boldsymbol{\Gamma}}] \rangle$$

*for $1 \leq i, j \leq m$.*

*Proof.* The second statement is an immediate consequence of Proposition 3.3.6 and the fact that $\widehat{\boldsymbol{\Gamma}}$ is connective, that is, concentrated in non-negative degrees. Let us prove the first statement. Assume first that $M \in \operatorname{add} \widehat{\boldsymbol{\Gamma}}$. By taking a conflation

$$T_1 \rightarrowtail T_0 \twoheadrightarrow N \dashrightarrow \tag{3.8}$$

with $T_0, T_1 \in \operatorname{add} \widehat{\boldsymbol{\Gamma}}$ and applying the cohomological functor $\operatorname{Hom}_{\mathcal{C}(Q,F,W)}(M, -)$, we get a long exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Ext}_{\mathcal{H}}^{-1}(M, T_1) &\longrightarrow \operatorname{Ext}_{\mathcal{H}}^{-1}(M, T_0) \longrightarrow \operatorname{Ext}_{\mathcal{H}}^{-1}(M, N) \\ \longrightarrow \operatorname{Hom}_{\mathcal{H}}(M, T_1) &\longrightarrow \operatorname{Hom}_{\mathcal{H}}(M, T_0) \longrightarrow \operatorname{Hom}_{\mathcal{H}}(M, N). \end{aligned}$$

By Proposition 3.3.6 and because $M \in \operatorname{add} \widehat{\boldsymbol{\Gamma}}$, the negative extensions between $M$ and the $T_i$ can be computed in $\operatorname{per}(\widehat{\boldsymbol{\Gamma}})$. Since $\operatorname{Ext}_{\widehat{\boldsymbol{\Gamma}}}^{-p}(\widehat{\boldsymbol{\Gamma}}, \widehat{\boldsymbol{\Gamma}}) \cong H^{-p}(\widehat{\boldsymbol{\Gamma}})$ and $\widehat{\boldsymbol{\Gamma}}$ is proper, the terms $\operatorname{Ext}_{\mathcal{H}}^{-p}(M, T_i)$ in the long exact sequence vanish for $p$ large enough. We conclude that the same holds for $\operatorname{Ext}_{\mathcal{H}}^{-p}(M, N)$. A similar argument shows that $\operatorname{Ext}_{\mathcal{H}}^{-p}(N, M)$ also vanishes for $p$ large enough, so we deduce that $[M, N]_{\mathcal{H}}$ is a finite sum. Finally, the general case can be similarly reduced to the case where $M \in \operatorname{add} \widehat{\boldsymbol{\Gamma}}$ by taking a two-term resolution of $M$ as in (3.8). $\surd$

Let $(Q, F, W)$ be a non-degenerate ice quiver with potential. We attach an ice quiver with potential $(Q_t, F_t, W_t)$ to any vertex $t \in \mathbb{T}_n$ of the $n$-regular tree by considering the sequence of mutations from the initial root to $t$. By Proposition 3.4.3, if the compatible pair associated with $(Q, F)$ is defined, then so is the compatible pair associated with $(Q_t, F_t)$, and the latter can be computed from the former by applying the corresponding sequence of mutations of compatible pairs.

**Theorem 3.4.6.** *Let $(Q, F, W)$ be a non-degenerate ice quiver with potential such that $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper. Let $\mathcal{H} = \mathcal{H}(Q, F, W)$ be the associated Higgs category. Then the compatible pair associated with $(Q_t, F_t)$ is defined for all $t \in \mathbb{T}_n$ and its Poisson coefficient matrix $\Lambda_t = (\lambda_{ij}^t)$ is given by*

$$\lambda_{ij}^t = [M_i^t, M_j^t]_{\mathcal{H}},$$

*where $M_i^t \in \mathcal{H}$ is the reachable rigid indecomposable object such that $CC(M_i^t)$ is the $i$-th cluster variable $x_{i;t}$ in the seed of $\mathcal{A}_{Q,F}$ associated with $t$. In particular, this endows $\mathcal{A}_{Q,F}$ with a $\Lambda$-cluster algebra structure.*

*Proof.* Fix $t \in \mathbb{T}_n$. By Proposition 3.4.3 and Lemma 3.4.4, the compatible pair associated with $(Q_t, F_t)$ is defined. Moreover, combined with Lemma 3.4.5, we have $\lambda_{ij}^t = [e_i\widehat{\mathbf{\Gamma}}_t, e_j\widehat{\mathbf{\Gamma}}_t]_{\mathcal{H}_t}$, where $\widehat{\mathbf{\Gamma}}_t = \widehat{\mathbf{\Gamma}}(Q_t, F_t, W_t)$ and $\mathcal{H}_t$ is the associated Higgs category. If $\Phi_t : \mathcal{H}_t \to \mathcal{H}$ is the equivalence given by the sequence of mutations from the root of $\mathbb{T}_n$ to $t$, then it sends $e_i\widehat{\mathbf{\Gamma}}_t$ to $M_i^t$. Since it comes from an equivalence between the corresponding relative cluster categories, $\Phi_t$ preserves negative extensions, so we conclude that $\lambda_{ij}^t$ can be computed as above. $\surd$

**Remark 3.4.7.** If $\widehat{\mathbf{\Gamma}}$ is proper and concentrated in degree zero, then $\mathcal{H}$ is a Frobenius *exact* category by [Wu23b, Theorem 6.2]. In particular, the first connecting morphism in the long exact sequence in the proof of Lemma 3.4.5 is the zero map. By this observation together with the fact that $\widehat{\mathbf{\Gamma}}$ has no cohomology in strictly negative degrees, we deduce that

$$[M, N]_{\mathcal{H}} = \dim \mathrm{Hom}_{\mathcal{H}}(M, N) - \dim \mathrm{Hom}_{\mathcal{H}}(N, M)$$

for $M, N \in \mathcal{H}$. As a corollary, we conclude that the $\Lambda$-cluster algebra structure on $\mathcal{A}_{Q,F}$ arising from Theorem 3.4.6 coincides with the one given by [GP25, Theorem 6.22] (see also [GLS13]). In fact, by including the terms with negative extensions in the long exact sequences in the proof of [GP25, Theorem 6.22], their argument can be easily adapted to the case where $\widehat{\mathbf{\Gamma}}$ is not concentrated in degree zero. This gives an alternative proof that the matrices in Theorem 3.4.6 yield compatible pairs and that they respect the mutation rule for Poisson coefficient matrices.

As recalled in Section 3.2, attached to the $\Lambda$-cluster algebra structure of $\mathcal{A}_{Q,F}$ yielded by Theorem 3.4.6, there are Cao's tropical and $F$-invariants, see (3.2) and (3.3). We conclude this section with a homological interpretation of these invariants.

**Proposition 3.4.8.** *Let $(Q, F, W)$ be a non-degenerate ice quiver with potential such that $\widehat{\mathbf{\Gamma}}(Q, F, W)$ is proper. Let $\mathcal{H} = \mathcal{H}(Q, F, W)$ be the associated Higgs category. For reachable rigid objects $M, N \in \mathcal{H}$, we have*

$$\langle CC(M), CC(N)\rangle_{\mathrm{trop}} = \dim \mathrm{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}}$$

*and*

$$(CC(M) \mid\mid CC(N))_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N).$$

*Proof.* Suppose $CC(M)$ is a cluster monomial for the seed attached to $t \in \mathbb{T}_n$. In particular, the $F$-polynomial $F^t_{CC(M)}$ is constant and equal to 1. Thus, by the definition of the tropical invariant, we have

$$\langle CC(M), CC(N)\rangle_{\mathrm{trop}} = (\mathbf{g}^t_{CC(M)})^T \Lambda_t \mathbf{g}^t_{CC(N)}.$$

We can compute the $g$-vectors above using Theorem 3.3.10. By the description of $\Lambda_t$ in Theorem 3.4.6, we deduce that

$$\langle CC(M), CC(N)\rangle_{\mathrm{trop}} = [M, T_0]_{\mathcal{H}} - [M, T_1]_{\mathcal{H}},$$

where $T_0$ and $T_1$ are such that there exists a conflation

$$T_1 \rightarrowtail T_0 \twoheadrightarrow N \dashrightarrow$$

in $\mathcal{H}$ and $T_0, T_1$ belong to the cluster-tilting subcategory $\operatorname{add} T$ associated with $t$. Applying $\operatorname{Hom}_{\mathcal{C}(Q,F,W)}(M, -)$ and $\operatorname{Hom}_{\mathcal{C}(Q,F,W)}(-, M)$ to this conflation, we get long exact sequences

$$\cdots \to \operatorname{Ext}^{-1}_{\mathcal{H}}(M, N) \to \operatorname{Hom}_{\mathcal{H}}(M, T_1) \to \operatorname{Hom}_{\mathcal{H}}(M, T_0) \to \operatorname{Hom}_{\mathcal{H}}(M, N) \to 0$$

and

$$\begin{aligned} \cdots \to \operatorname{Ext}^{-1}_{\mathcal{H}}(N, M) \to \operatorname{Hom}_{\mathcal{H}}(N, M) \to \operatorname{Hom}_{\mathcal{H}}(T_0, M) \to \operatorname{Hom}_{\mathcal{H}}(T_1, M) \\ \to \operatorname{Ext}^1_{\mathcal{H}}(N, M) \to 0, \end{aligned}$$

where we used that $\operatorname{Ext}^1_{\mathcal{H}}(M, T_1) = \operatorname{Ext}^1_{\mathcal{H}}(T_0, M) = 0$ since $M, T_0, T_1 \in \operatorname{add} T$. Taking the difference of the Euler characteristics of these exact sequences, we deduce that

$$[M, T_0]_{\mathcal{H}} - [M, T_1]_{\mathcal{H}} = \dim \operatorname{Ext}^1_{\mathcal{H}}(N, M) + [M, N]_{\mathcal{H}}.$$

By the 2-Calabi–Yau property, $\dim \operatorname{Ext}^1_{\mathcal{H}}(N, M) = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N)$, whence the formula for the tropical invariant. Finally, we have

$$\begin{aligned} (CC(M) \mid\mid CC(N))_F &= \langle CC(M), CC(N)\rangle_{\mathrm{trop}} + \langle CC(N), CC(M)\rangle_{\mathrm{trop}} \\ &= (\dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}}) + (\dim \operatorname{Ext}^1_{\mathcal{H}}(N, M) + [N, M]_{\mathcal{H}}) \\ &= 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) \end{aligned}$$

again by the 2-Calabi–Yau property and by the skew-symmetry of $[-, -]_{\mathcal{H}}$. $\surd$

**Remark 3.4.9.** Keep the notation and hypotheses of the proposition above. By Proposition 3.3.7, we have

$$\mathrm{Ext}^1_{\mathcal{H}}(M,N) \cong \mathrm{Ext}^1_{\overline{\mathcal{C}}}(p^*(M),p^*(N)),$$

where $\overline{\mathcal{C}} = \mathcal{C}(\overline{Q},\overline{W})$ and $(\overline{Q},\overline{W})$ is the quiver with potential obtained from $(Q,F,W)$ by deleting the frozen vertices. Moreover, by [Pla11a, Proposition 4.15] and since $p^*(M)$ and $p^*(N)$ are rigid, the dimension of the extension group above coincides with the $E$-invariant $E^{\mathrm{sym}}(\mathcal{M},\mathcal{N})$ of the corresponding pair of decorated representations $\mathcal{M}$ and $\mathcal{N}$ of $(\overline{Q},\overline{W})$ (in the sense of [DWZ10]). Hence, by Proposition 3.4.8, we have

$$(CC(M) \mid\mid CC(N))_F = 2 \cdot E^{\mathrm{sym}}(\mathcal{M},\mathcal{N}).$$

This recovers [Cao25, Theorem 6.11] in our setting. We remark that the $F$-invariant above can be computed on the level of the cluster algebra with trivial coefficients $\mathcal{A}_{\overline{Q}}$ by [Cao25, Section 4.4], which is the setting in the cited theorem. Furthermore, the term 2 appears because our compatible pairs are of type $2I_n$.

## 3.5 Background on monoidal categorification

We provide the necessary background on the monoidal categorification of cluster algebras via quantum affine algebras, following the work of Kashiwara–Kim–Oh–Park [KKOP20, KKOP24a, KKOP25]. After recalling some relevant properties of their Λ- and $\mathfrak{d}$-invariants, we state their result on Λ-monoidal categorification using the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$. We then describe the ice quiver of an initial monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$ and upgrade it to an ice quiver with potential that we will use in the subsequent sections to construct the corresponding additive categorification. For a particular case already considered by Hernandez–Leclerc in [HL16], we also present the modules that constitute this initial monoidal seed.

### 3.5.1 Quantum affine algebras and the Λ-invariant

We follow the conventions of [KKOP20, KKOP24a], where the reader can find more details and references.

We denote by $\mathbf{k}$ the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\bigcup_{m>0}\mathbb{C}((q^{1/m}))$, where $q$ is an indeterminate. Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra. We denote by $U_q'(\mathfrak{g})$ the associated quantum affine algebra, which is a Hopf algebra over $\mathbf{k}$. We work with the category $\mathscr{C}_{\mathfrak{g}}$ of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. It is a $\mathbf{k}$-linear non-semisimple abelian length category, and the coproduct of $U_q'(\mathfrak{g})$ endows $\mathscr{C}_{\mathfrak{g}}$ with the structure of a monoidal category, which is rigid in the sense that any $V \in \mathscr{C}_\mathfrak{g}$ has a left dual $\mathscr{D}^{-1}(V)$ and a right dual $\mathscr{D}(V)$. We extend this notation and define $\mathscr{D}^n(V)$ for any $n \in \mathbb{Z}$ in the natural way.

Although $\mathscr{C}_\mathfrak{g}$ is not braided as a monoidal category, any two simple objects $V, W \in \mathscr{C}_\mathfrak{g}$ determine a *universal R-matrix* (introduced in [Dri87]), which yields a canonical map $V \otimes W \to W \otimes V$ that is generically an isomorphism. Using the renormalizing coefficient of $V$ and $W$ coming from this universal $R$-matrix, Kashiwara–Kim–Oh–Park define in [KKOP20] the $\Lambda$-*invariant* $\Lambda(V, W) \in \mathbb{Z}$ and the $\mathfrak{d}$-*invariant*

$$\mathfrak{d}(V, W) = \frac{1}{2}(\Lambda(V, W) + \Lambda(W, V)).$$

We remark that $\mathfrak{d}(V, W) \in \mathbb{Z}_{\geq 0}$ by Proposition 3.16 and Corollary 3.19 in [KKOP20].

The following result describes the $\Lambda$-invariant in terms of the $\mathfrak{d}$-invariant and the functors $\mathscr{D}^{\pm 1}$. Notice the similarity with the formula for the tropical invariant in Proposition 3.4.8.

**Proposition 3.5.1.** *Let $V$ and $W$ be simple modules in $\mathscr{C}_\mathfrak{g}$. Then*

$$\begin{aligned}\Lambda(V, W) &= \mathfrak{d}(V, W) + \sum_{n \geq 1} (-1)^{n-1} \left[\mathfrak{d}(V, \mathscr{D}^{-n}(W)) - \mathfrak{d}(V, \mathscr{D}^{n}(W))\right] \\ &= \mathfrak{d}(V, W) + \sum_{n \geq 1} (-1)^{n-1} \left[\mathfrak{d}(V, \mathscr{D}^{-n}(W)) - \mathfrak{d}(W, \mathscr{D}^{-n}(V))\right]\end{aligned}$$

*Proof.* The first equality is [KKOP20, Proposition 3.22]. For the second we use the symmetry of the $\mathfrak{d}$-invariant and [KKOP20, Lemma 3.7], which implies that $\mathfrak{d}(V', W') = \mathfrak{d}(\mathscr{D}^{-1}(V'), \mathscr{D}^{-1}(W'))$ for any simple objects $V', W' \in \mathscr{C}_\mathfrak{g}$. We obtain that $\mathfrak{d}(V, \mathscr{D}^n(W)) = \mathfrak{d}(W, \mathscr{D}^{-n}(V))$ and hence the desired equality. $\surd$

There is an important class of simple $U_q'(\mathfrak{g})$-modules in $\mathscr{C}_\mathfrak{g}$ called the *Kirillov-Reshetikhin* (KR) *modules*. We recall their definition in the case where $\mathfrak{g}$ is *untwisted of simply-laced type*, that is, where it is of type $\mathsf{A}^{(1)}\mathsf{D}^{(1)}\mathsf{E}^{(1)}$ in Kac's classification [Kac90, Chapter 4]. In this case, let $\Delta$ denote the corresponding Dynkin diagram of *finite* type $\mathsf{ADE}$ obtained by removing an extending vertex from the Dynkin diagram of $\mathfrak{g}$. Consider the infinite set of variables $\mathscr{Y} = \{Y_{i,a} \mid i \in \Delta_0, a \in \mathbf{k}^\times\}$. By the work of Chari–Pressley (see [CP94, Theorem 12.2.6]) and Frenkel–Reshetikhin [FR99], the simple objects in $\mathscr{C}_\mathfrak{g}$ can be parametrized by the highest dominant monomial of their $q$-characters, which is a monomial with non-negative exponents in the variables of $\mathscr{Y}$. For such a monomial $m$, denote by $L(m)$ the corresponding simple module. For $i \in \Delta_0$, $a \in \mathbf{k}^\times$ and $r \geq 1$, the associated KR-module is defined as

$$W_{r,a}^{(i)} = L(Y_{i,a} Y_{i,aq^2} \cdots Y_{i,aq^{2(r-1)}}).$$

We have an explicit description of the duality functors on such modules. Denote by $\mathbb{D} : \mathscr{Y} \to \mathscr{Y}$ the bijection sending $Y_{i,a}$ to $Y_{i^*,aq^h}$, where $i \mapsto i^*$ is the involution on $\Delta_0$ induced by the longest element of the Weyl group of $\Delta$ and $h$ is the Coxeter number of $\Delta$. We extend $\mathbb{D}$ to a bijection on the set of monomials in the variables of $\mathscr{Y}$. By [FO21, Proposition 5.5] (and the references therein), we have $\mathscr{D}^n(L(m)) \cong L(\mathbb{D}^n(m))$ for any dominant monomial $m$ in the variables of $\mathscr{Y}$ and $n \in \mathbb{Z}$. In particular, observe that $\mathscr{D}^{-1}(W^{(i)}_{r,a}) \cong W^{(i^*)}_{r,aq^{-h}}$.

### 3.5.2 Λ-monoidal categorification

Let $\mathcal{C}$ be a full subcategory of $\mathscr{C}_{\mathfrak{g}}$ that contains the trivial module and is stable under tensor products, subquotients, and extensions. Denote by $K_0(\mathcal{C})$ the Grothendieck group of $\mathcal{C}$. It becomes a ring under the product induced by the monoidal structure. We say $\mathcal{C}$ is a *monoidal categorification* of a cluster algebra $\mathcal{A}$ if it is endowed with a $\mathbb{Z}$-algebra isomorphism

$$\varphi : K_0(\mathcal{C}) \longrightarrow \mathcal{A}$$

such that any cluster monomial in $\mathcal{A}$ is the image of the class of a simple module. In this case, a simple object $V \in \mathcal{C}$ is *reachable* if $\varphi([V])$ is a cluster monomial. We stress that $\varphi$ is part of the data of the monoidal categorification. For simplicity, we will denote $\varphi(V) \coloneqq \varphi([V])$ for $V \in \mathscr{C}_{\mathfrak{g}}$. We allow the cluster algebra to have infinite rank and denote its set of indices by $K = K^{\mathrm{un}} \sqcup K^{\mathrm{fr}}$.

For a seed $\mathcal{S} = (\mathbf{x}_t, \widetilde{B}_t)$ in $\mathcal{A}$, write $\mathbf{x}_t = \{x_{i;t}\}_{i\in K}$. It determines (up to isomorphism) a set $\mathcal{V}_t = \{V_{i;t}\}_{i\in K}$ of simple modules in $\mathcal{C}$ defined by $\varphi(V_{i;t}) = x_{i;t}$. Note that $V_{i;t}$ and $V_{j;t}$ *strongly commute* for any $i, j \in K$ in the sense that their tensor product is simple (since $\varphi(V_{i;t} \otimes V_{j;t})$ is a cluster monomial). In particular, each $V_{i;t}$ is a *real* simple module, that is, its tensor square is simple. We call the pair $\mathscr{S} = (\mathcal{V}_t, \widetilde{B}_t)$ a *(reachable) monoidal seed* in $\mathcal{C}$. For $v \in K^{\mathrm{un}}$, we denote by $\mu_v(\mathscr{S})$ the monoidal seed that corresponds to the mutation of $\mathcal{S}$ in the direction of $v$. More generally, we define a monoidal seed in $\mathcal{C}$ as a pair $(\mathcal{V}, \widetilde{B})$ where $\widetilde{B}$ is a $K \times K^{\mathrm{un}}$ extended exchange matrix and $\mathcal{V}$ is a strongly commuting family of real simple modules in $\mathcal{C}$ indexed by $K$.

Any monoidal seed $\mathscr{S} = (\mathcal{V}_t, \widetilde{B}_t)$ in $\mathcal{C}$ determines a square matrix $\Lambda_t = (\lambda^t_{ij})_{i,j\in K}$ given by the Λ-invariant

$$\lambda^t_{ij} = \Lambda(V_{i;t}, V_{j;t}).$$

Since $\mathcal{V}_t$ is a strongly commuting family of real simple modules, we have $\mathfrak{d}(V_{i;t}, V_{j;t}) = 0$ by [KKOP20, Corollary 3.17] and so $\Lambda_t$ is skew-symmetric. We say that the monoidal categorification is a Λ-*monoidal categorification* if the pair $(\widetilde{B}_t, \Lambda_t)$ is compatible for some reachable monoidal seed. In this case, by [KKOP20, Proposition 6.4] (see also [Cao25, Proposition 5.13]), this compatibility condition holds for *any* reachable monoidal seed and, if we denote $\mu_v(\mathscr{S}) = (\mathcal{V}_{t'}, \widetilde{B}_{t'})$ and the corresponding skew-symmetric matrix by $\Lambda_{t'}$, then $(\widetilde{B}_t, \Lambda_t)$ and $(\widetilde{B}_{t'}, \Lambda_{t'})$ are related by mutation of compatible pairs in the direction of $v \in K^{\mathrm{un}}$. Consequently, $\mathcal{A}$ is a $\Lambda$-cluster algebra.

**Remark 3.5.2.** The notion of monoidal categorification was first introduced in [HL10], while that of $\Lambda$-monoidal categorification in [KKOP20] (building on [KKKO18]). Observe that the definition in [KKOP20] is slightly more restrictive than the one given above, which is based on [Cao25, Section 5].

The following is one of the main results in [Cao25] and a major motivation for the definition of the tropical and $F$-invariants.

**Theorem 3.5.3** ([Cao25, Corollary 5.17])**.** *Let $\mathcal{C}$ be a $\Lambda$-monoidal categorification of a $\Lambda$-cluster algebra $\mathcal{A}$. Let $\varphi : K_0(\mathcal{C}) \to \mathcal{A}$ be the associated isomorphism. For reachable simple objects $V, W \in \mathcal{C}$, we have*

$$\langle \varphi(V), \varphi(W) \rangle_{\mathrm{trop}} = \Lambda(V, W)$$

*and*

$$(\varphi(V) \mid\mid \varphi(W))_F = 2 \cdot \mathfrak{d}(V, W).$$

Let us now briefly recall the $\Lambda$-monoidal categorification obtained in [KKOP24a, KKOP25]. By [KKOP22, Theorem 3.6], one can associate a simply-laced finite type Dynkin diagram $\Delta$ to $U_q'(\mathfrak{g})$. If $\mathfrak{g}$ is untwisted of simply-laced type, then $\Delta$ is the corresponding Dynkin diagram of finite type. In [KKOP24b], the notion of *complete duality datum* for $U_q'(\mathfrak{g})$ is introduced, which depends on $\Delta$ and is intimately connected to the quantum affine Schur–Weyl duality of [KKK18]. A *complete PBW-pair* is a pair $(\mathcal{D}, \underline{w}_0)$ where $\mathcal{D}$ is a complete duality datum for $U_q'(\mathfrak{g})$ and $\underline{w}_0 = (i_1, \ldots, i_{l(w_0)}) \in \Delta^{l(w_0)}$ is a reduced expression of the longest element $w_0$ of the Weyl group of type $\Delta$. By [KKOP24b], any such pair determines a family $\{\mathsf{S}_s\}_{s\in\mathbb{Z}}$ of simple modules in $\mathscr{C}_\mathfrak{g}$ called the *affine cuspidal modules.*

For $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ such that $a \le b$, we denote

$$[a, b] = \{s \in \mathbb{Z} \mid a \le s \le b\}.$$

Given such an interval and a complete PBW-pair $(\mathcal{D}, \underline{w}_0)$, Kashiwara–Kim–Oh–Park [KKOP24a] define $\mathscr{C}_\mathfrak{g}^{[a,b],\mathcal{D},\underline{w}_0}$ as the smallest full subcategory of $\mathscr{C}_\mathfrak{g}$ that is closed under taking tensor products, subquotients and extensions, and contains the trivial module and the affine cuspidal modules $\mathsf{S}_s$ associated with $(\mathcal{D}, \underline{w}_0)$ for $s \in [a, b]$.

**Theorem 3.5.4** ([KKOP24a, KKOP25])**.** *The Grothendieck ring of $\mathscr{C}_\mathfrak{g}^{[a,b],\mathcal{D},\underline{w}_0}$ has a $\Lambda$-cluster algebra structure and $\mathscr{C}_\mathfrak{g}^{[a,b],\mathcal{D},\underline{w}_0}$ is a $\Lambda$-monoidal categorification of this $\Lambda$-cluster algebra.*

We describe an initial monoidal seed for the $\Lambda$-monoidal categorification above in the next section.

### 3.5.3 Descriptions for the initial monoidal seed

From now on, $\Delta$ denotes the simply-laced finite type Dynkin diagram associated to $U_q'(\mathfrak{g})$. For two vertices $i, j \in \Delta_0$, we write $i \sim j$ to denote that $i$ and $j$ are adjacent in $\Delta$. Let $w_0$ the longest element of the Weyl group of type $\Delta$ and denote by $i \mapsto i^*$ the involution on $\Delta_0$ defined by $w_0(\alpha_i) = -\alpha_{i^*}$, where $\alpha_i$ is the simple root associated with $i \in \Delta_0$. For a reduced expression $\underline{w}_0 = (i_1, \ldots, i_{l(w_0)})$ for $w_0$, we define an infinite sequence $\widehat{\underline{w}}_0 = (i_s)_{s\in\mathbb{Z}}$ extending $\underline{w}_0$ and such that $i_{s+l(w_0)} = i_s^*$ for all $s \in \mathbb{Z}$. We define

$$s^- = \max\{t \in \mathbb{Z} \mid t < s, i_t = i_s\} \quad \text{and} \quad s^+ = \min\{t \in \mathbb{Z} \mid t > s, i_t = i_s\}$$

for $s \in \mathbb{Z}$. We now use the sequence $\widehat{\underline{w}}_0$ to attach an ice quiver (with potential) to any integer interval bounded on the right.

Fix $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z}$ with $a \leq b$. We define a quiver $Q^{[a,b]}(\underline{w}_0)$ as follows. Its vertex set is the interval $[a, b]$ and there is an arrow $s \to t$ if and only if one of the conditions below holds:

(1) $t = s^-$;

(2) $i_s \sim i_t$ and $s^- < t^- < s < t$;

(3) $i_s \sim i_t$, $s < t$ and $s^-, t^- < a$.

We declare a vertex $s$ to be frozen if $s^- < a$, and we denote by $F^{[a,b]}(\underline{w}_0)$ the full subquiver on frozen vertices. This yields an ice quiver $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$. Note that $F^{[a,b]}(\underline{w}_0)$ is not empty precisely when $a \in \mathbb{Z}$, and an arrow $s \to t$ is frozen if and only if condition (3) holds. In this case, we also define a potential $W^{[a,b]}(\underline{w}_0)$ on $Q^{[a,b]}(\underline{w}_0)$ as the sum of all simple cordless cycles. More precisely, for any arrow $s \to t$ with $s < t$ and $t$ unfrozen, there is a unique cycle in $Q^{[a,b]}(\underline{w}_0)$ of the form

$$\begin{tikzcd} & s \arrow[dr] & \\ t_l \arrow[ur] & \cdots \arrow[l] & t_1 \arrow[l] \end{tikzcd}$$

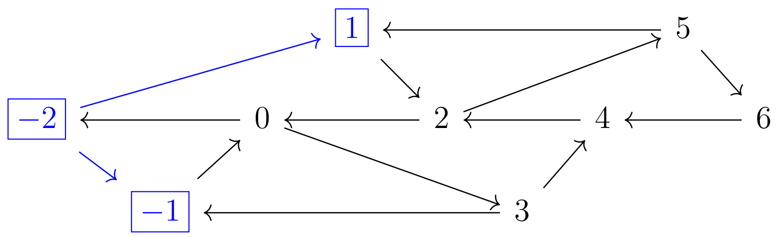
where $t_1 = t$ and $t_{i+1} = t_i^-$ for $1 \leq i < l$. Then $W^{[a,b]}(\underline{w}_0)$ is the sum of all such cycles.

**Remark 3.5.5.** When drawing the ice quiver $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$, we use the following conventions. We draw the vertices on horizontal lines indexed by $\Delta_0$. We place a vertex $s$ on the line $i_s$ and, if $s < t$, then $s$ is to the left of $t$. Thus, the only horizontal arrows in the picture are those of the form $s \to s^-$, which point to the left. We depict a frozen vertex inside a blue box, and frozen arrows are also blue.

**Example 3.5.6.** For $\Delta = \mathsf{A}_3$ and $\underline{w}_0 = (1,2,3,2,1,2)$, we have

$$\widehat{\underline{w}}_0 = (\dots,3,2,1,\underbrace{2,3,2,1,2,3,2,1,2}_{[-2,6]},3,2,1,2,3,2,\dots).$$

The ice quiver $(Q^{[-2,6]}(\underline{w}_0), F^{[-2,6]}(\underline{w}_0))$ is given by




The potential $W^{[-2,6]}(\underline{w}_0)$ is the sum of the six simple cordless cycles.

By the next result, the ice quiver above describes the $\Lambda$-cluster algebra structure from Theorem 3.5.4 for right-bounded intervals.

**Proposition 3.5.7** ([KKOP24a, KKOP25])**.** *Let $(\mathcal{D}, \underline{w}_0)$ be a complete PBW-pair. Let $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z}$ be such that $a \leq b$. Then $K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0})$ has a $\Lambda$-seed whose extended exchange matrix is the one associated to the ice quiver $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$.*

**Remark 3.5.8.** Our quivers are not exactly the ones appearing in [KKOP24a, KKOP25]. Indeed, they do not impose the condition (3) in our definition of $Q^{[a,b]}(\underline{w}_0)$. Therefore, our quiver may have additional arrows, but they are all between frozen vertices and do not influence the extended exchange matrix in the proposition above. We remark that ice quivers with potential similar to ours have appeared in [BIRS09] and [BIRS11]. On the other hand, they consider a signed version of the potential we consider, and so does [Con26]. We adopt the same signs as in [HL16].

The simple modules in the monoidal seed that corresponds to the $\Lambda$-seed in Proposition 3.5.7 are examples of *affine determinantial modules*. We give a more explicit description of this monoidal seed in the case where $\mathfrak{g}$ is untwisted of simply-laced type and $(\mathcal{D}, \underline{w}_0)$ is an adapted PBW-pair. In this case, these simple modules will be KR-modules. The general setting is treated in [KKOP24a].

We first need to recall some definitions. Let $Q$ be a quiver. We say a sequence $\mathbf{v} = (v_1, \dots, v_r)$ of vertices of $Q$ is a *source* (resp. *sink*) sequence if $v_i$ is a source (resp. sink) in $\mu_{v_{i-1}} \cdots \mu_{v_2}\mu_{v_1}(Q)$ for all $1 \leq i \leq r$. Alternatively, we say that $\mathbf{v}$ is *adapted* to $Q$ if it is a source sequence.

Fix $\epsilon : \Delta_0 \to \{0,1\}$ such that $\epsilon_i \neq \epsilon_j$ if $i \sim j$, where $\epsilon_i := \epsilon(i)$. A *height function* is a function $\xi : \Delta_0 \to \mathbb{Z}$ such that $\xi_i := \xi(i)$ has the same parity as $\epsilon_i$ and $|\xi_i - \xi_j| = 1$ for $i \sim j$. It determines an orientation $Q_\xi$ of $\Delta$ by defining an arrow $i \to j$ if $\xi_j = \xi_i + 1$. We say $i \in \Delta_0$ is a *source* (resp. *sink*) of $\xi$ if and only if $i$ is a source (resp. sink) of $Q_\xi$. If $i$ is a source of $\xi$, we define the *reflection* $s_i\xi : \Delta_0 \to \mathbb{Z}$ of $\xi$ at $i$ to be the function such that $(s_i\xi)_j = \xi_j$ for $j \neq i$ and $(s_i\xi)_i = \xi_i + 2$. Note that $s_i\xi$ is indeed a height function and $Q_{s_i\xi}$ is the reflection $\mu_i(Q_\xi)$ of $Q_\xi$ at the source $i$. Similarly, if $i$ is a sink of $\xi$, we define the *reflection* of $\xi$ at $i$ as a height function $s_i^{-1}\xi : \Delta_0 \to \mathbb{Z}$ given by $(s_i^{-1}\xi)_j = \xi_j$ for $j \neq i$ and $(s_i^{-1}\xi)_i = \xi_i - 2$.

**Remark 3.5.9.** We follow a convention opposite to that of [FO21] and [KKOP24a] to pass between height functions and quivers. In particular, a source (resp. sink) of a height function in our sense is the same as a sink (resp. source) in their sense.

Let $Q_{\mathrm{HL}}$ be the quiver whose vertex set is the subset of $\Delta_0 \times \mathbb{Z}$ of pairs $(i,p)$ such that $p$ has the same parity as $\epsilon_i$. We define an arrow from $(i,p)$ to $(j,r)$ if

(1) $i \sim j$ and $r = p+1$;

(2) $i = j$ and $r = p-2$.

For a height function $\xi$, we let $Q_{\mathrm{HL}}^{<\xi}$ be the full subquiver of $Q_{\mathrm{HL}}$ on the vertices $(i,p)$ such that $p < \xi_i$.

**Lemma 3.5.10.** *Let $\xi : \Delta_0 \to \mathbb{Z}$ be a height function. Let $\underline{w}_0 = (i_1, \dots, i_{l(w_0)})$ be a reduced expression of the longest element that is adapted to $Q_\xi$. Then there is a unique isomorphism $Q_{\mathrm{HL}}^{<\xi} \to Q^{[-\infty,0]}(\underline{w}_0)$ sending a vertex of the form $(i,p)$ to a vertex in the $i$-th row of $Q^{[-\infty,0]}(\underline{w}_0)$.*

*Proof.* It follows from [KKOP24a, Proposition 7.27] and the description of admissible sequences in [KKOP24a, Proposition 6.11]. √

**Remark 3.5.11.** Note that any quiver $Q^{[-\infty,b]}(\underline{w}_0)$ is isomorphic to $Q^{[-\infty,0]}(\underline{w}_0')$ for some reduced expression $\underline{w}_0'$. To obtain $\underline{w}_0'$, one can shift the sequence $\widehat{\underline{w}}_0$ by $b$ units and take the corresponding reduced expression. This shift operation corresponds to applying reflections to $\xi$ in the lemma above.

For the rest of this section, we assume $\mathfrak{g}$ is untwisted of simply-laced type. By [KKOP24a, Theorem 6.12], a height function $\xi$ determines a complete duality datum $\mathcal{D}_\xi$ for $U_q'(\mathfrak{g})$. We will say that a complete PBW-pair $(\mathcal{D}, \underline{w}_0)$ is *adapted to an orientation of* $\Delta$ if there exists a height function $\xi : \Delta_0 \to \mathbb{Z}$ such that $\mathcal{D} = \mathcal{D}_\xi$ and $\underline{w}_0$ is adapted to $Q_\xi$. In this case, we describe a monoidal seed of $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathcal{D},\underline{w}_0}$.

For a vertex $(i,p)$ in $Q_{\mathrm{HL}}^{<\xi}$, we attach the KR-module $\mathcal{M}_{(i,p)}^{<\xi} = W_{r,q^p}^{(i)}$, where $r \geq 1$ is the largest integer such that $p + 2(r-1) < \xi_i$. This gives a triple $\mathscr{S}^{<\xi} = (\{\mathcal{M}_{(i,p)}^{<\xi}\}, Q_{\mathrm{HL}}^{<\xi}, \varnothing)$, where the set $\{\mathcal{M}_{(i,p)}^{<\xi}\}$ is indexed by the vertices of $Q_{\mathrm{HL}}^{<\xi}$. The empty set in the triple represents the set of frozen vertices of $Q_{\mathrm{HL}}^{<\xi}$. In the next result, we will view $\mathscr{S}^{<\xi}$ as a monoidal seed whose extended exchange matrix comes from the ice quiver $(Q_{\mathrm{HL}}^{<\xi}, \varnothing)$.

**Theorem 3.5.12** ([KKOP24a, Section 8])**.** *Suppose $\mathfrak{g}$ is untwisted of simply-laced type. Let $\xi : \Delta_0 \to \mathbb{Z}$ be a height function. Let $\mathcal{D}_\xi$ be the corresponding complete duality datum and choose a reduced expression $\underline{w}_0$ for the longest element adapted to $Q_\xi$. Then $\mathscr{S}^{<\xi}$ is a reachable monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathcal{D}_\xi,\underline{w}_0}$ for the $\Lambda$-monoidal categorification in Theorem 3.5.4.*

**Remark 3.5.13.** Let $\xi : \Delta_0 \to \mathbb{Z}$ be the unique height function that only assumes the values 1 and 2. As explained in [KKOP24a, p. 894], the category $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathcal{D}_\xi,\underline{w}_0}$ above coincides with the category $\mathscr{C}^-$ in [HL16]. Moreover, the assignment of KR-modules in the monoidal seed $\mathscr{S}^{<\xi}$ is the one given previously by Hernandez–Leclerc.

**Remark 3.5.14.** For a height function $\xi : \Delta_0 \to \mathbb{Z}$, let $\mathcal{D}_\xi$ and $\underline{w}_0$ be as in Theorem 3.5.12. For an integer $a \leq 0$, a reachable monoidal seed for $\mathscr{C}_{\mathfrak{g}}^{[a,0],\mathcal{D}_\xi,\underline{w}_0}$ can be obtained by restricting $\mathscr{S}^{<\xi}$ to the subquiver that corresponds to $Q^{[a,0]}(\underline{w}_0)$ under the isomorphism of Lemma 3.5.10 and freezing the vertices that correspond to $F^{[a,0]}(\underline{w}_0)$. This follows from [KKOP24a, Lemma 7.15] (see also the proof of their Proposition 8.11). Moreover, by applying reflections to $\xi$ as in Remark 3.5.11, we can replace the right boundary of the interval in the theorem and in this remark by any $b \in \mathbb{Z}$.

## 3.6 Maximal green sequences

When $\mathfrak{g}$ is untwisted of simply-laced type, we will explain how to realize the left duality functor $\mathscr{D}^{-1}$ on KR-modules over $U_q'(\mathfrak{g})$ using an infinite version of a maximal green sequence. After recalling the definition and some properties of maximal green sequences, we show how to construct them for triangle products of Dynkin quivers following [GK21]. We then exploit an idea of [HL16] to obtain the desired realization of $\mathscr{D}^{-1}$.

### 3.6.1 Definition and effect on additive categorification

We recall the definition of (maximal) green sequences, which was first given in [Kel11b]. Let $Q$ be a finite quiver without loops or 2-cycles. Its *framed quiver* $\widehat{Q}$ (resp. *coframed quiver* $\check{Q}$) is obtained from $Q$ by adding a vertex $v'$ for any $v \in Q_0$ and an arrow $v \to v'$ (resp. $v' \to v$). The new vertices are declared to be frozen. If $R$ is a quiver obtained from $\widehat{Q}$ by a sequence of mutations (at unfrozen vertices), then we say that an unfrozen vertex $v$ is *green* in $R$ if there is no arrow from frozen vertices to $v$, and *red* otherwise. A sequence $\mathbf{v} = (v_1, \ldots, v_r)$ of vertices in $Q$ is called a *green sequence* if $v_i$ is green in $\mu_{v_{i-1}} \cdots \mu_{v_2}\mu_{v_1}(\widehat{Q})$ for all $1 \leq i \leq r$. Such a sequence is *maximal* if all unfrozen vertices of $\mu_{\mathbf{v}}(\widehat{Q}) \coloneqq \mu_{v_r} \cdots \mu_{v_2}\mu_{v_1}(\widehat{Q})$ are red.

**Proposition 3.6.1** ([BDP14, Proposition 2.10])**.** *Suppose $Q$ admits a maximal green sequence $\mathbf{v}$. Then there is a unique isomorphism from $\mu_{\mathbf{v}}(\widehat{Q})$ to $\check{Q}$ that fixes the frozen vertices. It sends $i \in Q_0$ to $\sigma_{\mathbf{v}}(i) \in Q_0$ for a unique permutation $\sigma_{\mathbf{v}}$ of $Q_0$.*

The next results provide a way to construct maximal green sequences that are either source or sink sequences.

**Lemma 3.6.2** ([BDP14, Lemma 2.20])**.** *Let $Q$ be an acyclic quiver with $n$ vertices. If $\mathbf{v} = (v_1, \ldots, v_n)$ is a source sequence for $Q$ such that $Q_0 = \{v_1, \ldots, v_n\}$, then $\mathbf{v}$ is a maximal green sequence and $\sigma_{\mathbf{v}} = \mathrm{id}$.*

**Proposition 3.6.3.** *Suppose $Q$ is an orientation of an* ADE *Dynkin diagram $\Delta$. Let $\mathbf{v} = (v_1, \ldots, v_{l(w_0)})$ be a sink sequence for $Q$ that is also a reduced expression of the longest element $w_0$ of $\Delta$. Then $\mathbf{v}$ is a maximal green sequence and $\sigma_{\mathbf{v}}$ coincides with the involution induced by $w_0$.*

*Proof.* A proof that $\mathbf{v}$ is a maximal green sequence is given in the proof of [BDP14, Theorem 4.4]. Their argument also shows that $\sigma_{\mathbf{v}}$ agrees with the permutation on the orbits of the Auslander–Reiten translation induced by the shift functor in the bounded derived category of $Q$, which is well-known to coincide with the involution induced by $w_0$. $\checkmark$

Now let $(Q, W)$ be a Jacobi-finite and non-degenerate quiver with potential. Denote by $\widehat{\boldsymbol{\Gamma}} = \widehat{\boldsymbol{\Gamma}}(Q, W)$ the corresponding completed (absolute) Ginzburg dg algebra. For a sequence of vertices $\mathbf{v} = (v_1, \ldots, v_r)$ of $Q$, we denote by $(\mu_{\mathbf{v}}(Q), \mu_{\mathbf{v}}(W))$ the quiver with potential obtained by mutation following $\mathbf{v}$. We write $\mu_{\mathbf{v}}(\widehat{\boldsymbol{\Gamma}})$ for $\widehat{\boldsymbol{\Gamma}}(\mu_{\mathbf{v}}(Q), \mu_{\mathbf{v}}(W))$ and we let

$$\Phi_{\mathbf{v}} : \mathcal{C}(\mu_{\mathbf{v}}(Q), \mu_{\mathbf{v}}(W)) \to \mathcal{C}(Q, W)$$

be the composition of equivalences $\Phi_{v_1} \cdots \Phi_{v_r}$ as in Section 3.3.3. The following result describes the action of $\Phi_{\mathbf{v}}$ when $\mathbf{v}$ is a maximal green sequence.

**Theorem 3.6.4** ([Kel11b],[Kel12, Section 7])**.** *Suppose $Q$ admits a maximal green sequence $\mathbf{v}$. Then the equivalence $\Phi_{\mathbf{v}}$ sends $e_i\mu_{\mathbf{v}}(\widehat{\boldsymbol{\Gamma}})$ to $\Sigma^{-1}e_{\sigma_{\mathbf{v}}(i)}\widehat{\boldsymbol{\Gamma}}$. In particular, we have $\Phi_{\mathbf{v}}(\mu_{\mathbf{v}}(\widehat{\boldsymbol{\Gamma}})) \cong \Sigma^{-1}\widehat{\boldsymbol{\Gamma}}$.*

### 3.6.2 Triangle products of quivers

Let $Q$ and $Q'$ be two quivers. We define their *tensor product* $Q \otimes Q'$ to be the quiver with vertex set $Q_0 \times Q'_0$ and with the following arrows. For any arrow $\alpha : i \to j$ in $Q$ and any $i' \in Q'_0$, there is an arrow $(\alpha, i') : (i, i') \to (j, i')$ in $Q \otimes Q'$, and similarly, for any arrow $\alpha' : i' \to j'$ in $Q'$ and any $i \in Q_0$, there is an arrow $(i, \alpha') : (i, i') \to (i, j')$ in $Q \otimes Q'$. We also define the *triangle product* $Q \boxtimes Q'$ as the quiver obtained from $Q \otimes Q'$ by adding an arrow $(\alpha, \alpha')^{\mathrm{op}} : (j, j') \to (i, i')$ for all pairs of arrows $\alpha : i \to j$ in $Q$ and $\alpha' : i' \to j'$ in $Q'$.

Following [GK21], we provide a method for constructing maximal green sequences for $Q \boxtimes Q'$. Let $\mathbf{v} = (v_1, \ldots, v_r)$ and $\mathbf{w} = (w_1, \ldots, w_s)$ be sequences of vertices in $Q$ and $Q'$, respectively. We define $\mathbf{v} \boxtimes \mathbf{w}$ as the sequence

$$((v_1, w_1), (v_1, w_2), \ldots, (v_1, w_s), (v_2, w_1), \ldots, (v_2, w_s), \ldots\ldots, (v_r, w_1), \ldots, (v_r, w_s)),$$

which is a sequence of vertices in $Q \boxtimes Q'$.

**Theorem 3.6.5** ([GK21, Theorem 3.8])**.** *Let $Q$ and $Q'$ be finite quivers. Suppose that $Q$ admits a sink maximal green sequence $\mathbf{v}$ and $Q'$ admits a source maximal green sequence $\mathbf{w}$. Then $\mathbf{v} \boxtimes \mathbf{w}$ is a maximal green sequence for $Q \boxtimes Q'$ and $\sigma_{\mathbf{v}\boxtimes\mathbf{w}} = \sigma_{\mathbf{v}} \times \sigma_{\mathbf{w}} = \sigma_{\mathbf{v}} \times \mathrm{id}$.*

In particular, by Lemma 3.6.2 and Proposition 3.6.3, we can apply the theorem above when $Q$ is a Dynkin quiver and $Q'$ is acyclic. It is conjectured in [GK21] that this is precisely the case for which the result above applies.

**Example 3.6.6.** Consider the quivers

$$Q = \ 1 \longrightarrow 2 \longrightarrow 3 \quad \text{and} \quad Q' = \ 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4.$$

The triangle product $Q \boxtimes Q'$ is given as follows:

$$\begin{tikzcd}
(1,1) \arrow[dr] & & (1,2) \arrow[ll] \arrow[dr] & & (1,3) \arrow[ll] \arrow[dr] & & (1,4) \arrow[ll] \arrow[dr] & & \\
& (2,1) \arrow[ur] \arrow[dr] & & (2,2) \arrow[ll] \arrow[ur] \arrow[dr] & & (2,3) \arrow[ll] \arrow[ur] \arrow[dr] & & (2,4) \arrow[ll] \arrow[dr] & \\
& & (3,1) \arrow[ur] & & (3,2) \arrow[ll] \arrow[ur] & & (3,3) \arrow[ll] \arrow[ur] & & (3,4) \arrow[ll]
\end{tikzcd}$$

By Proposition 3.6.3, $\mathbf{v} = (3, 2, 1, 3, 2, 3)$ is a sink maximal green sequence for $Q$. By Lemma 3.6.2, $\mathbf{w} = (4, 3, 2, 1)$ is a source maximal green sequence for $Q'$. By Theorem

3.6.5, the sequence

$$\begin{aligned}\mathbf{v} \boxtimes \mathbf{w} = ((3,4),(3,3),(3,2),(3,1),(2,4),(2,3),(2,2),(2,1),\\ (1,4),(1,3),(1,2),(1,1),(3,4),(3,3),(3,2),(3,1),\\ (2,4),(2,3),(2,2),(2,1),(3,4),(3,3),(3,2),(3,1))\end{aligned}$$

is a maximal green sequence for $Q \boxtimes Q'$. Moreover, $\sigma_{\mathbf{v}\boxtimes\mathbf{w}}$ is an involution fixing the vertices $(2,i)$ and exchanging $(1,i)$ with $(3,i)$ for $1 \leq i \leq 4$.

### 3.6.3 Monoidal duality from a maximal green sequence

In this section, we suppose $\mathfrak{g}$ is untwisted of simply-laced type. Fix a height function $\xi : \Delta_0 \to \mathbb{Z}$. Let $\mathcal{D}_\xi$ be the complete duality datum associated with $\xi$ and fix a reduced word $\underline{w}_0$ for the longest element adapted to $Q_\xi$. By Theorem 3.5.12, the category $\mathscr{C}_{\mathfrak{g}}^{[-\infty,0],\mathcal{D}_\xi,\underline{w}_0}$ has an initial monoidal seed $\mathscr{S}^{<\xi}$ consisting of KR-modules $\mathcal{M}^{<\xi}_{(i,p)}$, where $(i,p)$ is a vertex of $Q^{<\xi}_{\mathrm{HL}}$. Our goal in this section is to define a sequence of mutations of $\mathscr{S}^{<\xi}$ such that the modules in the resulting monoidal seed are precisely the left duals $\mathscr{D}^{-1}(\mathcal{M}^{<\xi}_{(i,p)})$.

By the definition of the quiver $Q^{<\xi}_{\mathrm{HL}}$, it is clear that it is isomorphic to the triangle product of $Q_\xi$ with the infinite quiver

$$\cdots \longleftarrow 3 \longleftarrow 2 \longleftarrow 1.$$

By combining a sink maximal green sequence of $Q_\xi$ with the sequence $(1,2,3,\dots)$ for the quiver above, we will define an infinite version of the sequence of Theorem 3.6.5. We first consider a simpler case. For $j \in \Delta_0$, define the sequence

$$\mathbf{v}_j = ((j,\xi_j-2),(j,\xi_j-4),(j,\xi_j-6),\dots) \tag{3.9}$$

of vertices of $Q^{<\xi}_{\mathrm{HL}}$. Note that this sequence contains all the vertices in the row corresponding to $j$.

**Remark 3.6.7.** In the next results, we will apply some infinite mutation sequences from [HL16] on the level of monoidal seeds. The final result should be interpreted as a limit, as in loc. cit., which can be computed on each vertex in finitely many steps. We remark that their mutation sequences do not have arrows "going to infinity", so that the limit is indeed well-defined and can be lifted to the categorical level. However, the resulting monoidal seed is not reachable.

**Lemma 3.6.8.** *If $j$ is a sink of $\xi$, then the monoidal seed obtained by mutating $\mathscr{S}^{<\xi}$ along the sequence $\mathbf{v}_j$ is isomorphic to $\mathscr{S}^{<s_j^{-1}\xi}$.*

*Proof.* Let us explain how to prove the lemma using the arguments in [HL16, Section 3.2.3]. Suppose first that $\mathfrak{g}$ is of type $\mathsf{A}_2$, and label the vertices of $\Delta$ by 1 and 2. Without loss of generality, assume that $j = 2$. We can thus depict $\mathscr{S}^{<\xi}$ as

$$\begin{tikzcd}
\cdots \arrow[dr] & & W^{(1)}_{3,q^{\xi_1-6}} \arrow[ll] \arrow[dr] & & W^{(1)}_{2,q^{\xi_1-4}} \arrow[ll] \arrow[dr] & & W^{(1)}_{1,q^{\xi_1-2}} \arrow[ll] \arrow[dr] & \\
& \cdots \arrow[ur] & & W^{(2)}_{3,q^{\xi_2-6}} \arrow[ll] \arrow[ur] & & W^{(2)}_{2,q^{\xi_2-4}} \arrow[ll] \arrow[ur] & & W^{(2)}_{1,q^{\xi_2-2}} \arrow[ll]
\end{tikzcd}$$

where we write over each vertex $(i,p)$ of $Q^{<\xi}_{\mathrm{HL}}$ the KR-module $\mathcal{M}^{<\xi}_{(i,p)}$. The sequence $\mathbf{v}_2$ reads the bottom row from right to left. By the mutation sequence in [HL16, Section 6.1] and by the argument in [HL16, Section 3.2.3] on $T$-systems (which admit an interpretation as short exact sequences [Nak03, Her06]), the monoidal seed $\mu_{\mathbf{v}_2}(\mathscr{S}^{<\xi})$ obtained after mutating along $\mathbf{v}_2$ is

$$\begin{tikzcd}
\cdots \arrow[drrr] & & W^{(1)}_{3,q^{\xi_1-6}} \arrow[ll] \arrow[drrr] & & W^{(1)}_{2,q^{\xi_1-4}} \arrow[ll] \arrow[drrr] & & W^{(1)}_{1,q^{\xi_1-2}} \arrow[ll] & \\
& \cdots \arrow[ul] & & W^{(2)}_{3,q^{\xi_2-8}} \arrow[ll] \arrow[ul] & & W^{(2)}_{2,q^{\xi_2-6}} \arrow[ll] \arrow[ul] & & W^{(2)}_{1,q^{\xi_2-4}} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

Note that the spectral parameters of the KR-modules on the bottom row are shifted by $q^{-2}$. Sliding the bottom row to the left, we see that $\mu_{\mathbf{v}_2}(\mathscr{S}^{<\xi})$ coincides with

$$\begin{tikzcd}
& \cdots \arrow[dr] & & W^{(1)}_{3,q^{\xi_1-6}} \arrow[ll] \arrow[dr] & & W^{(1)}_{2,q^{\xi_1-4}} \arrow[ll] \arrow[dr] & & W^{(1)}_{1,q^{\xi_1-2}} \arrow[ll] \\
\cdots \arrow[ur] & & W^{(2)}_{3,q^{\xi_2-8}} \arrow[ll] \arrow[ur] & & W^{(2)}_{2,q^{\xi_2-6}} \arrow[ll] \arrow[ur] & & W^{(2)}_{1,q^{\xi_2-4}} \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

which is isomorphic to $\mathscr{S}^{<s_2^{-1}\xi}$, as desired. The proof when $\mathfrak{g}$ is of type $\mathsf{A}_1$ is similar. Finally, as explained in [HL16, Section 3.2.3], the general case can be reduced to type $\mathsf{A}_2$ by looking at the neighbors of $j$ in $\Delta$ one at a time. $\surd$

Denote $\widehat{\underline{w}}_0 = (i_s)_{s\in\mathbb{Z}}$. Note that $(i_0, i_{-1}, \ldots, i_{-l(w_0)+1})$ is a sink sequence for $Q_\xi$ that is also a reduced expression of $w_0$. We set

$$\mathbf{v}_{\mathscr{D}^{-1}} = (\mathbf{v}_{i_0}, \mathbf{v}_{i_{-1}}, \ldots, \mathbf{v}_{i_{-l(w_0)+1}})$$

to be the concatenation of the infinite sequences as defined in (3.9). Denote $\mathscr{D}^{-1}(\mathscr{S}^{<\xi}) = (\{\mathscr{D}^{-1}(\mathcal{M}^{<\xi}_{(i,p)})\}, Q^{<\xi}_{\mathrm{HL}}, \varnothing)$ the monoidal seed obtained by applying $\mathscr{D}^{-1}$ to $\mathscr{S}^{<\xi}$.

**Proposition 3.6.9.** *Let $\xi : \Delta_0 \to \mathbb{Z}$ be a height function and let $\mathbf{v}_{\mathscr{D}^{-1}}$ be the sequence of vertices of $Q^{<\xi}_{\mathrm{HL}}$ defined above. Then there is an isomorphism of monoidal seeds $\mu_{\mathbf{v}_{\mathscr{D}^{-1}}}(\mathscr{S}^{<\xi}) \to \mathscr{D}^{-1}(\mathscr{S}^{<\xi})$ sending a vertex of $Q^{<\xi}_{\mathrm{HL}}$ of the form $(i, \xi_i - 2r)$ to $(i^*, \xi_{i^*} - 2r)$.*

*Proof.* Let $\xi' = s^{-1}_{i_{-l(w_0)+1}} \cdots s^{-1}_{i_{-1}} s^{-1}_{i_0} \xi$. By repeated applications of Lemma 3.6.8, we obtain that $\mu_{\mathbf{v}_{\mathscr{D}^{-1}}}(\mathscr{S}^{<\xi})$ is isomorphic to $\mathscr{S}^{<\xi'}$. By Lemma 3.5.10 (and Remark 3.5.11), $Q^{<\xi'}_{\mathrm{HL}}$ is isomorphic to the quiver $Q^{[-\infty,-l(w_0)]}(\underline{w}_0)$. By the construction of the infinite sequence $\widehat{\underline{w}}_0$, there is an isomorphism from this last quiver to $Q^{[-\infty,0]}(\underline{w}_0)$ sending the $r$-th vertex on the row corresponding to $i \in \Delta_0$ (counted from right to left) to the $r$-th vertex on the row corresponding to $i^*$. By a second application of Lemma 3.5.10, we obtain the desired isomorphism on the level of the quivers of the monoidal seeds. Now, again by Lemma 3.6.8, notice that the module in $\mu_{\mathbf{v}_{\mathscr{D}^{-1}}}(\mathscr{S}^{<\xi})$ associated with a vertex of the form $(i, \xi_i - 2r)$ is $W^{(i)}_{r,q^{\xi_i-2r-2t_i}}$, where $t_i$ is the number of times the vertex $i$ appears in $(i_0, i_{-1}, \ldots, i_{-l(w_0)+1})$. Since $\underline{w}_0$ is adapted to $Q_\xi$, we have $t_i = (h + \xi_i - \xi_{i^*})/2$ by [Béd99, Corollary 2.20(a)], where $h$ is the Coxeter number of $\Delta$, so this KR-module is

$$W^{(i)}_{r,q^{\xi_{i^*}-2r-h}} \cong \mathscr{D}^{-1}\left(W^{(i^*)}_{r,q^{\xi_{i^*}-2r}}\right) = \mathscr{D}^{-1}\left(\mathcal{M}^{<\xi}_{(i^*,\xi_{i^*}-2r)}\right),$$

finishing the proof. $\surd$

**Remark 3.6.10.** We can iterate the result above as follows. Denote by $\mathbf{v}^*_{\mathscr{D}^{-1}}$ the sequence obtained from $\mathbf{v}_{\mathscr{D}^{-1}}$ by replacing each vertex of the form $(i, \xi_i - 2r)$ by $(i^*, \xi_{i^*} - 2r)$. If we define $\mathbf{v}_{\mathscr{D}^{-m}}$ as the concatenation of the first $m$ terms of the sequence

$$\mathbf{v}_{\mathscr{D}^{-1}}, \mathbf{v}^*_{\mathscr{D}^{-1}}, \mathbf{v}_{\mathscr{D}^{-1}}, \mathbf{v}^*_{\mathscr{D}^{-1}}, \ldots,$$

then $\mu_{\mathbf{v}_{\mathscr{D}^{-m}}}(\mathscr{S}^{<\xi})$ is isomorphic to $\mathscr{D}^{-m}(\mathscr{S}^{<\xi})$.

**Remark 3.6.11.** For $m \geq 1$, the computation of a module in $\mu_{\mathbf{v}_{\mathscr{D}^{-m}}}(\mathscr{S}^{<\xi})$ associated with a given vertex depends only on finitely many entries of the sequence $\mathbf{v}_{\mathscr{D}^{-m}}$. This leads to the following observation. For $r \geq 1$, define $\mathbf{v}^{\leq r}_{\mathscr{D}^{-m}}$ to be the finite subsequence of $\mathbf{v}_{\mathscr{D}^{-m}}$ consisting of the vertices of the form $(i, \xi_i - 2s)$ with $1 \leq s \leq r$. For a fixed vertex $(i, \xi_i - 2s)$, we can find $r \geq s$ large enough so that $\mathscr{D}^{-m}(\mathcal{M}^{<\xi}_{(i,\xi_i-2s)})$ is the module in $\mu_{\mathbf{v}^{\leq r}_{\mathscr{D}^{-m}}}(\mathscr{S}^{<\xi})$ corresponding to the vertex $(i, \xi_i - 2s)$ (if $m$ is even) or $(i^*, \xi_{i^*} - 2s)$ (if $m$ is odd).

**Remark 3.6.12.** Let $a \leq 0$ be such that every row of $Q^{[a,0]}(\underline{w}_0)$ has exactly $r+1$ vertices for some $r \geq 0$. Thus, $Q^{[a,0]}(\underline{w}_0)$ is isomorphic to the triangle product of $Q_\xi$ and a linear orientation of $\mathsf{A}_{r+1}$. Identifying $Q^{<\xi}_{\mathrm{HL}}$ with $Q^{[-\infty,0]}(\underline{w}_0)$ by Lemma 3.5.10, we can view the sequence $\mathbf{v}^{\leq r}_{\mathscr{D}^{-1}}$ from Remark 3.6.11 as a sequence of unfrozen vertices in $Q^{[a,0]}(\underline{w}_0)$. By Lemma 3.6.2, Proposition 3.6.3, and Theorem 3.6.5, $\mathbf{v}^{\leq r}_{\mathscr{D}^{-1}}$ is a maximal green sequence for the quiver obtained from $Q^{[a,0]}(\underline{w}_0)$ by removing the frozen vertices, and the associated permutation coincides with the one described in Proposition 3.6.9.

## 3.7 The inverse dualizing bimodules and their properties

In this section, we recall the definition and main properties of the relative inverse dualizing bimodule and the relative derived preprojective algebra. More details can be found in [Kel11a], [Yeu22], [Wu23b], and [KW23a]. In these references, the definitions are given for general dg categories, but we will present them in the generality that we shall later need.

We aim to provide the main tools to prove Theorem 3.8.7 and Corollary 3.8.8. The latter states that the relative Ginzburg dg algebras arising in our setting are proper and hence we can apply Theorem 3.4.6, yielding a $\Lambda$-cluster algebra structure.

Properness will be shown for a quasi-isomorphic algebra, namely a relative derived preprojective algebra, see Theorem 3.7.6. For such algebras, we observe in Lemma 3.7.7 that properness follows if the relative inverse dualizing bimodule is nilpotent. The latter property will be obtained through a reduction to a quiver of type $\mathsf{A}_1$ and the gluing result of Section 3.7.3, which is of independent interest.

*Notation:* For a small $k$-category $\mathcal{A}$, we denote by $\operatorname{Mod}\mathcal{A}$ its category of *right* modules and by $\operatorname{mod}\mathcal{A}$ the full subcategory of $\mathcal{A}$-modules $M : \mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$ such that $M(x)$ is finite-dimensional for all $x \in \mathcal{A}$. Let $\mathcal{D}(\mathcal{A})$, $\operatorname{per}(\mathcal{A})$, and $\operatorname{pvd}(\mathcal{A})$ denote the derived, the perfect derived, and the perfectly valued derived categories of $\mathcal{A}$. The suspension functors of these triangulated categories will be denoted by $\Sigma$. We call $\mathcal{A}$ *proper* if it is Hom-finite and equivalent to a category with finitely many objects. We denote by $\mathcal{A}^e = \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}$ the enveloping category of $\mathcal{A}$. We say $\mathcal{A}$ is *smooth* if the regular $\mathcal{A}$-bimodule $\mathcal{A}$ belongs to $\operatorname{per}(\mathcal{A}^e)$. When $\mathcal{A}$ is proper, smoothness is equivalent to finiteness of the global dimension since we assume $k$ is algebraically closed (hence perfect). If $\operatorname{per}(\mathcal{A}) = \operatorname{pvd}(\mathcal{A})$, we denote this category by $\mathcal{D}^b(\mathcal{A})$. This is the case, for example, if $\mathcal{A}$ is smooth and proper. We view $k$-algebras as $k$-categories with a single object. Morphisms of algebras are not necessarily unital.

### 3.7.1 The definition

Let $\mathcal{A}$ be a small $k$-category. For $M \in \mathcal{D}(\mathcal{A}^e)$, we define its bimodule dual $M^\vee = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(M, \mathcal{A}^e)$. We view it as an object of $\mathcal{D}(\mathcal{A}^e)$ (that is, as a *right* dg $\mathcal{A}^e$-module) by composing with the duality $(\mathcal{A}^e)^{\mathrm{op}} \to \mathcal{A}^e$ sending $f \otimes g$ to $g \otimes f$ for any morphisms $f, g$ in $\mathcal{A}$. We define the *absolute inverse dualizing bimodule* of $\mathcal{A}$ as the bimodule dual of the regular $\mathcal{A}$-bimodule:

$$\Omega_{\mathcal{A}} = \mathcal{A}^{\vee} = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^e) \in \mathcal{D}(\mathcal{A}^e).$$

**Proposition 3.7.1** ([Kel08, Lemma 4.1]). *Suppose $\mathcal{A}$ is smooth. For $L \in \mathrm{pvd}(\mathcal{A})$ and $M \in \mathcal{D}(\mathcal{A})$, there is a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M \otimes^{\mathbf{L}}_{\mathcal{A}} \Omega_{\mathcal{A}}, L).$$

**Remark 3.7.2.** If $\mathcal{A}$ is proper and smooth, then $\mathbb{S}_{\mathcal{A}} = - \otimes^{\mathbf{L}}_{\mathcal{A}} D\mathcal{A}^{\mathrm{op}}$ defines a Serre functor on the bounded derived category $\mathcal{D}^b(\mathcal{A})$, that is, $\mathbb{S}_{\mathcal{A}}$ is an autoequivalence such that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(L, M) \cong D\mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(M, \mathbb{S}_{\mathcal{A}}(L))$$

for $L, M \in \mathcal{D}^b(\mathcal{A})$. By the proposition above, there is an isomorphism of functors $\mathbb{S}_{\mathcal{A}}^{-1} \cong - \otimes^{\mathbf{L}}_{\mathcal{A}} \Omega_{\mathcal{A}}$ and, consequently, $\mathbb{S}_{\mathcal{A}} \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\Omega_{\mathcal{A}}, -)$.

Let $\mathcal{B} \to \mathcal{A}$ be a linear functor between small $k$-categories. Since we work over a field, we have canonical isomorphisms in the derived category of bimodules:

$$\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} = \mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} = \mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e.$$

We will use these identifications freely. There is a canonical map $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}$ which corresponds to the functor $\mathcal{B} \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^e, \mathcal{A}) = \mathcal{A}$, viewed as a morphism of $\mathcal{B}$-bimodules, under the derived tensor-Hom adjunction. Alternatively, it coincides with the composition

$$\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \longrightarrow H^0(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}) = \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A},$$

where the second map is induced by the composition law in the category $\mathcal{A}$. We define the *relative inverse dualizing bimodule* $\Omega_{\mathcal{A},\mathcal{B}} \in \mathcal{D}(\mathcal{A}^e)$ as the bimodule dual of the cone of the canonical map $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \longrightarrow \mathcal{A}$.

**Lemma 3.7.3** ([Kel11a, Lemma 2.9]). *If $M \in \mathrm{per}(\mathcal{B}^e)$, then there is a canonical isomorphism*

$$(M \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e)^{\vee} \cong M^{\vee} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e.$$

*In particular, if $\mathcal{B}$ is smooth, then $(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A})^{\vee} \cong \Omega_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e$.*

**Remark 3.7.4.** By the lemma above, if $\mathcal{B}$ is smooth, then $\Omega_{\mathcal{A},\mathcal{B}}$ is the cocone of a canonical morphism $\Omega_{\mathcal{A}} \to \Omega_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e$. Consequently, if $\mathcal{A}$ and $\mathcal{B}$ are both smooth and proper, then we can compute $M \otimes^{\mathbf{L}}_{\mathcal{A}} \Omega_{\mathcal{A},\mathcal{B}}$ as the cocone of a canonical map

$$\mathbb{S}_{\mathcal{A}}^{-1}(M) \longrightarrow \mathbb{S}_{\mathcal{B}}^{-1}(M|_{\mathcal{B}}) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}$$

and $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\Omega_{\mathcal{A},\mathcal{B}}, M)$ as the cone of a canonical map

$$\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, \mathbb{S}_{\mathcal{B}}(M|_{\mathcal{B}})) \longrightarrow \mathbb{S}_{\mathcal{A}}(M)$$

for any $M \in \mathcal{D}^b(\mathcal{A})$. The resulting complexes are again in $\mathcal{D}^b(\mathcal{A})$.

### 3.7.2 Relative derived preprojective algebras

We keep the notation from the previous section. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both smooth. Let $\omega \in \mathcal{D}(\mathcal{A}^e)$ be a cofibrant replacement of $\Sigma^2\Omega_{\mathcal{A},\mathcal{B}}$. The *relative derived preprojective category* $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$ is the tensor dg category

$$T_{\mathcal{A}}(\omega) = \bigoplus_{n\geq 0} \omega^{\otimes_{\mathcal{A}} n} = \mathcal{A} \oplus \omega \oplus (\omega \otimes_{\mathcal{A}} \omega) \oplus \cdots .$$

Its objects are the same as those of $\mathcal{A}$, and the morphism complexes are the evaluations of the dg $\mathcal{A}$-bimodule above. Composition is given by concatenation and the $\mathcal{A}$-bimodule structure. We remark that $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$ is defined up to quasi-equivalence. When $\mathcal{A}$ and $\mathcal{B}$ are algebras, we refer to $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$ as the associated *relative derived preprojective algebra.*

**Remark 3.7.5.** If $\mathcal{A}$ has finitely many objects, we can turn $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$ into a dg algebra by taking the direct sum of its morphism complexes. The resulting dg algebra will be called the *total dg algebra* of $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$. It is derived Morita equivalent to $\mathbf{\Pi}_3(\mathcal{A},\mathcal{B})$.

We now show how to realize a class of relative derived preprojective algebras as relative Ginzburg dg algebras.

Let $(Q', F)$ be a finite ice quiver. Let $A$ be the finite-dimensional quotient of $kQ'$ by an admissible ideal $I$, and suppose that $A$ has global dimension at most 2. Let $B = kF$ and assume $I \cap B = 0$, so that we can view $B$ as a subalgebra of $A$. Let $R \subseteq I$ be the union over all $i, j \in Q'_0$ of a set of representatives of a basis of $e_j(I/(IJ + JI))e_i$, where $J \subseteq kQ'$ denotes the ideal generated by the arrows. Suppose further that $R$ generates $I$ as an ideal. We define a new quiver $Q$ by adding additional arrows $\rho_r : j \to i$ for each relation $r \in R$ from $i$ to $j$. We equip it with the potential

$$W = \sum_{r\in R} r\rho_r.$$

In this way, we have an ice quiver with potential $(Q, F, W)$. Its relative Ginzburg dg algebra relates to the relative derived preprojective algebra of $A$ and $B$ as follows.

**Theorem 3.7.6.** *With the notation and hypotheses above, the relative derived preprojective algebra* $\mathbf{\Pi}_3(A, B)$ *is quasi-isomorphic to the non-completed relative Ginzburg dg algebra* $\mathbf{\Gamma}(Q, F, W)$.

*Proof.* The absolute case, that is, when $F$ is empty, is [Kel11a, Theorem 6.10] (see also [Kel18, Section 3]). The key step in the proof is to replace $A$ by a quasi-isomorphic cofibrant dg algebra $\tilde{A}$ for which the derived preprojective algebra can be computed using an explicit description of $\Omega_{\tilde{A}}$. This replacement is of the form $\tilde{A} = (k\tilde{Q}, d)$, where $\tilde{Q}$ is a finite non-positively graded quiver containing $Q'$. In the relative case, we can view $B$ as a dg subalgebra of $\tilde{A}$ via the inclusion $F \subseteq \tilde{Q}$, then one can similarly compute $\mathbf{\Pi}_3(\tilde{A}, B)$ using an explicit description of $\Omega_{\tilde{A},B}$ (see, for instance, [Wu23b, Section 3.6]). $\surd$

We conclude this section with a useful observation. We say that $M \in \mathcal{D}(\mathcal{A}^e)$ is *nilpotent* if there exists $n \geq 1$ such that $M^{\otimes^{\mathbf{L}}_{\mathcal{A}} n} = 0$ in $\mathcal{D}(\mathcal{A}^e)$.

**Lemma 3.7.7.** *Let $\mathcal{B} \to \mathcal{A}$ be a linear functor between small $k$-categories. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both smooth and proper. If $\Omega_{\mathcal{A},\mathcal{B}}$ is nilpotent, then $\mathbf{\Pi}_3(\mathcal{A}, \mathcal{B})$ is proper.*

*Proof.* Observe that $\Omega_{\mathcal{A},\mathcal{B}}$ and its derived tensor powers are in $\mathrm{pvd}(\mathcal{A}^e)$. The lemma then follows from the construction of $\mathbf{\Pi}_3(\mathcal{A}, \mathcal{B})$. $\surd$

### 3.7.3 A gluing result for relative inverse dualizing bimodules

Let $\mathcal{A}$ be a small $k$-category. Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ be strictly full subcategories which together contain all objects of $\mathcal{A}$. Denote $\mathcal{C} = \mathcal{A}_1 \cap \mathcal{A}_2$. We will assume that one of the following conditions holds:

$$\mathrm{Hom}(\mathcal{A}_1 \setminus \mathcal{C}, \mathcal{A}_2) = \mathrm{Hom}(\mathcal{A}_2 \setminus \mathcal{C}, \mathcal{A}_1) = 0 \tag{3.10}$$

or

$$\mathrm{Hom}(\mathcal{A}_2, \mathcal{A}_1 \setminus \mathcal{C}) = \mathrm{Hom}(\mathcal{A}_1, \mathcal{A}_2 \setminus \mathcal{C}) = 0. \tag{3.11}$$

We have the following “gluing” result.

**Proposition 3.7.8.** *Consider $\mathcal{A}_i, \mathcal{A}$ and $\mathcal{C}$ as above such that (3.10) or (3.11) holds. Let $\mathcal{B} \subseteq \mathcal{A}_1$ be a not necessarily full subcategory. If $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$ and $\mathcal{C}$ are smooth, then there is a distinguished triangle*

$$\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2^e} \mathcal{A}^e \longrightarrow \Omega_{\mathcal{A},\mathcal{B}} \longrightarrow \Omega_{\mathcal{A}_1,\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \longrightarrow \Sigma\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2^e} \mathcal{A}^e$$

*in $\mathcal{D}(\mathcal{A}^e)$.*

*Proof.* The inclusions $\mathcal{C} \subseteq \mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ induce the commutative square in $\mathcal{D}(\mathcal{A}^e)$ on the top left corner in the diagram below. It can be completed to the full diagram by the $3 \times 3$

lemma for triangulated categories (see [BBD82, Proposition 1.1.11]):

$$
\begin{tikzcd}
\mathcal{C} \otimes^{\mathbf{L}}_{\mathcal{C}^e} \mathcal{A}^e \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \arrow[r] \arrow[d, "f'"] & X \arrow[r] \arrow[d] & \Sigma\mathcal{C} \otimes^{\mathbf{L}}_{\mathcal{C}^e} \mathcal{A}^e \arrow[d] \\
\mathcal{A}_2 \otimes^{\mathbf{L}}_{\mathcal{A}_2^e} \mathcal{A}^e \arrow[r, "g'"] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \Sigma\mathcal{A}_2 \otimes^{\mathbf{L}}_{\mathcal{A}_2^e} \mathcal{A}^e \arrow[d] \\
V \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma V \\
\Sigma\mathcal{C} \otimes^{\mathbf{L}}_{\mathcal{C}^e} \mathcal{A}^e \arrow[r] & \Sigma\mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \arrow[r] & \Sigma X &
\end{tikzcd}
$$

Here, all squares are commutative, and the first three rows and columns are distinguished triangles in $\mathcal{D}(\mathcal{A}^e)$. We now prove that $Z \cong 0$ using the conditions (3.10) or (3.11).

First, let us assume that (3.10) holds. It suffices to show that $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} Z \cong 0$ for all $x \in \mathcal{A}$, where we write $x^\wedge$ for the representable $\mathcal{A}$-module $\mathcal{A}(-,x)$. We argue separately depending on whether $x$ is in $\mathcal{A}_1$ or $\mathcal{A}_2$. Suppose first that $x \in \mathcal{A}_1$. Let us start by showing that $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} f'$ is an isomorphism. Using that $\mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \cong \mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A}$, note that $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} f'$ identifies with the canonical map

$$x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A} \longrightarrow x^\wedge$$

induced by the $\mathcal{A}$-module action on $x^\wedge$. Since $\mathcal{A}_1$ is a full subcategory of $\mathcal{A}$ and $x \in \mathcal{A}_1$, the restriction of $x^\wedge$ to $\mathcal{D}(\mathcal{A}_1)$ is the representable $\mathcal{A}_1$-module $\mathcal{A}_1(-,x)$. Thus, $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A} \cong x^\wedge$ and the above map becomes the identity on $x^\wedge$, which is indeed an isomorphism. Let us now prove that $x \otimes^{\mathbf{L}}_{\mathcal{A}} f$ is an isomorphism. Consider the canonical map

$$\varphi : x^\wedge \otimes^{\mathbf{L}}_{\mathcal{C}} \mathcal{A}_2 \longrightarrow x^\wedge|_{\mathcal{A}_2}$$

in $\mathcal{D}(\mathcal{A}_2)$. Then $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} f$ identifies with $\varphi \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}$. Our problem is reduced to showing that $\varphi$ is an isomorphism. Since there are no morphisms from an object in $\mathcal{A}_2 \setminus \mathcal{C}$ to an object in $\mathcal{A}_1$ by the condition (3.10), we can find a projective resolution of $x^\wedge|_{\mathcal{A}_2}$ whose terms are direct sums of direct summands of representable modules of the form $\mathcal{A}_2(-,y)$ with $y \in \mathcal{C}$. This resolution can be taken to be finite because $\mathcal{A}_2$ is smooth, so it suffices to verify that the canonical map

$$\mathcal{A}_2(-,y) \otimes^{\mathbf{L}}_{\mathcal{C}} \mathcal{A}_2 \longrightarrow \mathcal{A}_2(-,y)$$

is an isomorphism for any $y \in \mathcal{C}$. This can be done as before. Now that we know that $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} f$ and $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} f'$ are isomorphisms, we have $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} V \cong x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} W \cong 0$, hence

$x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} Z \cong 0$, as desired. If $x \in \mathcal{A}_2$, a symmetric argument shows that $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} g$ and $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} g'$ are isomorphisms, so $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} X \cong x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} Y \cong 0$ and again we have $x^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} Z \cong 0$. This concludes the proof that $Z \cong 0$ when (3.10) holds.

On the other hand, if (3.11) holds, we can show instead that $Z \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{A}(x, -) \cong 0$ for all $x \in \mathcal{A}$. This can be proven as in the previous paragraph using a symmetric argument. Consequently, we have $Z \cong 0$ independently of which condition holds. We deduce that $V \cong W$ and $X \cong Y$.

As we did in the beginning, the inclusions $\mathcal{B} \subseteq \mathcal{A}_1 \subseteq \mathcal{A}$ induce the commutative square in $\mathcal{D}(\mathcal{A}^e)$ on the top left corner in the diagram below. The full diagram is obtained by applying the octahedral axiom:

$$\begin{tikzcd}
\mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e \arrow[r] \arrow[d, equal] & \mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d] & \Sigma\mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e \arrow[d, equal] \\
\mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & \Sigma\mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e \\
& W \arrow[r, equal] \arrow[d] & W \arrow[d] & \\
& \Sigma\mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1^e} \mathcal{A}^e \arrow[r] & \Sigma M &
\end{tikzcd}$$

Here, the squares are commutative, and the rows and columns are distinguished triangles in $\mathcal{D}(\mathcal{A}^e)$. Observe that the map $f'$ and its cone $W$ are the same as in the previous diagram. Applying the bimodule dual functor $(-)^\vee = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(-, \mathcal{A}^e)$ to the triangle in red, we get the distinguished triangle

$$W^\vee \longrightarrow N^\vee \longrightarrow M^\vee \longrightarrow \Sigma W^\vee,$$

which we now show is the triangle from the statement. By definition, $N^\vee = \Omega_{\mathcal{A},\mathcal{B}}$. If we write $C_1$ for the cone of the canonical map $\mathcal{A}_1 \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}_1 \to \mathcal{A}_1$, then

$$M^\vee \cong (C_1 \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A}^e)^\vee \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}_1^e}(C_1, \mathcal{A}_1^e) \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A}^e = \Omega_{\mathcal{A}_1,\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A}^e,$$

where the second isomorphism follows by Lemma 3.7.3. Note that $C_1$ is perfect because $\mathcal{A}_1$ and $\mathcal{B}$ are smooth, so the hypothesis of this lemma is satisfied. Finally, if $C_2$ is the cone of the canonical map $\mathcal{A}_2 \otimes^{\mathbf{L}}_{\mathcal{C}} \mathcal{A}_2 \to \mathcal{A}_2$, then

$$W^\vee \cong V^\vee \cong (C_2 \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}^e)^\vee \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}_2^e}(C_2, \mathcal{A}_2^e) \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}^e = \Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}^e,$$

where the first isomorphism is a consequence of $V \cong W$ and the third one follows again by Lemma 3.7.3 since $\mathcal{A}_2$ and $\mathcal{C}$ are smooth. This finishes the proof. $\surd$

It will be important to understand the (co)induction functor from $\mathcal{A}_i$ to $\mathcal{A}$ when (3.10) or (3.11) holds. For $i = 1, 2$ and an $\mathcal{A}_i$-module $M : \mathcal{A}_i^{\mathrm{op}} \to \operatorname{Mod} k$, we define an $\mathcal{A}$-module $\mu_i(M) : \mathcal{A}^{\mathrm{op}} \to \operatorname{Mod} k$ on objects by

$$\mu_i(M)(x) = \begin{cases} M(x) & \text{if } x \in \mathcal{A}_i, \\ 0 & \text{if } x \notin \mathcal{A}_i, \end{cases}$$

and on morphisms by $\mu_i(M)(f) = M(f)$ for any morphism $f$ in $\mathcal{A}_i$. This assignment is indeed functorial because either $\operatorname{Hom}(\mathcal{A} \setminus \mathcal{A}_i, \mathcal{A}_i) = 0$ or $\operatorname{Hom}(\mathcal{A}_i, \mathcal{A} \setminus \mathcal{A}_i) = 0$. It defines an exact and fully faithful functor $\mu_i : \operatorname{Mod} \mathcal{A}_i \to \operatorname{Mod} \mathcal{A}$ whose image are the $\mathcal{A}$-modules that vanish on $\mathcal{A} \setminus \mathcal{A}_i$.

**Lemma 3.7.9.** *Let $M$ be an $\mathcal{A}_i$-module. If (3.10) holds, then there are natural isomorphisms*

$$M \otimes^{\mathbf{L}}_{\mathcal{A}_i} \mathcal{A} \cong M \otimes_{\mathcal{A}_i} \mathcal{A} \cong \mu_i(M).$$

*Similarly, if (3.11) holds, then there are natural isomorphisms*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{A}_i}(\mathcal{A}, M) \cong \operatorname{Hom}_{\mathcal{A}_i}(\mathcal{A}, M) \cong \mu_i(M).$$

*Proof.* Suppose (3.10) holds. To prove the first isomorphism above, it suffices to show that $\mathcal{A}(x, -)$ is projective as a left $\mathcal{A}_i$-module for any $x \in \mathcal{A}$. Indeed, its restriction to $\mathcal{A}_i$ vanishes if $x \in \mathcal{A} \setminus \mathcal{A}_i$ by our hypothesis, and it is the representable left $\mathcal{A}_i$-module associated with $x$ if $x \in \mathcal{A}_i$ since $\mathcal{A}_i \subseteq \mathcal{A}$ is full. The same argument also shows that $M \otimes_{\mathcal{A}_i} \mathcal{A}$ vanishes on $\mathcal{A} \setminus \mathcal{A}_i$ and that its restriction to $\mathcal{A}_i$ is $M \otimes_{\mathcal{A}_i} \mathcal{A}_i = M$. We immediately conclude that $M \otimes_{\mathcal{A}_i} \mathcal{A} \cong \mu_i(M)$. The proof of the second statement is analogous. $\surd$

**Lemma 3.7.10.** *Suppose (3.10) or (3.11) holds. If $\mathcal{A}_1$ and $\mathcal{A}_2$ have finite global dimension, then the same holds for $\mathcal{A}$. In this case,* $\operatorname{gldim}(\mathcal{A}) \leq \max\{d_1, d_2\}$ *where* $d_i = \operatorname{gldim}(\mathcal{A}_i)$.

*Proof.* If (3.10) holds, then by Lemma 3.7.9 the exact functor $\mu_i$ is isomorphic to $- \otimes_{\mathcal{A}_i} \mathcal{A}$, which preserves projective modules since it is left adjoint to the restriction functor from $\mathcal{A}$ to $\mathcal{A}_i$. Similarly, if (3.11) holds, the same lemma implies that $\mu_i$ preserves injective modules. Together with our hypothesis that the projective and injective dimensions of any $\mathcal{A}_i$-module are at most $d_i$, we deduce that, to show that $\mathcal{A}$ has global dimension at most $\max\{d_1, d_2\}$, it suffices to prove that any $\mathcal{A}$-module $M$ is an extension of modules in the image of $\mu_1$ and $\mu_2$. Indeed, defining $M_1 = \mu_1(M|_{\mathcal{A}_1})$ and $M_2$ on objects by

$$M_2(x) = \begin{cases} M(x) & \text{if } x \in \mathcal{A}_2 \setminus \mathcal{C}, \\ 0 & \text{if } x \in \mathcal{A}_1, \end{cases}$$

with the action on morphisms coming from $M$, it is clear that $M$ is an extension of $M_2$ by $M_1$ (if (3.10) holds) or an extension of $M_1$ by $M_2$ (if (3.11) holds). $\surd$

### 3.7.4 Tensor products (of path algebras)

Let $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_2$ be small $k$-categories. Let $\mathcal{B}_2 \to \mathcal{A}_2$ be a linear functor. The next result describes the relative inverse dualizing bimodule $\Omega_{\mathcal{A},\mathcal{B}}$ for the induced functor $\mathcal{B} \to \mathcal{A}$, where

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \text{ and } \mathcal{B} = \mathcal{A}_1 \otimes \mathcal{B}_2.$$

**Lemma 3.7.11.** *If $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_2$ are smooth, then we have an isomorphism $\Omega_{\mathcal{A},\mathcal{B}} \cong \Omega_{\mathcal{A}_1} \otimes \Omega_{\mathcal{A}_2,\mathcal{B}_2}$ in $\mathcal{D}(\mathcal{A}^e)$.*

*Proof.* We can compute $\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} = \mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}$ by taking a cofibrant resolution of $\mathcal{B}$ in $\mathcal{D}(\mathcal{B}^e)$. Since $\mathcal{B} = \mathcal{A}_1 \otimes \mathcal{B}_2$, such a resolution can be obtained by tensoring a cofibrant resolution of $\mathcal{A}_1$ in $\mathcal{D}(\mathcal{A}_1^e)$ with a cofibrant resolution of $\mathcal{B}_2$ in $\mathcal{D}(\mathcal{B}_2^e)$. Now, for an $\mathcal{A}_1$-bimodule $M$ and a $\mathcal{B}_2$-bimodule $N$, we have an evident natural isomorphism

$$\mathcal{A} \otimes_{\mathcal{B}} (M \otimes N) \otimes_{\mathcal{B}} \mathcal{A} \cong M \otimes (\mathcal{A}_2 \otimes_{\mathcal{B}_2} N \otimes_{\mathcal{B}_2} \mathcal{A}_2)$$

of $\mathcal{A}$-bimodules. We deduce that

$$\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \cong \mathcal{A}_1 \otimes (\mathcal{A}_2 \otimes^{\mathbf{L}}_{\mathcal{B}_2} \mathcal{A}_2)$$

in $\mathcal{D}(\mathcal{A}^e)$. Under this identification, the canonical morphism $f : \mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$ becomes the tensor product of $\mathcal{A}_1$ with the canonical morphism $f_2 : \mathcal{A}_2 \otimes^{\mathbf{L}}_{\mathcal{B}_2} \mathcal{A}_2 \to \mathcal{A}_2$. This yields the isomorphism $\mathrm{cone}(f) \cong \mathcal{A}_1 \otimes \mathrm{cone}(f_2)$. By the definition of the inverse dualizing bimodules, we need to show that

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathrm{cone}(f), \mathcal{A}^e) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}_1^e}(\mathcal{A}_1, \mathcal{A}_1^e) \otimes \mathbf{R}\mathrm{Hom}_{\mathcal{A}_2}(\mathrm{cone}(f_2), \mathcal{A}_2^e).$$

This follows from the hypothesis and the natural isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(V \otimes W, \mathcal{A}^e) \xleftarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathcal{A}_1^e}(V, \mathcal{A}_1^e) \otimes \mathbf{R}\mathrm{Hom}_{\mathcal{A}_2^e}(W, \mathcal{A}_2^e)$$

in $\mathcal{D}(\mathcal{A}^e)$ for $V \in \mathrm{per}(\mathcal{A}_1^e)$ and $W \in \mathrm{per}(\mathcal{A}_2^e)$. $\surd$

Keeping the notations above, we have the following immediate corollary.

**Corollary 3.7.12.** *Suppose $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}_2$ are smooth. If $\Omega_{\mathcal{A}_1}$ or $\Omega_{\mathcal{A}_2,\mathcal{B}_2}$ is nilpotent, then the same holds for $\Omega_{\mathcal{A},\mathcal{B}}$.*

We are interested in the case of tensor products of finite-dimensional path algebras. Let us recall some of their properties. Let $Q$ and $Q'$ be two finite acyclic quivers. The tensor product $A = kQ \otimes kQ'$ is a finite-dimensional algebra of global dimension at most 2. By [Kel13, Section 5], it can be described as a quotient of the path algebra of $Q \otimes Q'$ (as in Section 3.6.2) by the commutation relations

$$(j,\alpha')(\alpha,i') - (\alpha,j')(i,\alpha') \tag{3.12}$$

for all arrows $\alpha : i \to j$ and $\alpha' : i' \to j'$ in $Q$ and $Q'$. For $i_0 \in Q'$, consider the subalgebra $B = kQ \otimes e_{i_0}(kQ')e_{i_0} \cong kQ$ of $A$. By Theorem 3.7.6, we deduce that $\mathbf{\Pi}_3(A,B)$ is quasi-isomorphic to the relative Ginzburg dg algebra of the triangle product $Q \boxtimes Q'$, where we freeze the full subquiver on the vertices of the form $(i,i_0)$ and endow it with the potential

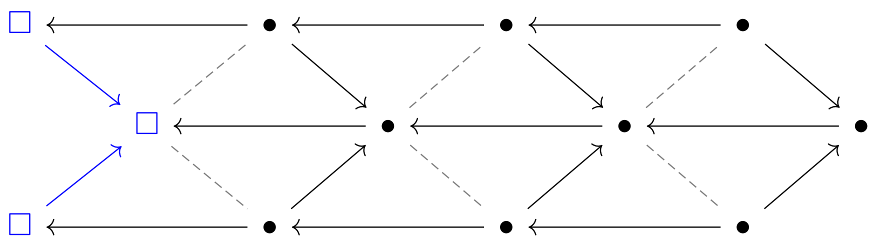
$$\sum (j,\alpha')(\alpha,i')(\alpha,\alpha')^{\mathrm{op}} - (\alpha,j')(i,\alpha')(\alpha,\alpha')^{\mathrm{op}}, \tag{3.13}$$

where the sum varies over all arrows $\alpha : i \to j$ in $Q$ and $\alpha' : i' \to j'$ in $Q'$.

**Example 3.7.13.** Consider the quivers

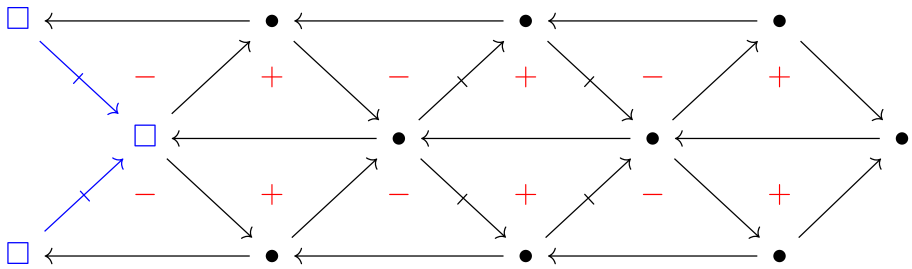
$$Q = \ 1 \longrightarrow 2 \longleftarrow 3 \quad \text{and} \quad Q' = \ \boxed{1} \longleftarrow 2 \longleftarrow 3 \longleftarrow 4.$$

The algebra $A = kQ \otimes kQ'$ is the path algebra of the quiver



modulo the relations which make all squares commute. For $i_0 = 1$, the subalgebra $B = kQ \otimes e_1(kQ')e_1$ consists of the path algebra of the frozen subquiver depicted in blue. Then

$\mathbf{\Pi}_3(A, B)$ is quasi-isomorphic to the relative Ginzburg dg algebra of the ice quiver




equipped with the potential consisting of the signed sum of all 3-cycles, where the signs are indicated in red.

**Remark 3.7.14.** The case we are interested in is when $Q$ is a Dynkin quiver, $Q'$ is a linear orientation of a Dynkin diagram of type $\mathsf{A}_n$, and $i_0$ is the unique sink of $Q'$, as in the example above. In this case, when viewing $\mathbf{\Pi}_3(A, B)$ as a relative Ginzburg dg algebra, we can (and will) ignore the signs in the potential (3.13). Indeed, we can always find an automorphism of the path algebra of $Q \boxtimes Q'$ that changes the signs of some of the arrows of the form $(\alpha, i')$ or $(\alpha, \alpha')^{\mathrm{op}}$, and sends the potential (3.13) to itself but where we replaced all minus signs by plus ones. For instance, in the example, it suffices to change the sign of the marked arrows.

### 3.7.5 Truncated Auslander algebras of Dynkin quivers

Let $Q$ be a Dynkin quiver. Let $\mathrm{ind}(kQ) \subseteq \mathrm{mod}\, kQ$ be the full subcategory of indecomposable modules. We say that a full subcategory $\mathcal{A} \subseteq \mathrm{ind}(kQ)$ is *left-closed* if it satisfies the following condition: for any $N \in \mathcal{A}$ and $M \in \mathrm{ind}(kQ)$ such that $\mathrm{Hom}_{kQ}(M, N) \neq 0$, we have $M \in \mathcal{A}$. Intuitively, $\mathcal{A}$ is obtained from $\mathrm{ind}(kQ)$ by removing indecomposable modules at the right side of the Auslander–Reiten quiver of $kQ$. If $X$ is the direct sum of a set of representatives for the isomorphism classes of indecomposable modules in $\mathcal{A}$, we call $\mathrm{End}_{kQ}(X)$ a *left-truncated Auslander algebra*. Note that $\mathcal{A}$ and $\mathrm{End}_{kQ}(X)$ are Morita equivalent.

We recall the following description of $\mathcal{A}$ as a mesh category. Let $\Gamma_{\mathcal{A}}$ be the Gabriel quiver of $\mathcal{A}$, which identifies with a subquiver of the Auslander–Reiten quiver $\Gamma$ of $kQ$. Since $\mathcal{A}$ is left-closed, any mesh in $\Gamma$ of the form

$$\begin{tikzcd}
 & y_1 \arrow[dr, "\alpha_1"] & \\
\tau(x) \arrow[ur, "\sigma(\alpha_1)"] \arrow[dr, "\sigma(\alpha_n)"'] & \vdots & x \\
 & y_n \arrow[ur, "\alpha_n"'] &
\end{tikzcd}$$

with $x \in \Gamma_{\mathcal{A}}$ is fully contained in $\Gamma_{\mathcal{A}}$. By [Rie80], $\mathcal{A}$ is equivalent to the quotient of the path category of $\Gamma_{\mathcal{A}}$ modulo the ideal generated by the *mesh relations*

$$r_x = \sum_{i=1}^{n} \sigma(\alpha_i)\alpha_i \tag{3.14}$$

for all $x \in \Gamma_{\mathcal{A}}$ that is not projective.

We will now present some lemmas concerning the global dimension of $\mathcal{A}$ and the action of a relative inverse dualizing bimodule on its derived category. Let us first introduce some notation.

For $M \in \mathcal{A}$, we write $M^{\wedge} = \mathrm{Hom}_{kQ}(-, M)|_{\mathcal{A}}$ and $M^{\vee} = D\mathrm{Hom}_{kQ}(M, -)|_{\mathcal{A}}$ for the corresponding representable and corepresentable $\mathcal{A}$-modules. Recall that $M^{\wedge}$ has a unique simple quotient $S_M$, and any simple $\mathcal{A}$-module is of this form. We denote by $\tau$ the Auslander–Reiten translation of $\operatorname{mod} kQ$ (and not $\mathcal{D}^b(kQ)$).

**Lemma 3.7.15.** *Suppose $\mathcal{A} \subseteq \mathrm{ind}(kQ)$ is full and left-closed. Then $\mathcal{A}$ has global dimension at most 2.*

*Proof.* For $N \in \mathcal{A}$, we can naturally view $S_N$ as a simple module over $\mathrm{ind}(kQ)$ and, as such, it admits a projective resolution

$$0 \longrightarrow \mathrm{Hom}_{kQ}(-, L) \longrightarrow \mathrm{Hom}_{kQ}(-, M) \longrightarrow \mathrm{Hom}_{kQ}(-, N) \longrightarrow S_N \longrightarrow 0$$

where $M \to N$ is a right minimal almost split map and $L$ is its kernel (see [ASS06, Theorem 6.11]). Since $\mathcal{A}$ is left-closed, the indecomposable direct summands of $M$ belong to $\mathcal{A}$, and the same holds for $L$ if it is nonzero. Therefore, the restriction of the resolution above to $\mathcal{A}$ is still a projective resolution. We obtain that every simple $\mathcal{A}$-module has projective dimension at most 2, hence $\mathcal{A}$ has global dimension at most 2. $\surd$

**Lemma 3.7.16** ([Wu23b, Proposition 8.6])**.** *Let $\mathcal{A} \subseteq \mathrm{ind}(kQ)$ be a full and left-closed subcategory. Denote by $\mathcal{B}$ the full subcategory of $\mathcal{A}$ consisting of the indecomposable projective modules in $\mathcal{A}$. For $M \in \mathcal{A}$, we have*

$$M^{\wedge} \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^2 \Omega_{\mathcal{A},\mathcal{B}} \cong \begin{cases} (\tau^{-1}M)^{\wedge} & \text{if } \tau^{-1}M \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We assume first that $\mathcal{A}$ contains all indecomposable projective $kQ$-modules. By Remark 3.7.4, $M^{\wedge} \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^2 \Omega_{\mathcal{A},\mathcal{B}}$ is the cocone of the canonical map

$$\Sigma^2 \mathbb{S}_{\mathcal{A}}^{-1}(M^{\wedge}) \longrightarrow \Sigma^2 \mathbb{S}_{\mathcal{B}}^{-1}(M^{\wedge}|_{\mathcal{B}}) \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}.$$

We adapt the argument given in [Wu23b, Proposition 8.6] to compute it. Choose an injective resolution

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow 0$$

of the $kQ$-module $M$. If we denote by $P_i = \mathbb{S}_{kQ}^{-1}(I_i)$ the projective module corresponding to $I_i$, then [Wu23b] shows that

$$\Sigma^2 \mathbb{S}_{\mathcal{B}}^{-1}(M^\wedge|_{\mathcal{B}}) \otimes_{\mathcal{B}}^{\mathbf{L}} \mathcal{A} \quad \cong \quad 0 \longrightarrow P_0^\wedge \longrightarrow P_1^\wedge \longrightarrow 0$$

where the complex on the right has $P_0^\wedge$ in degree $-2$. To compute $\Sigma^2 \mathbb{S}_{\mathcal{A}}^{-1}(M^\wedge)$, we need an injective resolution of $M^\wedge$. It is shown in [Wu23b] that the $\operatorname{ind}(kQ)$-module $\operatorname{Hom}_{kQ}(-, M)$ has the injective resolution

$$0 \longrightarrow \operatorname{Hom}(-, M) \longrightarrow D\operatorname{Hom}(P_0, -) \longrightarrow D\operatorname{Hom}(P_1, -) \longrightarrow D\operatorname{Hom}(\tau^{-1}M, -) \longrightarrow 0.$$

Notice that the restriction of the last term to $\mathcal{A}$ vanishes if $\tau^{-1}M \notin \mathcal{A}$ since $\mathcal{A}$ is left-closed. Restricting to $\mathcal{A}$ and forgetting the last term if $\tau^{-1}M \notin \mathcal{A}$, we obtain an injective resolution for $M^\wedge$, whence

$$\Sigma^2 \mathbb{S}_{\mathcal{A}}(M^\wedge) \quad \cong \quad 0 \longrightarrow P_0^\wedge \longrightarrow P_1^\wedge \longrightarrow N^\wedge \longrightarrow 0$$

where the complex on the right has $P_0^\wedge$ in degree $-2$, and where $N = \tau^{-1}M$ if $\tau^{-1}M \in \mathcal{A}$ or $N = 0$ otherwise. The canonical map then becomes the evident map between the complexes above, and we conclude that $M^\wedge \otimes_{\mathcal{A}}^{\mathbf{L}} \Sigma^2 \Omega_{\mathcal{A},\mathcal{B}} \cong N^\wedge$, as claimed.

Let us prove the general case. Assume that the left-closed subcategory $\mathcal{A} \subseteq \operatorname{ind}(kQ)$ does not contain an indecomposable projective module $P$ associated with a vertex $i \in Q_0$. Then any representation $M \in \mathcal{A}$ must vanish at the vertex $i$, otherwise we would have a nonzero map $P \to M$. Thus, we can view $\mathcal{A}$ as a subcategory of $\operatorname{ind}(kQ')$ where $Q'$ is the full subquiver of $Q$ whose vertices correspond to the indecomposable projective modules in $\mathcal{A}$. Note that $Q'$ is not necessarily connected, but its connected components are Dynkin quivers. We deduce that $\mathcal{A}$ is a disjoint union of subcategories where each one is equivalent to a left-closed subcategory of $\operatorname{ind}(kQ'')$ (for some Dynkin quiver $Q''$) that contains all indecomposable projective $kQ''$-modules. We conclude the proof by applying the previous argument to each component of this disjoint union. $\surd$

**Lemma 3.7.17.** *Let $\mathcal{A} \subseteq \operatorname{ind}(kQ)$ be a full and left-closed subcategory. Denote by $\mathcal{B}$ the full subcategory of $\mathcal{A}$ consisting of the indecomposable projective modules in $\mathcal{A}$. For $M \in \mathcal{A}$, we have*

$$\mathbf{R}\operatorname{Hom}_{\mathcal{A}}(\Sigma^2 \Omega_{\mathcal{A},\mathcal{B}}, S_M) \cong \begin{cases} S_{\tau M} & \text{if } M \notin \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By Remark 3.7.4, $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\Sigma^2\Omega_{\mathcal{A},\mathcal{B}}, S_M)$ is the cone of the canonical map

$$\Sigma^{-2}\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, \mathbb{S}_{\mathcal{B}}(S_M|_{\mathcal{B}})) \longrightarrow \Sigma^{-2}\mathbb{S}_{\mathcal{A}}(S_M).$$

If $M \notin \mathcal{B}$, that is, if $M$ is not a projective $kQ$-module, then $S_M|_{\mathcal{B}} = 0$ and we are reduced to computing the term on the right. If

$$0 \longrightarrow \tau M \longrightarrow L_1 \oplus \cdots \oplus L_m \longrightarrow M \longrightarrow 0$$

is the almost split sequence ending at $M$, where each $L_i$ is indecomposable, then $\tau M, L_i \in \mathcal{A}$ because $\mathcal{A}$ is left-closed. Thus, $S_M$ has a projective resolution given by

$$0 \longrightarrow (\tau M)^{\wedge} \longrightarrow L_1^{\wedge} \oplus \cdots \oplus L_m^{\wedge} \longrightarrow M^{\wedge} \longrightarrow S_M \longrightarrow 0$$

and we deduce that $\Sigma^{-2}\mathbb{S}_{\mathcal{A}}(S_M)$ is the complex

$$0 \longrightarrow (\tau M)^{\vee} \longrightarrow L_1^{\vee} \oplus \cdots \oplus L_m^{\vee} \longrightarrow M^{\vee} \longrightarrow 0,$$

where $(\tau M)^{\vee}$ is in degree zero. By [ASS06, Theorem 6.11], this complex is concentrated in degree zero, where its cohomology is $S_{\tau M}$. We conclude that $\Sigma^{-2}\mathbb{S}_{\mathcal{A}}(S_M) \cong S_{\tau M}$, as desired.

Now, assume $M \in \mathcal{B}$. Let $L_1 \oplus \cdots \oplus L_m$ be the radical of $M$, where each $L_i$ is indecomposable. By [ASS06, Theorem 6.11] and the fact that $\mathcal{A}$ is left-closed, a projective resolution of $S_M$ is given by

$$0 \longrightarrow L_1^{\wedge} \oplus \cdots \oplus L_m^{\wedge} \longrightarrow M^{\wedge} \longrightarrow S_M \longrightarrow 0.$$

Since $kQ$ is hereditary, each $L_i$ is again projective and the restriction of the resolution above to $\mathcal{B}$ is a projective resolution of $S_M|_{\mathcal{B}}$. Using that $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, -)$ preserves corepresentable modules, we see that both $\Sigma^{-2}\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{A}, \mathbb{S}_{\mathcal{B}}(S_M|_{\mathcal{B}}))$ and $\Sigma^{-2}\mathbb{S}_{\mathcal{A}}(S_M)$ are given by the complex

$$0 \longrightarrow L_1^{\vee} \oplus \cdots \oplus L_m^{\vee} \longrightarrow M^{\vee} \longrightarrow 0,$$

where $M^{\vee}$ is in degree 2, and the canonical map between them is the identity. We obtain that $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\Sigma^2\Omega_{\mathcal{A},\mathcal{B}}, S_M) = 0$, finishing the proof. $\surd$

## 3.8 Properness of the relative Ginzburg algebra

As in Section 3.5.3, let $\underline{w}_0$ be a reduced expression of the longest element and let $[a, b]$ be a finite integer interval. We let $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ and $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ denote the non-completed and completed relative Ginzburg dg algebras associated with the ice quiver with potential $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$. We pose the following conjecture.

**Conjecture 3.8.1.** The relative Ginzburg dg algebras $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ and $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ are proper.

Our main goal in this section is to prove the conjecture for when $\underline{w}_0$ is adapted to an orientation of $\Delta$. The key observation is that, for certain intervals $[a, b]$, $Q^{[a,b]}(\underline{w}_0)$ is a triangle product as in Section 3.6.2. In this so-called "regular" case, we reduce the conjecture to the case when $\Delta = \mathsf{A}_1$, for which we can easily verify it directly. For arbitrary intervals, we exploit the fact that $\underline{w}_0$ is adapted to decompose $Q^{[a,b]}(\underline{w}_0)$ into two subquivers: a regular one and another one isomorphic to the Gabriel quiver of a truncated Auslander algebra as in Section 3.7.5. With the aid of Proposition 3.7.8, we settle the conjecture in the adapted case. In fact, we prove the stronger Theorem 3.8.7, which we will need for proving Theorem 3.9.2. We conclude this section by exploring what happens when we apply braid and commutation moves to $\underline{w}_0$, following the ideas in [Con26, Section 6.1]. This allows us to obtain more cases of the conjecture.

## 3.8.1 The regular case

For $\ell \geq 1$, let $(Q^\circ_\ell, F^\circ_\ell)$ be the following ice quiver:

$$\boxed{1} \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow \ell.$$

We remark that it is isomorphic to $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0))$ if $\Delta = \mathsf{A}_1$ and $\ell = b - a + 1$. Define

$$A^\circ_\ell = kQ^\circ_\ell \text{ and } B^\circ_\ell = e_1 A^\circ_\ell e_1 \cong k \tag{3.15}$$

Note that $B^\circ_\ell$ is the subalgebra corresponding to the frozen vertex.

For $1 \leq x \leq y \leq \ell$, denote by $V_{[x,y]}$ the indecomposable $A^\circ_\ell$-module concentrated on the vertices $x, x+1, \ldots, y$ of $Q^\circ_\ell$. We also extend the notation for arbitrary $x, y \leq \ell$, where we interpret $V_{[x,y]}$ as $V_{[1,y]}$ if $x < 1$ and as zero if $y < 1$ or $x > y$. We write $V_{[x]}$ for $V_{[x,x]}$.

**Lemma 3.8.2.** *For $1 \leq x \leq y \leq \ell$, we have an isomorphism*

$$V_{[x,y]} \otimes^{\mathbf{L}}_{A^\circ_\ell} \Sigma\Omega_{A^\circ_\ell, B^\circ_\ell} \cong V_{[x-1,y-1]}$$

*in $\mathcal{D}^b(A^\circ_\ell)$.*

*Proof.* Denote $A = A^\circ_\ell$ and $B = B^\circ_\ell$. By Remark 3.7.4, $V_{[x,y]} \otimes^{\mathbf{L}}_A \Sigma\Omega_{A,B}$ is the cocone of the canonical map

$$\Sigma\mathbb{S}_A^{-1}(V_{[x,y]}) \longrightarrow \Sigma\mathbb{S}_B^{-1}(V_{[x,y]}\big|_B) \otimes^{\mathbf{L}}_B A.$$

If $x > 1$, then the restriction of $V_{[x,y]}$ to $B$ is zero, and we are reduced to computing the term on the left. Knowing that $\Sigma\mathbb{S}_A^{-1}$ is the inverse Auslander–Reiten translation on $\mathcal{D}^b(A)$ and that $A$ is the path algebra of a linearly oriented Dynkin quiver of type $\mathsf{A}_\ell$, one easily computes that $\Sigma\mathbb{S}_A^{-1}(V_{[x,y]}) \cong V_{[x-1,y-1]}$. For the case $x = 1$, we use the short exact sequence

$$0 \longrightarrow V_{[2,y]} \longrightarrow V_{[1,y]} \longrightarrow V_{[1]} \longrightarrow 0.$$

By the previous case, $V_{[2,y]} \otimes_A^{\mathbf{L}} \Sigma\Omega_{A,B} \cong V_{[1,y-1]} = V_{[x-1,y-1]}$, so it suffices to show that $V_{[1]} \otimes_A^{\mathbf{L}} \Sigma\Omega_{A,B} = 0$ to conclude the proof. Indeed, an easy computation shows that the canonical map above identifies with the identity map of $\Sigma V_{[1,\ell]}$, whose cocone is zero. $\checkmark$

**Corollary 3.8.3.** *The relative inverse dualizing bimodule $\Omega_{A_\ell^\circ,B_\ell^\circ}$ is nilpotent.*

*Proof.* Denote $A = A_\ell^\circ$ and $B = B_\ell^\circ$. We equivalently prove that $\Sigma\Omega_{A,B}$ is nilpotent. Let $G : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ be the functor $G = - \otimes_A^{\mathbf{L}} \Sigma\Omega_{A,B}$. By Lemma 3.8.2, $G^\ell$ vanishes on all simple $A$-modules, hence it vanishes on $\mathcal{D}^b(A)$. We conclude that $(\Sigma\Omega_{A,B})^{\otimes_A^{\mathbf{L}}\ell} \cong G^\ell(A) \cong 0$, as desired. $\checkmark$

**Corollary 3.8.4.** *Let $Q$ be a Dynkin quiver. Write $A = kQ \otimes A_\ell^\circ$ and $B = kQ \otimes B_\ell^\circ$, where $A_\ell^\circ$ and $B_\ell^\circ$ are as in (3.15). Then $\Omega_{A,B}$ is nilpotent. Consequently, $\mathbf{\Pi}_3(A,B)$ is proper.*

*Proof.* The first statement follows from Corollaries 3.7.12 and 3.8.3. The second follows from Lemma 3.7.7. $\checkmark$

For the next result, we denote by $\tau = \mathbb{S}_{kQ}\Sigma^{-1}$ the Auslander–Reiten translation on the bounded derived category of the Dynkin quiver $Q$.

**Corollary 3.8.5.** *Let $Q$ be a Dynkin quiver. Write $A = kQ \otimes A_\ell^\circ$ and $B = kQ \otimes B_\ell^\circ$, where $A_\ell^\circ$ and $B_\ell^\circ$ are as in (3.15). For $M \in \mathcal{D}^b(kQ)$ and $1 \leq x \leq y \leq \ell$, we have an isomorphism*

$$(M \otimes V_{[x,y]}) \otimes_A^{\mathbf{L}} \Sigma^2\Omega_{A,B} \cong \tau^{-1}M \otimes V_{[x-1,y-1]}.$$

*in $\mathcal{D}^b(A)$.*

*Proof.* By Lemma 3.7.11, we have $\Sigma^2\Omega_{A,B} \cong \Sigma\Omega_{kQ} \otimes \Sigma\Omega_{A_\ell^\circ,B_\ell^\circ}$. Therefore,

$$(M \otimes V_{[x,y]}) \otimes_A^{\mathbf{L}} \Sigma^2\Omega_{A,B} \cong (M \otimes_{kQ}^{\mathbf{L}} \Sigma\Omega_{kQ}) \otimes (V_{[x,y]} \otimes_{A_\ell^\circ}^{\mathbf{L}} \Sigma\Omega_{A_\ell^\circ,B_\ell^\circ}).$$

The result then follows by Remark 3.7.2 and Lemma 3.8.2. $\checkmark$

### 3.8.2 Coincidence of extensions in the adapted case

Let $(Q, F, W)$ and $(Q', F', W')$ be two ice quivers with potential. Let $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, F, W)$ and $\mathbf{\Gamma}' = \mathbf{\Gamma}(Q', F', W')$ be the associated relative Ginzburg dg algebras. Let $X \subseteq Q_0$ and $X' \subseteq Q'_0$ be subsets such that there is a bijection $f : X \to X'$. We say that $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ *agree on extensions* (along $f$) if we have an isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}}(e_s\mathbf{\Gamma}, e_t\mathbf{\Gamma}) \cong \mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}'}(e_{f(s)}\mathbf{\Gamma}', e_{f(t)}\mathbf{\Gamma}')$$

in the derived category of vector spaces for all $s, t \in X$.

**Remark 3.8.6.** In what follows, if $Q_0$ and $Q'_0$ are both defined as subsets of a common set, we will take $X = X' = Q_0 \cap Q'_0$ and $f = \mathrm{id}$. This applies in particular when we want to compare extensions of $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ and $\mathbf{\Gamma}^{[a',b']}(\underline{w}_0)$ for different values of $a$, $a'$, $b$ and $b'$.

The following is the main result of this subsection.

**Theorem 3.8.7.** *Suppose $\underline{w}_0$ is adapted to an orientation of $\Delta$. For any $a_1, a_2, b \in \mathbb{Z}$ with $a_1, a_2 \leq b$, the relative Ginzburg dg algebras $\mathbf{\Gamma}^{[a_1,b]}(\underline{w}_0)$ and $\mathbf{\Gamma}^{[a_2,b]}(\underline{w}_0)$ agree on extensions.*

**Corollary 3.8.8.** *Suppose $\underline{w}_0$ is adapted to an orientation of $\Delta$. For any $a, b \in \mathbb{Z}$ with $a \leq b$, both $\mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ and $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ are proper dg algebras.*

*Proof of Corollary 3.8.8.* By Lemma 3.5.10 (and Remark 3.5.11), all cycles in $W^{[a,b]}(\underline{w}_0)$ have length 3, so it is enough to prove that $\mathbf{\Gamma} = \mathbf{\Gamma}^{[a,b]}(\underline{w}_0)$ is proper by Lemma 3.3.3 and Remark 3.3.4. Note that

$$H^n(\mathbf{\Gamma}) \cong \bigoplus_{a \leq s,t \leq b} e_t H^n(\mathbf{\Gamma}) e_s \cong \bigoplus_{a \leq s,t \leq b} \mathrm{Ext}^n_{\mathbf{\Gamma}}(e_s\mathbf{\Gamma}, e_t\mathbf{\Gamma}).$$

for any $n \in \mathbb{Z}$. By Theorem 3.8.7, $\mathbf{\Gamma}$ agrees on extensions with $\mathbf{\Gamma}^{[a',b]}(\underline{w}_0)$ for any $a' \leq b$. Thus, by the formula above for the cohomology of $\mathbf{\Gamma}$, it suffices to find $a' \leq a$ such that $\mathbf{\Gamma}^{[a',b]}(\underline{w}_0)$ is proper. We take $a' = b - m \cdot 2l(w_0) + 1$ for some $m \geq 0$ such that $a' \leq a$. Since the length of the interval $[a', b]$ is a multiple of $2l(w_0)$ and $\underline{w}_0$ is adapted, all rows of $Q^{[a',b]}(\underline{w}_0)$ have the same number of vertices by [Béd99, Corollary 2.20]. Denote this number by $\ell$. By Lemma 3.5.10, we conclude that $Q^{[a',b]}(\underline{w}_0)$ is isomorphic to the triangle product between an orientation of $\Delta$ and $Q^\circ_\ell$. By the discussion in Section 3.7.4 and Corollary 3.8.4, $\mathbf{\Gamma}^{[a',b]}(\underline{w}_0)$ is proper, as desired. $\checkmark$

The proof of Theorem 3.8.7 consists of three main steps: Lemma 3.8.9, Proposition 3.8.10, and Proposition 3.8.14.

*Convention:* From now on, we fix an adapted expression $\underline{w}_0$ and omit it from the notation. We also fix $a, b \in \mathbb{Z}$ with $a \leq b$. We use throughout that $Q^{[-\infty,b]}$ is isomorphic to a subquiver of $Q_{\mathrm{HL}}$ by Lemma 3.5.10.

**Lemma 3.8.9.** *Let $N = l(w_0)$ be the length of $w_0$. If $b-a+1 < N$, then $\mathbf{\Gamma}^{[a,b]}$ and $\mathbf{\Gamma}^{[b-N+1,b]}$ agree on extensions.*

*Proof.* Denote by $\overline{Q}^{[a,b]}$ the quiver obtained from $Q^{[a,b]}$ by removing the horizontal arrows. Since $\underline{w}_0$ is adapted and since the length of the interval $[a,b]$ is at most $N$, $\overline{Q}^{[a,b]}$ is isomorphic to the Gabriel quiver of a left-truncated Auslander algebra $A$ in such a way that each row of $\overline{Q}^{[a,b]}$ corresponds to an orbit of the Auslander–Reiten translation in the associated left-closed subcategory $\mathcal{A}$ (see [Béd99, OS19a]). Note that $A$ is the quotient of $k\overline{Q}^{[a,b]}$ by the mesh relations. Let $B \subseteq A$ be the subalgebra that corresponds to the indecomposable projective modules in $\mathcal{A}$. By Theorem 3.7.6, $\mathbf{\Gamma} = \mathbf{\Gamma}^{[a,b]}$ is quasi-isomorphic to $\mathbf{\Pi}_3(A,B)$. Consequently,

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}}(e_s\mathbf{\Gamma}, e_t\mathbf{\Gamma}) \cong \bigoplus_{r\geq 0} \mathbf{R}\mathrm{Hom}_A(e_sA, F^r(e_tA))$$

for any $s,t \in [a,b]$, where $F$ is the functor $-\otimes^{\mathbf{L}}_A \Sigma^2\Omega_{A,B}$. By Lemma 3.7.16, $F(e_tA)$ is isomorphic to $e_{t^+}A$ if $t^+ \leq b$, or to zero otherwise. Moreover, we have

$$\mathbf{R}\mathrm{Hom}_A(e_sA, e_{t'}A) = \mathrm{Hom}_A(e_sA, e_{t'}A) \cong e_{t'}Ae_s,$$

where this last space is the quotient of the space of linear combinations of all paths from $s$ to $t'$ in $\overline{Q}^{[a,b]}$ by the mesh relations.

Now, denote $a' = b-N+1$. Replacing the interval $[a,b]$ by $[a',b]$ in the previous paragraph, we define similarly a quiver $\overline{Q}^{[a',b]}$ and algebras $B' \subseteq A'$ such that $\mathbf{\Gamma}' = \mathbf{\Gamma}^{[a',b]}$ is quasi-isomorphic to $\mathbf{\Pi}_3(A',B')$. The same argument also shows that $\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}'}(e_s\mathbf{\Gamma}', e_t\mathbf{\Gamma}')$ for $s,t \in [a',b]$ is concentrated in degree zero and its zeroth cohomology is the quotient by the mesh relations of the space of linear combinations of all paths from $s$ to a vertex $t' \in [t,b]$ in the same row of $t$. To conclude, note that $\overline{Q}^{[a,b]}$ is a full subquiver of $\overline{Q}^{[a',b]}$ and any path in $\overline{Q}^{[a',b]}$ whose start and end points are in $\overline{Q}^{[a,b]}$ is completely contained in $\overline{Q}^{[a,b]}$. It follows immediately from these properties and the discussion above that $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ agree on extensions. $\surd$

For $j \in \Delta_0$, let $m_j$ be the number of indices $s \in [a,b]$ such that $i_s = j$. We define the *regular width* of $Q^{[a,b]}$ as

$$\ell = \ell(Q^{[a,b]}) = \min\{m_j \mid j \in \Delta_0\}.$$

Suppose $\ell \geq 1$. For $\tilde{\ell} \geq \ell$, we define the *$\tilde{\ell}$-extension of* $Q^{[a,b]}$ as the full subquiver $Q^{[a,b],\tilde{\ell}}$ of $Q^{[-\infty,b]}$ whose vertices are the integers in $[a,b]$ and all $\tilde{\ell}-\ell$ vertices immediately to the left of a frozen vertex of $Q^{[a,b]}$. More precisely, for any $s \in F^{[a,b]}$, the quiver $Q^{[a,b],\tilde{\ell}}$ also includes the vertices $s_1, s_2, \ldots, s_{\tilde{\ell}-\ell}$, where $s_1 = s^-$ and $s_{j+1} = s_j^-$ for $1 \leq j < \tilde{\ell}-\ell$. We define the frozen subquiver $F^{[a,b],\tilde{\ell}} \subseteq Q^{[a,b],\tilde{\ell}}$ as the full subquiver of all vertices $s$ such that $s^-$ does not belong to $Q^{[a,b],\tilde{\ell}}$, and we equip $Q^{[a,b],\tilde{\ell}}$ with the potential $W^{[a,b],\tilde{\ell}}$ consisting of the sum of all 3-cycles. This defines an ice quiver with potential and a relative Ginzburg dg algebra $\mathbf{\Gamma}^{[a,b],\tilde{\ell}} = \mathbf{\Gamma}(Q^{[a,b],\tilde{\ell}}, F^{[a,b],\tilde{\ell}}, W^{[a,b],\tilde{\ell}})$. We remark that $\mathbf{\Gamma}^{[a,b]} = \mathbf{\Gamma}^{[a,b],\tilde{\ell}}$ for $\tilde{\ell} = \ell$.

The next step in the proof of Theorem 3.8.7 is the following.

**Proposition 3.8.10.** *Assume $\ell = \ell(Q^{[a,b]}) \geq 1$ and let $\tilde{\ell}_1, \tilde{\ell}_2 \geq \ell$ be two integers. Then $\mathbf{\Gamma}^{[a,b],\tilde{\ell}_1}$ and $\mathbf{\Gamma}^{[a,b],\tilde{\ell}_2}$ agree on extensions.*

The main tool to prove the result above will be Proposition 3.7.8. To apply it, we need to introduce some categories attached to $Q^{[a,b],\tilde{\ell}}$. We refer the reader to Example 3.8.12 for a concrete instance of the constructions we now define.

From now on, assume $\ell \geq 1$ and fix $\tilde{\ell} \geq \ell$. We define $Q_{\mathrm{reg}}^{[a,b],\tilde{\ell}}$ to be the full subquiver of $Q^{[a,b],\tilde{\ell}}$ which contains the $\tilde{\ell}$ leftmost vertices of each row. We call it the *regular part* of $Q^{[a,b],\tilde{\ell}}$, and its vertices are called *regular*. We define the *Auslander part* of $Q^{[a,b],\tilde{\ell}}$ to be its full subquiver consisting of the vertices that are not regular and the rightmost regular vertex of each row. We call these vertices the *Auslander vertices*. Note that the Auslander part does not depend on $\tilde{\ell}$, so we will denote it by $Q_{\mathrm{Aus}}^{[a,b]}$. We remark that the intersection of $Q_{\mathrm{reg}}^{[a,b],\tilde{\ell}}$ with $Q_{\mathrm{Aus}}^{[a,b]}$ is isomorphic to $F^{[a,b],\tilde{\ell}}$ and $F^{[a,b]}$.

**Lemma 3.8.11.** *Let $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$ be the subquiver of $Q_{\mathrm{Aus}}^{[a,b]}$ obtained by removing the horizontal arrows. There is an orientation $\vec{\Delta}^{[a,b]}$ of $\Delta$ and a left-closed full subcategory $\mathcal{A}_{\mathrm{Aus}}^{[a,b]} \subseteq \operatorname{ind}(k\vec{\Delta}^{[a,b]})$ whose Gabriel quiver is isomorphic to $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$. We can take this isomorphism in such a way that the row of $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$ associated with $j \in \Delta_0$ corresponds to the orbit under the Auslander–Reiten translation of the indecomposable projective $k\vec{\Delta}^{[a,b]}$-module associated with $j$.*

*Proof.* Let $\overline{Q}_{\mathrm{HL}}$ be the subquiver of $Q_{\mathrm{HL}}$ obtained by removing the arrows of the form $(i,p) \to (i,p-2)$. For a height function $\xi : \Delta_0 \to \mathbb{Z}$, the Auslander–Reiten quiver of $kQ_\xi$ is isomorphic to the full subquiver of $\overline{Q}_{\mathrm{HL}}$ with vertices of the form $(i, \xi(i) + 2r)$ for $i \in \Delta_0$ and $r \in \mathbb{Z}$ such that $0 \leq 2r < \xi(i^*) - \xi(i) + h$, where $h$ is the Coxeter number of $\Delta$ (see [FO21, Section 2.3]). Moreover, by fixing $i$ and varying $r$, we obtain the vertices corresponding to the orbit under the Auslander–Reiten translation of the indecomposable projective $kQ_\xi$-module associated with $i$. We remark that $\xi(i^*) - \xi(i) + h$ is an even integer.

Since $\underline{w}_0$ is adapted, we can identify $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$ as a full subquiver of $\overline{Q}_{\mathrm{HL}}$ by Lemma 3.5.10. Moreover, the full subquiver consisting of the leftmost vertices of each row in $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$ is connected and determines an orientation $\vec{\Delta}^{[a,b]}$ of $\Delta$. A similar statement holds if we replace "leftmost" by "rightmost". Choose a height function $\xi$ such that $Q_\xi = \vec{\Delta}^{[a,b]}$ and $(i, \xi(i))$ is the leftmost vertex of the $i$-th row of $\overline{Q}_{\mathrm{Aus}}^{[a,b]}$. Let $(i, t_i)$ denote the rightmost vertex of the $i$-th row. To prove the lemma, we need to show that $t_i < \xi(i^*)+h$. By construction, note that there is $i_0 \in \Delta_0$ whose corresponding row in $\overline{Q}_{\text{Aus}}^{[a,b]}$ has precisely one vertex. We may assume that $\xi(i_0) = 0$. If $d(i, i_0)$ denotes the minimal distance in $\Delta$ between $i$ and $i_0$, then notice that $t_i \leq d(i, i_0)$ since the subquiver of $\overline{Q}_{\text{Aus}}^{[a,b]}$ on the vertices $(i, t_i)$ is connected. Similarly, $\xi(i) \geq -d(i, i_0)$. Therefore, we have $t_i - \xi(i^*) \leq d(i, i_0) + d(i^*, i_0)$. By direct inspection, one sees that the inequality $d(i, j) + d(i^*, j) < h$ holds for all vertices $i$ and $j$ in an arbitrary Dynkin diagram $\Delta$ of type $\mathsf{ADE}$. This concludes the proof. $\surd$

We note that the frozen subquiver $F^{[a,b],\tilde{\ell}}$ is isomorphic to $\vec{\Delta}^{[a,b]}$ in the lemma above. Since $\underline{w}_0$ is adapted, it follows from Lemma 3.5.10 that $Q_{\text{reg}}^{[a,b],\tilde{\ell}}$ and is canonically isomorphic to the triangle product $\vec{\Delta}^{[a,b]} \boxtimes Q^{\circ}_{\tilde{\ell}}$, where $Q^{\circ}_{\tilde{\ell}}$ is the quiver from Section 3.8.1. This canonical isomorphism sends a row associated with a vertex $j \in \Delta_0$ to the row in the triangle product corresponding to $j$. We denote by $Q_{\text{tens}}^{[a,b],\tilde{\ell}}$ the subquiver of $Q_{\text{reg}}^{[a,b],\tilde{\ell}}$ that corresponds to $\vec{\Delta}^{[a,b]} \otimes Q^{\circ}_{\tilde{\ell}}$ under this isomorphism.

Let $Q_{\mathcal{A}}$ be the subquiver of $Q^{[a,b],\tilde{\ell}}$ that contains all vertices and whose arrows are precisely those in $\overline{Q}_{\text{Aus}}^{[a,b]}$ and $Q_{\text{tens}}^{[a,b],\tilde{\ell}}$. We define $\mathcal{A}$ as the quotient of the path category of $Q_{\mathcal{A}}$ by the mesh relations on the subquiver $\overline{Q}_{\text{Aus}}^{[a,b]}$ (coming from Lemma 3.8.11 and (3.14)) and by the commutation relations on the subquiver $Q_{\text{tens}}^{[a,b],\tilde{\ell}}$ (coming from (3.12)). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the full subcategories on the regular and Auslander vertices, respectively, and denote by $\mathcal{C}$ their intersection. Note that

$$\operatorname{Hom}(\mathcal{A}_1 \setminus \mathcal{C}, \mathcal{A}_2) = \operatorname{Hom}(\mathcal{A}_2 \setminus \mathcal{C}, \mathcal{A}_1) = 0$$

because there are no arrows in $Q_{\mathcal{A}}$ from a vertex in $Q_{\mathcal{A}} \setminus Q_{\text{tens}}^{[a,b],\tilde{\ell}} \cap \overline{Q}_{\text{Aus}}^{[a,b]}$ to a vertex in $Q_{\text{tens}}^{[a,b],\tilde{\ell}} \cap \overline{Q}_{\text{Aus}}^{[a,b]}$. We also deduce that $\mathcal{A}_1$ is Morita equivalent to $k\vec{\Delta}^{[a,b]} \otimes kQ^{\circ}_{\tilde{\ell}}$ (by Section 3.7.4), and $\mathcal{A}_2$ is equivalent to the category $\mathcal{A}_{\text{Aus}}^{[a,b]}$ in Lemma 3.8.11 (by Section 3.7.5). Finally, we define $\mathcal{B} \subseteq \mathcal{A}_1$ to be the full subcategory on the vertices of $F^{[a,b],\tilde{\ell}}$. We remark that the hypotheses of Proposition 3.7.8 are satisfied.

**Example 3.8.12.** Suppose $\Delta$ is of type $\mathsf{A}_3$ and take $\underline{w}_0 = (3, 1, 2, 3, 1, 2)$. For $[a, b] = [-1, 6]$, the ice quiver $(Q^{[-1,6]}, F^{[-1,6]})$ is given by

$$\begin{tikzcd}
 & & \boxed{2} \arrow[dr] & & 5 \arrow[ll] \arrow[dr] & \\
 & \boxed{0} \arrow[ur] \arrow[dr] & & 3 \arrow[ll] \arrow[ur] \arrow[dr] & & 6 \arrow[ll] \\
\boxed{-1} \arrow[ur] & & 1 \arrow[ll] \arrow[ur] & & 4 \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

The regular width is $\ell = 2$. The red subquiver is $Q_{\text{reg}}^{[-1,6],\ell} \cap Q_{\text{Aus}}^{[-1,6]}$. On its left we have $Q_{\text{reg}}^{[-1,6],\ell}$, consisting of the vertices $-1$, $0$, $1$, $2$, $3$ and $5$. On its right we have $Q_{\text{Aus}}^{[-1,6]}$, consisting of the vertices 1, 3, 4, 5 and 6. The subquiver $Q_{\mathrm{tens}}^{[-1,6],\ell}$ is obtained from $Q_{\mathrm{reg}}^{[-1,6],\ell}$ by removing the arrows $0 \to 1$ and $2 \to 3$, and $\overline{Q}_{\mathrm{Aus}}^{[-1,6]}$ is obtained from $Q_{\mathrm{Aus}}^{[-1,6]}$ by removing the arrows $4 \to 1$ and $6 \to 3$. The quiver $Q_{\mathcal{A}}$ is given by

$$\begin{tikzcd}
 & & {\color{blue}2} \arrow[dr, dash, dashed] & & {\color{red}5} \arrow[ll] \arrow[dr] & \\
 & {\color{blue}0} \arrow[ur, blue] \arrow[dr, dash, dashed] & & {\color{red}3} \arrow[ll] \arrow[ur, red] \arrow[dr] \arrow[rr, dash, dashed] & & 6 \\
{\color{blue}-1} \arrow[ur, blue] & & {\color{red}1} \arrow[ll] \arrow[ur, red] \arrow[rr, dash, dashed] & & 4 \arrow[ur] &
\end{tikzcd}$$

The category $\mathcal{A}$ is the path category of $Q_{\mathcal{A}}$ modulo the relations indicated by the dashed lines: the three squares (anti)commute and the composition $1 \to 3 \to 4$ vanishes. The subcategory $\mathcal{B} \subseteq \mathcal{A}$ has the blue vertices as objects, while the objects of $\mathcal{C} \subseteq \mathcal{A}$ are the red ones.

Below, we depict the ice quiver $(Q^{[-1,6],\tilde{\ell}}, F^{[-1,6],\tilde{\ell}})$ for $\tilde{\ell} = 3$:

$$\begin{tikzcd}
 & & \boxed{-2} \arrow[dr] & & 2 \arrow[ll] \arrow[dr] & & {\color{red}5} \arrow[ll] \arrow[dr] & \\
 & \boxed{-3} \arrow[ur, blue] \arrow[dr] & & 0 \arrow[ll] \arrow[ur] \arrow[dr] & & {\color{red}3} \arrow[ll] \arrow[ur, red] \arrow[dr] & & 6 \arrow[ll] \\
\boxed{-4} \arrow[ur, blue] & & -1 \arrow[ll] \arrow[ur] & & {\color{red}1} \arrow[ll] \arrow[ur, red] & & 4 \arrow[ll] \arrow[ur] &
\end{tikzcd}$$

The associated quivers and categories have a similar description.

**Lemma 3.8.13.** *The total dg algebra of $\mathbf{\Pi}_3(\mathcal{A}, \mathcal{B})$ is quasi-isomorphic to $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}$.*

*Proof.* By Lemmas 3.7.10 and 3.7.15, $\mathcal{A}$ has global dimension at most 2. The statement then follows from Theorem 3.7.6 and the description of the categories $\mathcal{A}$ and $\mathcal{B}$ in terms of quivers and relations. See also Remark 3.7.14. $\checkmark$

We are ready to prove Proposition 3.8.10.

*Proof of Proposition 3.8.10.* Let $\tilde{\ell} \geq \ell$. Let $\mathbf{\Gamma} = \mathbf{\Gamma}^{[a,b],\tilde{\ell}}$ and $\mathbf{\Pi} = \mathbf{\Pi}_3(\mathcal{A}, \mathcal{B})$ be as above. By Lemma 3.8.13 and the construction of $\mathbf{\Pi}$ as a tensor dg category over $\mathcal{A}$, we have

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}}(e_s\mathbf{\Gamma}, e_t\mathbf{\Gamma}) \cong \mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}}(s^\wedge, t^\wedge) \cong \bigoplus_{r \geq 0} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(s^\wedge, F^r(t^\wedge))$$

for any two vertices $s, t \in Q_0^{[-\infty,b]}$ that belong to $Q^{[a,b],\tilde{\ell}}$, where $F$ denotes the functor $-\otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^2 \Omega_{\mathcal{A},\mathcal{B}}$ and we write $x^\wedge = \mathcal{A}(-, x)$ for a vertex $x$ in $Q^{[a,b],\tilde{\ell}}$. To prove the proposition, it suffices to explicitly compute the complex above for fixed $s, t \in Q_0^{[-\infty,b]}$ and show that it does not depend on $\tilde{\ell}$. We will describe $F^r(t^\wedge)$ for this computation, but first, we need some notation.

Fix a vertex $t \in Q_0^{[-\infty,b]}$ that belongs to $Q^{[a,b],\tilde{\ell}}$. If $t$ is not Auslander, let $d \geq 2$ be the integer such that $t$ is the $d$-th regular vertex of its row, counted from right to left. If $t$ is Auslander, we set $d = 1$. We define $t_{\mathrm{Aus}}$ as the smallest Auslander vertex such that $t \leq t_{\mathrm{Aus}}$ and $i_t = i_{t_{\mathrm{Aus}}}$. Let $M_t \in \mathrm{ind}(k\vec{\Delta}^{[a,b]})$ be the indecomposable representation which corresponds to $t_{\mathrm{Aus}}$ under the isomorphism in Lemma 3.8.11. For an integer $s$, we will denote $s^+$ by $\tau^{-1}(s)$. Let $r_0 \geq 0$ be the largest integer such that $\tau^{-r_0}(t_{\mathrm{Aus}})$ is in $Q^{[a,b]}_{\mathrm{Aus}}$. We remark that $d$, $t_{\mathrm{Aus}}$, $M_t$ and $r_0$ do not depend on $\tilde{\ell}$.

Recall that the subcategory $\mathcal{A}_1 \subseteq \mathcal{A}$ is Morita equivalent to $k\vec{\Delta}^{[a,b]} \otimes kQ^\circ_{\tilde{\ell}}$. More precisely, we can view any $k\vec{\Delta}^{[a,b]} \otimes kQ^\circ_{\tilde{\ell}}$-module as a representation of the quiver $Q^{[a,b]}_{\mathrm{tens}}$, which can in turn be viewed as an $\mathcal{A}_1$-module by the description of $\mathcal{A}_1$ as a quotient of a path category. Additionally, we identify $\mathcal{A}_1$-modules with $\mathcal{A}$-modules vanishing on $\mathcal{A} \setminus \mathcal{A}_1$ via the functor $\mu_1$, as explained in Section 3.7.3. In what follows, we use these identifications to view objects in $\mathcal{D}^b(k\vec{\Delta}^{[a,b]} \otimes kQ^\circ_{\tilde{\ell}})$ as objects in $\mathcal{D}^b(\mathcal{A})$. We also denote by $\tau$ the Auslander–Reiten translation on $\mathcal{D}^b(k\vec{\Delta}^{[a,b]})$ and by $V_{[x,y]}$ the indecomposable $kQ^\circ_{\tilde{\ell}}$-module from Section 3.8.1.

We will show by induction on $r \geq 0$ that $F^r(t^\wedge)$ admits a filtration

$$\begin{tikzcd}[column sep=small]
0 = N_0 \arrow[rr] & & N_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & N_{n-1} \arrow[rr] & & N_n \arrow[dl] \arrow[rr] & & N_{n+1} = F^r(t^\wedge) \arrow[dl] \\
& X_1 \arrow[ul, dashed] & & & & X_2 \arrow[ul, dashed] & & X_n \arrow[ul, dashed] &
\end{tikzcd}$$

in $\mathcal{D}^b(\mathcal{A})$ whose set $\mathcal{X}_r = \{X_1, \ldots, X_n\}$ of successive quotients is the following. If $r \leq r_0$, then $\mathcal{X}_r$ consists of the objects

$$\tau^{-r}(t_{\mathrm{Aus}})^\wedge, \quad \tau^{-r}M_t \otimes V_{[\tilde{\ell}-x]}\ (1 \leq x \leq r), \quad \text{and} \quad \tau^{-r}M_t \otimes V_{[\tilde{\ell}-r-d+1,\tilde{\ell}-r-1]}.$$

Otherwise, if $r > r_0$, it consists of the objects

$$\tau^{-r}M_t \otimes V_{[\tilde{\ell}-x]}\ (r - r_0 \leq x \leq r) \quad \text{and} \quad \tau^{-r}M_t \otimes V_{[\tilde{\ell}-r-d+1,\tilde{\ell}-r-1]}.$$

Let us start with the base case $r = 0$. If $t$ is Auslander, then $t = t_{\mathrm{Aus}}$, $d = 1$ and $M_t \otimes V_{[\tilde{\ell}-d+1,\tilde{\ell}-1]} = 0$, from where our claim easily follows. If $t$ is not Auslander, note that $t_{\mathrm{Aus}}$ is also regular (and thus belongs to $\mathcal{C}$), so $M_t$ is the indecomposable projective $k\vec{\Delta}^{[a,b]}$-module associated with $i_t \in \Delta_0$. Therefore, since there are no morphisms from $\mathcal{A} \setminus \mathcal{A}_1$ to $\mathcal{A}$ and from $\mathcal{A} \setminus \mathcal{C}$ to $\mathcal{C}$, it is not hard to see that $t^\wedge \cong M_t \otimes V_{[\tilde{\ell}-d+1,\tilde{\ell}]}$ and

$t^\wedge_{\mathrm{Aus}} \cong M_t \otimes V_{[\tilde{\ell}]}$. In particular, we have an exact sequence

$$0 \longrightarrow t^\wedge_{\mathrm{Aus}} \longrightarrow t^\wedge \longrightarrow M_t \otimes V_{[\tilde{\ell}-d+1,\tilde{\ell}-1]} \longrightarrow 0$$

in $\operatorname{mod} \mathcal{A}$, whence our claim.

Now we assume that our claim holds for some $r \geq 0$ and we prove that it also holds for $r+1$. We first compute $F(X)$ for $X \in \mathcal{X}_r$. We use Proposition 3.7.8 to obtain a distinguished triangle

$$X \otimes^{\mathbf{L}}_{\mathcal{A}_2} \Sigma^2\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A} \to F(X) \to X \otimes^{\mathbf{L}}_{\mathcal{A}_1} \Sigma^2\Omega_{\mathcal{A}_1,\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A} \to \Sigma X \otimes^{\mathbf{L}}_{\mathcal{A}_2} \Sigma^2\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}$$

in $\mathcal{D}^b(\mathcal{A})$. If $X$ is of the form $\tau^{-r}M_t \otimes V_{[x,y]}$ with $x, y < \tilde{\ell}$, note that $X|_{\mathcal{A}_2} = 0$, so the triangle above gives

$$F(X) \cong X \otimes^{\mathbf{L}}_{\mathcal{A}_1} \Sigma^2\Omega_{\mathcal{A}_1,\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{A}_1} \mathcal{A} \cong \tau^{-(r+1)}M_t \otimes V_{[x-1,y-1]},$$

where the second isomorphism follows from Corollary 3.8.5 and Lemma 3.7.9. This shows that $F(X) \in \mathcal{X}_{r+1}$. This recovers all objects of $\mathcal{X}_{r+1}$ except for $\tau^{-(r+1)}M_t \otimes V_{[\tilde{\ell}-1]}$ (if $r \leq r_0$) and $\tau^{-(r+1)}(t_{\mathrm{Aus}})^\wedge$ (if $r < r_0$). When $r \leq r_0$, it remains to compute $F(X)$ for $X = \tau^{-r}(t_{\mathrm{Aus}})^\wedge$. In this case, observe that the vertex $\tau^{-r}(t_{\mathrm{Aus}})$ corresponds to $\tau^{-r}M_t \in \operatorname{ind}(k\vec{\Delta}^{[a,b]})$ under the isomorphism in Lemma 3.8.11. We then have $X|_{\mathcal{A}_1} \cong \tau^{-r}M_t \otimes V_{[\tilde{\ell}]}$, hence the distinguished triangle above is isomorphic to

$$X \otimes^{\mathbf{L}}_{\mathcal{A}_2} \Sigma^2\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A} \longrightarrow F(X) \longrightarrow \tau^{-(r+1)}M_t \otimes V_{[\tilde{\ell}-1]} \longrightarrow \Sigma X \otimes^{\mathbf{L}}_{\mathcal{A}_2} \Sigma^2\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A}$$

where we computed the third term by Corollary 3.8.5 and Lemma 3.7.9 again. Finally, Lemma 3.7.16 implies that

$$X \otimes^{\mathbf{L}}_{\mathcal{A}_2} \Sigma^2\Omega_{\mathcal{A}_2,\mathcal{C}} \otimes^{\mathbf{L}}_{\mathcal{A}_2} \mathcal{A} \cong \begin{cases} \tau^{-(r+1)}(t_{\mathrm{Aus}})^\wedge & \text{if } r < r_0, \\ 0 & \text{if } r = r_0. \end{cases}$$

This gives a filtration of $F(X)$ by the remaining objects of $\mathcal{X}_{r+1}$ that we did not recover before. Since $F$ is a triangle functor, we conclude that the image under $F$ of the filtration of $F^r(t^\wedge)$ by $\mathcal{X}_r$ gives rise to a filtration of $F^{r+1}(t^\wedge)$ by $\mathcal{X}_{r+1}$, as desired. This finishes the induction.

For $s \in Q_0^{[a,b],\tilde{\ell}}$, we can finally compute $C^r_{s,t} = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(s^\wedge, F^r(t^\wedge))$. As a first step, note that all objects in $\mathcal{X}_r$ are concentrated in a single cohomological degree $m$, where $m \leq 0$ is the integer such that $\Sigma^m\tau^{-r}M_t$ is concentrated in degree zero. Observe that

$m = 0$ if $r \leq r_0$. Therefore, $C^r_{s,t}$ is concentrated in degree $m$ and we have an isomorphism

$$C^r_{s,t} \cong \bigoplus_{X \in \mathcal{X}_r} \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(s^\wedge, X)$$

in the derived category of vector spaces. Viewing $X$ as a functor from the opposite category of $\mathcal{A}$ to the category of complexes of vector spaces, each term in the direct sum is given by the evaluation $X(s)$. By the definition of $\mathcal{X}_r$, we thus get the following description of $C^r_{s,t}$. First, assume $s$ is Auslander. If $r > r_0$, then $C^r_{s,t} = 0$. If $r \leq r_0$, then $C^r_{s,t}$ is concentrated in degree 0 and

$$\dim H^0(C^r_{s,t}) = \dim \mathcal{A}_2(s, \tau^{-r}(t_{\mathrm{Aus}})).$$

Now, suppose $s$ is not Auslander. Let $d_s \geq 2$ be the integer such that $s$ is the $d_s$-th regular vertex of its row, counted from right to left. Then $C^r_{s,t}$ can only be nonzero if

$$\tilde{\ell} - r - d + 1 \leq \tilde{\ell} - d_s + 1 \leq \tilde{\ell} - \max\{1, r - r_0\} \iff \max\{1, r - r_0\} + 1 \leq d_s \leq r + d,$$

and, in this case, it is concentrated in degree $m$ and the dimension of $H^m(C^r_{s,t})$ is the multiplicity of the simple associated with $i_s \in \Delta_0$ in the $k\vec{\Delta}^{[a,b]}$-module $\Sigma^m \tau^{-r} M_t$. As we can see, in any case, $C^r_{s,t}$ admits a uniform description independent of $\tilde{\ell}$, concluding the proof. $\checkmark$

The final step for the proof of Theorem 3.8.7 is to truncate $Q^{[a,b],\tilde{\ell}}$ into a certain triangle product. To do so, we need to define another decomposition of $Q^{[a,b],\tilde{\ell}}$ similar to the one we used for the proof of Proposition 3.8.10, but we will swap "left" and "right".

As before, assume $\ell \geq 1$ and fix $\tilde{\ell} \geq \ell$. We define $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ to be the full subquiver of $Q^{[a,b],\tilde{\ell}}$ which contains the $\tilde{\ell}$ *rightmost* vertices of each row. We call it the *trans-regular part* of $Q^{[a,b],\tilde{\ell}}$, and its vertices are called *trans-regular*. We let $F^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}} \subseteq Q^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ be the full subquiver consisting of the leftmost vertex of each row, and consider the potential $W^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ consisting of the sum of all 3-cycles. This defines a relative Ginzburg dg algebra $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}} = (Q^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}, F^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}, W^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}})$. Our next goal will be to prove the following result.

**Proposition 3.8.14.** *Assume $\ell = \ell(Q^{[a,b]}) \geq 1$ and take $\tilde{\ell} \geq \ell$. Then $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}$ and $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ agree on extensions.*

Similar to the previous case, $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ is canonically isomorphic to the triangle product $F^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}} \boxtimes Q^\circ_{\tilde{\ell}}$ in such a way that the row associated with a vertex $j \in \Delta_0$ corresponds to the row in the triangle product associated to $j$. We denote by $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-tens}}$ the subquiver of $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$ that corresponds to $F^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}} \otimes Q^\circ_{\tilde{\ell}}$ under this isomorphism.

We define the *trans-Auslander part* $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ of $Q^{[a,b],\tilde{\ell}}$ to be its full subquiver consisting of the vertices that are not trans-regular and the leftmost trans-regular vertex of each row.

We call these vertices the *trans-Auslander vertices*. We remark that $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ is isomorphic to the Auslander part $Q^{[a,b],\tilde{\ell}}$. In particular, if we define $\overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ as the subquiver of the trans-Auslander part obtained by removing the horizontal arrows, then $\overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ is isomorphic to the quiver $\overline{Q}^{[a,b]}_{\mathrm{Aus}}$ in Lemma 3.8.11.

Let $Q_{\mathcal{A}^{\mathrm{t}}}$ be the subquiver of $Q^{[a,b],\tilde{\ell}}$ that contains all vertices and whose arrows are precisely those in $\overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ and $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-tens}}$. We define $\mathcal{A}^{\mathrm{t}}$ as the quotient of the path category of $Q_{\mathcal{A}^{\mathrm{t}}}$ by the mesh relations on the subquiver $\overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ (coming from Lemma 3.8.11 and (3.14)) and by the commutation relations on the subquiver $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-tens}}$ (coming from (3.12)). Let $\mathcal{A}^{\mathrm{t}}_1$ and $\mathcal{A}^{\mathrm{t}}_2$ be the full subcategories on the trans-Auslander and trans-regular vertices, respectively, and denote by $\mathcal{C}^{\mathrm{t}}$ their intersection. Note that

$$\mathrm{Hom}(\mathcal{A}^{\mathrm{t}}_2, \mathcal{A}^{\mathrm{t}}_1 \setminus \mathcal{C}^{\mathrm{t}}) = \mathrm{Hom}(\mathcal{A}^{\mathrm{t}}_1, \mathcal{A}^{\mathrm{t}}_2 \setminus \mathcal{C}^{\mathrm{t}}) = 0.$$

because there are no arrows in $Q_{\mathcal{A}^{\mathrm{t}}}$ from a vertex in $Q^{[a,b],\tilde{\ell}}_{\mathrm{t-tens}} \cap \overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$ to a vertex in $Q_{\mathcal{A}^{\mathrm{t}}} \setminus Q^{[a,b],\tilde{\ell}}_{\mathrm{t-tens}} \cap \overline{Q}^{[a,b],\tilde{\ell}}_{\mathrm{t-Aus}}$. Similarly to the previous case (but with a different index), $\mathcal{A}^{\mathrm{t}}_1$ is equivalent to the category $\mathcal{A}^{[a,b]}_{\mathrm{Aus}}$ in Lemma 3.8.11. Finally, we define $\mathcal{B}^{\mathrm{t}} \subseteq \mathcal{A}^{\mathrm{t}}_1$ to be the full subcategory on the vertices of $F^{[a,b],\tilde{\ell}}$. We remark that the hypotheses of Proposition 3.7.8 are satisfied, but this time (3.11) is the condition that holds and not (3.10).

We have an analogue of Lemma 3.8.13, which can be proved similarly.

**Lemma 3.8.15.** *The total dg algebras of* $\mathbf{\Pi}_3(\mathcal{A}^{\mathrm{t}}, \mathcal{B}^{\mathrm{t}})$ *and* $\mathbf{\Pi}_3(\mathcal{A}^{\mathrm{t}}_2, \mathcal{C}^{\mathrm{t}})$ *are quasi-isomorphic to* $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}$ *and* $\mathbf{\Gamma}^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$*, respectively.*

We can now prove Proposition 3.8.14.

*Proof of Proposition 3.8.14.* Following the constructions above, we denote $\mathbf{\Gamma} = \mathbf{\Gamma}^{[a,b],\tilde{\ell}}$ and $\mathbf{\Pi} = \mathbf{\Pi}_3(\mathcal{A}^{\mathrm{t}}, \mathcal{B}^{\mathrm{t}})$. By Lemma 3.8.15 and the construction of $\mathbf{\Pi}$ as a tensor dg category over $\mathcal{A}^{\mathrm{t}}$, we have

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}}(e_s\mathbf{\Gamma}, e_t\mathbf{\Gamma}) \cong \mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}}(s^\wedge, t^\wedge) \cong \bigoplus_{r\geq 0} \mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(s^\wedge, F^r(t^\wedge))$$

for any two vertices $s, t \in Q^{[a,b],\tilde{\ell}}$, where $F$ denotes the functor $- \otimes^{\mathbf{L}}_{\mathcal{A}^{\mathrm{t}}} \Sigma^2\Omega_{\mathcal{A}^{\mathrm{t}},\mathcal{B}^{\mathrm{t}}}$ and we write $x^\wedge = \mathcal{A}^{\mathrm{t}}(-, x)$ for a vertex $x$. By Serre duality on $\mathcal{D}^b(\mathcal{A}^{\mathrm{t}})$, we have

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(s^\wedge, F^r(t^\wedge)) \cong D\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(F^r(t^\wedge), s^\vee) \cong D\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(t^\wedge, G^r(s^\vee)),$$

where $D$ denotes duality over $k$, $s^\vee$ is the corepresentable module $D\mathcal{A}^{\mathrm{t}}(s, -)$, and $G = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(\Sigma^2\Omega_{\mathcal{A}^{\mathrm{t}},\mathcal{B}^{\mathrm{t}}}, -)$ is the right adjoint of $F$. When $t$ is trans-regular, observe that the complex above only depends on the restriction of $G^r(s^\vee)$ to $\mathcal{A}^{\mathrm{t}}_2$. Similarly, if $\mathbf{\Gamma}' = \mathbf{\Gamma}^{[a,b],\tilde{\ell}}_{\mathrm{t-reg}}$, then we obtain that

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}'}(e_s\mathbf{\Gamma}', e_t\mathbf{\Gamma}') \cong \bigoplus_{r\geq 0} D\mathbf{R}\mathrm{Hom}_{\mathcal{A}_2^{\mathrm{t}}}(t^\wedge|_{\mathcal{A}_2^{\mathrm{t}}}, G_2^r(s^\vee|_{\mathcal{A}_2^{\mathrm{t}}}))$$

for any $s, t \in Q_{\mathrm{t-reg}}^{[a,b],\tilde{\ell}}$, where $G_2 = \mathbf{R}\mathrm{Hom}_{\mathcal{A}_2^{\mathrm{t}}}(\Sigma^2\Omega_{\mathcal{A}_2^{\mathrm{t}},\mathcal{C}^{\mathrm{t}}}, -)$. Therefore, to prove that $\mathbf{\Gamma}$ and $\mathbf{\Gamma}'$ agree on extensions, it suffices to show that

$$G^r(s^\vee)|_{\mathcal{A}_2^{\mathrm{t}}} \cong G_2^r(s^\vee|_{\mathcal{A}_2^{\mathrm{t}}})$$

for any trans-regular vertex $s$ and $r \geq 0$. In fact, we will now prove a stronger result:

$$G(M)|_{\mathcal{A}_2^{\mathrm{t}}} \cong G_2(M|_{\mathcal{A}_2^{\mathrm{t}}}).$$

for any $M \in \mathcal{D}^b(\mathcal{A}^{\mathrm{t}})$.

Applying $\mathbf{R}\mathrm{Hom}_{\mathcal{A}^{\mathrm{t}}}(-, M)$ to the distinguished triangle in Proposition 3.7.8, we get the distinguished triangle

$$\mu_1^{\mathbf{R}}(G_1(M|_{\mathcal{A}_1^{\mathrm{t}}})) \longrightarrow G(M) \longrightarrow \mu_2^{\mathbf{R}}(G_2(M|_{\mathcal{A}_2^{\mathrm{t}}})) \longrightarrow \Sigma\mu_1^{\mathbf{R}}(G_1(M|_{\mathcal{A}_1^{\mathrm{t}}}))$$

in $\mathcal{D}^b(\mathcal{A}^{\mathrm{t}})$, where $G_1 = \mathbf{R}\mathrm{Hom}_{\mathcal{A}_1^{\mathrm{t}}}(\Sigma^2\Omega_{\mathcal{A}_1^{\mathrm{t}},\mathcal{B}^{\mathrm{t}}}, -)$ and $\mu_i^{\mathbf{R}} = \mathbf{R}\mathrm{Hom}_{\mathcal{A}_i^{\mathrm{t}}}(\mathcal{A}^{\mathrm{t}}, -)$ for $i = 1, 2$. Let us show that

$$\mu_1^{\mathbf{R}}(G_1(M|_{\mathcal{A}_1^{\mathrm{t}}}))\Big|_{\mathcal{A}_2^{\mathrm{t}}} = 0, \tag{3.16}$$

which will imply our claim. For a simple $\mathcal{A}_1^{\mathrm{t}}$-module $S$, Lemma 3.7.17 implies that $G_1(S)$ is still concentrated in degree zero and $G_1(S)|_{\mathcal{C}^{\mathrm{t}}} = 0$. By Lemma 3.7.9, $\mu_1^{\mathbf{R}}(G_1(S)) = \mu_1(G_1(S))$ where $\mu_1 : \mathrm{Mod}\,\mathcal{A}_1^{\mathrm{t}} \to \mathrm{Mod}\,\mathcal{A}^{\mathrm{t}}$ is the functor defined in Section 3.7.3. But $\mu_1(G_1(S))$ vanishes on $\mathcal{A}_2^{\mathrm{t}}$ since $G_1(S)$ vanishes on $\mathcal{C}^{\mathrm{t}}$. Since $M|_{\mathcal{A}_1^{\mathrm{t}}}$ belongs to $\mathcal{D}^b(\mathcal{A}_1^{\mathrm{t}})$, which is generated by the simple $\mathcal{A}_1^{\mathrm{t}}$-modules, we obtain (3.16), as desired. This concludes the proof. $\surd$

We can finally give a proof for Theorem 3.8.7.

*Proof of Theorem 3.8.7.* Let $a_1, a_2, b \in \mathbb{Z}$ with $a_1, a_2 \leq b$. To prove that $\mathbf{\Gamma}^{[a_1,b]}$ and $\mathbf{\Gamma}^{[a_2,b]}$ agree on extensions, we may assume that $a_1, a_2 \leq b - l(w_0) + 1$ by Lemma 3.8.9. In particular, the regular lengths $\ell_1 = \ell(Q^{[a_1,b]})$ and $\ell_2 = \ell(Q^{[a_2,b]})$ satisfy $\ell_1, \ell_2 \geq 1$. Take $\tilde{\ell} \geq \ell_1, \ell_2$ such that $Q_{\mathrm{t-reg}}^{[a_i,b],\tilde{\ell}}$ contains $Q^{[a_i,b]}$ for $i = 1, 2$. By Propositions 3.8.10 and 3.8.14, $\mathbf{\Gamma}^{[a_i,b]}$ and $\mathbf{\Gamma}_{\mathrm{t-reg}}^{[a_i,b],\tilde{\ell}}$ agree on extensions. But $Q_{\mathrm{t-reg}}^{[a_1,b],\tilde{\ell}} = Q_{\mathrm{t-reg}}^{[a_2,b],\tilde{\ell}}$, so $\mathbf{\Gamma}_{\mathrm{t-reg}}^{[a_1,b],\tilde{\ell}} = \mathbf{\Gamma}_{\mathrm{t-reg}}^{[a_2,b],\tilde{\ell}}$, concluding the proof. $\surd$

### 3.8.3 Commutation and braid moves

We will now describe how the ice quiver with potential $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$ changes when we apply a commutation move or a braid move to the reduced expression $\underline{w}_0$. We will follow the arguments in [Con26, Section 6.1] closely.

Let $\mathfrak{i} = (i_s)_{s\in\mathbb{Z}}$ be an infinite sequence of indices $i_s \in \Delta_0$ satisfying the following:

$$\text{for any } s \in \mathbb{Z}, \text{ there are } s' < s < s'' \text{ such that } i_{s'} = i_s = i_{s''}. \tag{3.17}$$

Note that the sequence $\widehat{\underline{w}}_0$ satisfies this condition. It allows us to define $s^-$ and $s^+$ for any $s \in \mathbb{Z}$ as in Section 3.5.3. Therefore, for a finite interval $[a, b]$, we can define an ice quiver with potential $(Q^{[a,b]}(\mathfrak{i}), F^{[a,b]}(\mathfrak{i}), W^{[a,b]}(\mathfrak{i}))$ as in Section 3.5.3 by replacing $\widehat{\underline{w}}_0$ by $\mathfrak{i}$.

Let $\mathfrak{i} = (i_s)_{s\in\mathbb{Z}}$ and $\mathfrak{i}' = (i'_s)_{s\in\mathbb{Z}}$ be two sequences of indices in $\Delta_0$ satisfying (3.17). We say that $\mathfrak{i}'$ is obtained from $\mathfrak{i}$ by a

- *commutation move* at position $s \in \mathbb{Z}$ if $i_s \not\sim i_{s+1}$ and we have $i'_s = i_{s+1}$, $i'_{s+1} = i_s$ and $i'_t = i_t$ for $t \neq s, s+1$.
- *braid move* at position $s \in \mathbb{Z}$ if $i_s \sim i_{s+1}$ and we have $i'_{s-1} = i'_{s+1} = i_s$, $i'_s = i_{s-1} = i_{s+1}$ and $i'_t = i_t$ for $t \neq s-1, s, s+1$.

These moves have the following effect on the associated ice quivers with potential.

**Lemma 3.8.16.** *Let $\mathfrak{i} = (i_s)_{s\in\mathbb{Z}}$ and $\mathfrak{i}' = (i'_s)_{s\in\mathbb{Z}}$ be two sequences of indices in $\Delta_0$ satisfying (3.17). Let $[a, b]$ be a finite interval. If $\mathfrak{i}'$ is obtained from $\mathfrak{i}$ by a commutation move at position $a \leq s < b$, then there exists an isomorphism of ice quivers with potential*

$$\varphi : (Q^{[a,b]}(\mathfrak{i}), F^{[a,b]}(\mathfrak{i}), W^{[a,b]}(\mathfrak{i})) \longrightarrow (Q^{[a,b]}(\mathfrak{i}'), F^{[a,b]}(\mathfrak{i}'), W^{[a,b]}(\mathfrak{i}')).$$

*Proof.* We define $\varphi$ on vertices by

$$\varphi(t) = \begin{cases} t, & \text{if } t \neq s, s+1, \\ s+1 & \text{if } t = s, \\ s & \text{if } t = s+1, \end{cases}$$

for $t \in [a, b]$. Since $i_s \not\sim i_{s+1}$, it is immediate from the definition of these ice quivers with potential that $\varphi$ yields the desired isomorphism. $\surd$

**Lemma 3.8.17.** *Let $\mathfrak{i} = (i_s)_{s\in\mathbb{Z}}$ and $\mathfrak{i}' = (i'_s)_{s\in\mathbb{Z}}$ be two sequences of indices in $\Delta_0$ satisfying (3.17). Let $[a, b]$ be a finite interval. If $\mathfrak{i}'$ is obtained from $\mathfrak{i}$ by a braid move at position $a < s < b$, then there exists an isomorphism of ice quivers with potential*

$$\varphi : \mu_{s+1}(Q^{[a,b]}(\mathfrak{i}), F^{[a,b]}(\mathfrak{i}), W^{[a,b]}(\mathfrak{i})) \longrightarrow (Q^{[a,b]}(\mathfrak{i}'), F^{[a,b]}(\mathfrak{i}'), W^{[a,b]}(\mathfrak{i}')).$$

*Proof.* To ease the reading, we will omit the index $[a, b]$ in the notation of the ice quivers and the potentials. We compute the mutation above following the proof of [Con26, Lemma 6.2].

By the definition of $Q(\mathfrak{i})$, its full subquiver containing all vertices connected to $s+1$ is of the following form:

$$\begin{tikzcd}
s-1 \arrow[dr, dashed, "\delta_1"'] & & s+1 \arrow[ll, "\gamma_1"'] \arrow[dr, "\gamma_4"] & (s+1)^+ \arrow[l, "\gamma_2"'] \\
& s \arrow[ur, "\gamma_3"] & & s^+ \arrow[ll, "\gamma_5"] \arrow[u, dashed, "\delta_2"']
\end{tikzcd}$$

Here, the arrow $\delta_1$ appears if $(s-1)^- < s^-$ or if both $s-1$ and $s$ are frozen vertices. The vertex $s^+$ appears if $s^+ \leq b$, and similarly for $(s+1)^+$. If both appear, the arrow $\delta_2$ is present precisely when $s^+ < (s+1)^+$. If $\delta_1$ does not appear or is not frozen, the proof reduces to that in [Con26], so we will assume from now on that $s-1$ and $s$ are frozen. We will also suppose that $s^+$ and $(s+1)^+$ appear but not $\delta_2$. The other cases can be treated similarly. With these assumptions, we have the following picture:

$$\begin{tikzcd}
\boxed{s-1} \arrow[dr, "\delta_1"'] & & s+1 \arrow[ll, "\gamma_1"'] \arrow[dr, "\gamma_4"] & (s+1)^+ \arrow[l, "\gamma_2"'] \\
& \boxed{s} \arrow[ur, "\gamma_3"] & & s^+ \arrow[ll, "\gamma_5"]
\end{tikzcd}$$

We can write

$$W(\mathfrak{i}) = \gamma_5\gamma_4\gamma_3 + \gamma_1\gamma_3\delta_1 + c_1 + c_2 + c_3 + W_0$$

where:

(1) $c_1$ is the sum of all simple cordless cycles containing the path $\gamma_1\gamma_2$;

(2) $c_2$ is zero or the unique simple cordless cycle containing the path $\gamma_4\gamma_2$ if it exists;

(3) $c_3$ is the sum of all simple cordless cycles containing the arrow $\gamma_5$ except for $\gamma_5\gamma_4\gamma_3$;

(4) $W_0$ is the sum of all simple cordless cycles that do not contain any arrow in the picture above.

Let us compute $\widetilde{\mu}_{s+1}(Q(\mathfrak{i}), F(\mathfrak{i}), W(\mathfrak{i})) = (\widetilde{\mu}_{s+1}Q(\mathfrak{i}), F(\mathfrak{i}), \widetilde{\mu}_{s+1}W(\mathfrak{i}))$. We see that $Q' =$

$\widetilde{\mu}_{s+1}Q(\mathfrak{i})$ differs from $Q(\mathfrak{i})$ only around $s+1$, where it is given by

$$\begin{tikzcd}
\boxed{s-1} \arrow[rr, "\gamma_1^*"] \arrow[dr, "\delta_1"', shift right] & & s+1 \arrow[rr, "\gamma_2^*"] \arrow[dl, "\gamma_3^*"'] & & (s+1)^+ \arrow[llll, bend right, "{[\gamma_1\gamma_2]}"'] \arrow[d, "{[\gamma_4\gamma_2]}"] \\
& \boxed{s} \arrow[ul, "{[\gamma_1\gamma_3]}"', shift right] \arrow[rrr, shift left, "{[\gamma_4\gamma_3]}"] & & & s^+ \arrow[ull, "\gamma_4^*"'] \arrow[lll, shift left, "\gamma_5"]
\end{tikzcd}$$

where the new red arrows are not frozen. Moreover, we have

$$W' = \widetilde{\mu}_{s+1}W(\mathfrak{i}) = \gamma_5[\gamma_4\gamma_3] + [\gamma_1\gamma_3]\delta_1 + \widetilde{c}_1 + \widetilde{c}_2 + c_3 + W^* + W_0$$

where $\widetilde{c}_1$ and $\widetilde{c}_2$ are obtained from $c_1$ and $c_2$ by replacing all instances of the paths $\gamma_1\gamma_2$ and $\gamma_4\gamma_2$ by $[\gamma_1\gamma_2]$ and $[\gamma_4\gamma_2]$, and

$$W^* = [\gamma_1\gamma_2]\gamma_2^*\gamma_1^* + [\gamma_1\gamma_3]\gamma_3^*\gamma_1^* + [\gamma_4\gamma_2]\gamma_2^*\gamma_4^* + [\gamma_4\gamma_3]\gamma_3^*\gamma_4^*.$$

Observe that $(Q', F(\mathfrak{i}), W')$ is not reduced, and the 2-cycles contained in the new potential are precisely $\gamma_5[\gamma_4\gamma_3]$ and $[\gamma_1\gamma_3]\delta_1$. We now follow the algorithm in the proof of [Pre20, Lemma 3.14] (which is based on [DWZ08, Lemma 4.7]). Consider the algebra endomorphism $f$ of the completed path algebra of $Q'$ that fixes the vertices and is defined on arrows as follows:

(1) $f(\gamma_5) = \gamma_5 + \gamma_3^*\gamma_4^*$;

(2) $f([\gamma_4\gamma_3]) = [\gamma_4\gamma_3] - \widetilde{c}_3$;

(3) $f(\alpha) = -\alpha$ for $\alpha = \delta_1, \gamma_1^*, \gamma_2^*, \gamma_4^*, [\gamma_1\gamma_3]$;

(4) $f(\beta) = \beta$ for any arrow $\beta \neq \gamma_5, [\gamma_4\gamma_3], \delta_1, \gamma_1^*, \gamma_2^*, \gamma_4^*, [\gamma_1\gamma_3]$ in $Q'$.

Here, $\widetilde{c}_3$ is the sum of paths such that $c_3$ is cyclically equivalent to $\gamma_5\widetilde{c}_3$. Note that no path in $\widetilde{c}_3$ contains the arrow $\gamma_5$ by the definition of $W(\mathfrak{i})$. Clearly, $f$ is an automorphism by [DWZ08, Proposition 2.4]. Moreover, it preserves the completed path algebra of the frozen subquiver, so it induces a right equivalence of $(Q', F(\mathfrak{i}), W')$ and $(Q', F(\mathfrak{i}), f(W'))$. One computes that, up to cyclic equivalence, we have

$$f(W') = \gamma_5[\gamma_4\gamma_3] + [\gamma_1\gamma_3]\delta_1 + \widetilde{c}_1 + \widetilde{c}_2 + \gamma_3^*\gamma_4^*\widetilde{c}_3 + (W^* - [\gamma_4\gamma_3]\gamma_3^*\gamma_4^*) + W_0.$$

The potential above has the form described in [Pre20, Lemma 3.14]. Thus, by the proof of Theorem 3.6 in loc. cit., the ice quiver of the reduction of $(Q', F(\mathfrak{i}), W')$ is obtained by deleting the arrows $\gamma_5$, $[\gamma_4\gamma_3]$ and $\delta_1$, and freezing the arrow $[\gamma_1\gamma_3]$. Graphically, after adjusting the positions of the vertices $s-1$ and $s$, it is given locally at $s+1$ by

$$\begin{tikzcd}
 & \boxed{s-1} \arrow[dr, "\gamma_1^*"] & & (s+1)^+ \arrow[ll, "{[\gamma_1\gamma_2]}"'] \arrow[d, "{[\gamma_4\gamma_2]}"] \\
\boxed{s} \arrow[ur, "{[\gamma_1\gamma_3]}"] & & s+1 \arrow[ll, "\gamma_3^*"] \arrow[ur, "\gamma_2^*"] & s^+ \arrow[l, "\gamma_4^*"]
\end{tikzcd}$$

We can take the potential of the reduction to be

$$\begin{aligned}
W_{\mathrm{red}} &= \widetilde{c}_1 + \widetilde{c}_2 + \gamma_3^*\gamma_4^*\widetilde{c}_3 + (W^* - [\gamma_4\gamma_3]\gamma_3^*\gamma_4^*) + W_0 \\
&= [\gamma_1\gamma_2]\gamma_2^*\gamma_1^* + [\gamma_1\gamma_3]\gamma_3^*\gamma_1^* + [\gamma_4\gamma_2]\gamma_2^*\gamma_4^* + \widetilde{c}_1 + \widetilde{c}_2 + \gamma_3^*\gamma_4^*\widetilde{c}_3 + W_0.
\end{aligned}$$

This reduced ice quiver with potential is $\mu_{s+1}(Q(\mathfrak{i}), F(\mathfrak{i}), W(\mathfrak{i}))$ by definition. Comparing it with $(Q(\mathfrak{i}'), F(\mathfrak{i}'), W(\mathfrak{i}'))$, it is clear that we have an isomorphism $\varphi$ as in the statement which is given on vertices by

$$\varphi(t) = \begin{cases} t, & \text{if } t \neq s-1, s, \\ s & \text{if } t = s-1, \\ s-1 & \text{if } t = s, \end{cases}$$

for $t \in [a, b]$. $\surd$

For the next result, we say that a sequence of commutation and braid moves is *compatible* with an interval $[a, b]$ if the position $s$ of each of its moves satisfies $a \leq s < b$ (if it is a commutation move) or $a < s < b$ (if it is a braid move).

**Proposition 3.8.18.** *Let $\mathfrak{i} = (i_s)_{s\in\mathbb{Z}}$ and $\mathfrak{i}' = (i'_s)_{s\in\mathbb{Z}}$ be two sequences of indices in $\Delta_0$ satisfying (3.17). Let $[a, b]$ be a finite interval. Suppose $\mathfrak{i}'$ is obtained from $\mathfrak{i}$ by a sequence of commutation and braid moves compatible with $[a, b]$. Then one of*

$$\widehat{\boldsymbol{\Gamma}}(Q^{[a,b]}(\mathfrak{i}), F^{[a,b]}(\mathfrak{i}), W^{[a,b]}(\mathfrak{i})) \quad \text{and} \quad \widehat{\boldsymbol{\Gamma}}(Q^{[a,b]}(\mathfrak{i}'), F^{[a,b]}(\mathfrak{i}'), W^{[a,b]}(\mathfrak{i}'))$$

*is proper if and only if the other is proper.*

*Proof.* This is a combination of Lemmas 3.3.8, 3.8.16, and 3.8.17. $\surd$

We can now return to the case when $\mathfrak{i} = \underline{\widehat{w}}_0$. Let $[a, b]$ be a finite integer interval. Let $r \geq 0$ be the largest integer such that $a' = a + r \cdot l(w_0)$ satisfies $a' \leq b$. We define the $[a, b]$-*residue* of $\underline{\widehat{w}}_0$ to be the element

$$w = s_{i_{a'}} s_{i_{a'+1}} \cdots s_{i_b}$$

of the Weyl group of $\Delta$, where $s_j$ denotes the simple reflection associated with $j \in \Delta_0$.

**Proposition 3.8.19.** *Let $\underline{w}_0$ be a reduced expression of $w_0$ and take a finite integer interval $[a,b]$. If the $[a,b]$-residue of $\widehat{\underline{w}}_0$ has a reduced expression adapted to an orientation of $\Delta$, then $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ is proper.*

*Proof.* Assume without loss of generality that $a = 1$. Let $w = s_{i_{a'}} s_{i_{a'+1}} \cdots s_{i_b}$ denote the $[1,b]$-residue of $\widehat{\underline{w}}_0$, where $a' = 1 + r \cdot l(w_0)$ as above. Let $(j_1, \ldots, j_t)$ be a reduced expression of $w$ adapted to an orientation $Q$ of $\Delta$. By [Béd99] (see also [OS19a]), there is a compatible reading of the Auslander–Reiten quiver of $Q$ starting with $(j_1, \ldots, j_t)$. Consequently, we can extend this sequence of indices to a reduced word $(j_1, \ldots, j_{l(w_0)})$ for $w_0$ that is adapted to $Q$. By performing a sequence $\mathfrak{s}$ of commutation and braid moves, we can transform $(i_{a'}, i_{a'+1}, \ldots, i_b)$ into $(j_1, \ldots, j_t)$, since both represent the same element $w$. Similarly, we can transform $(i_{b+1}, i_{b+2}, \ldots, i_{a'+l(w_0)-1})$ into $(j_{t+1}, j_{t+2}, \ldots, j_{l(w_0)})$ by a sequence $\mathfrak{s}'$ of moves, since both represent the element $w^{-1}w_0$. Now we define a reduced word $\underline{w}_0'$ for $w_0$ as

$$\underline{w}_0' = (j_1, \ldots, j_{l(w_0)}) \quad \text{or} \quad \underline{w}_0' = (j_1^*, \ldots, j_{l(w_0)}^*),$$

depending on whether $r$ is even or odd, respectively. This ensures that the subsequence of $\widehat{\underline{w}}_0'$ from entries $a'$ to $b$ is precisely $(j_1, \ldots, j_t)$. Note that $\underline{w}_0'$ is adapted to an orientation of $\Delta$. The next step is to extend $\mathfrak{s}$ and $\mathfrak{s}'$ to a sequence of braid and commutation moves on $\widehat{\underline{w}}_0$ as follows. We can divide the integers from 1 to $b$ into blocks of consecutive numbers whose sizes alternate between $t$ and $l(w_0) - t$. On each block of size $t$, we apply the sequence $\mathfrak{s}$, and on each block of size $l(w_0) - t$, we apply the sequence $\mathfrak{s}'$ (with the positions of the moves appropriately shifted). It is not hard to see that this indeed defines a sequence of commutation and braid moves starting from $\widehat{\underline{w}}_0$ and that the entries between positions 1 and $b$ of the resulting infinite sequence match the corresponding entries in $\widehat{\underline{w}}_0'$. But $\widehat{\mathbf{\Gamma}}^{[1,b]}(\underline{w}_0')$ is proper by Corollary 3.8.8, so the same holds for $\widehat{\mathbf{\Gamma}}^{[1,b]}(\underline{w}_0)$ by Proposition 3.8.18. $\surd$

**Corollary 3.8.20.** *Let $\underline{w}_0$ be a reduced expression of $w_0$ and take a finite integer interval $[a,b]$. If $l(w_0)$ divides $b - a + 1$, then $\widehat{\mathbf{\Gamma}}^{[a,b]}(\underline{w}_0)$ is proper.*

*Proof.* The hypothesis forces the $[a,b]$-residue of $\widehat{\underline{w}}_0$ to be the longest element, which has a reduced expression adapted to an orientation of $\Delta$. The result then follows from Proposition 3.8.19. $\surd$

## 3.9 Additive categorification of the monoidal Λ-matrix

In this section, we suppose that $\mathfrak{g}$ is untwisted of simply-laced type and that the reduced word $\underline{w}_0$ is adapted to an orientation of $\Delta$. We fix $a, b \in \mathbb{Z}$ with $a \le b$.

**Lemma 3.9.1.** *The ice quiver with potential* $(Q^{[a,b]}(\underline{w}_0), F^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$ *is rigid.*

*Proof.* Since $\underline{w}_0$ is adapted, we can view $Q^{[a,b]}(\underline{w}_0)$ as a subquiver of $Q_{\mathrm{HL}}$ by Lemma 3.5.10. Thus, the argument in [HL16, Proposition 4.17] shows that the quiver with potential $(Q^{[a,b]}(\underline{w}_0), W^{[a,b]}(\underline{w}_0))$ is rigid. They do not use the relation $\partial_\alpha(W^{[a,b]}(\underline{w}_0)) = 0$ for $\alpha \in F_1^{[a,b]}(\underline{w}_0)$, which is not present in the relative Jacobian algebra, so their proof also applies to the ice quiver with potential above. $\surd$

In particular, the ice quiver with potential above is non-degenerate. It is also Jacobi-finite since the completed relative Ginzburg dg algebra $\widehat{\boldsymbol{\Gamma}}^{[a,b]}(\underline{w}_0)$ is proper by Corollary 3.8.8. Therefore, by Section 3.3.4, we can use the associated Higgs category $\mathcal{H}^{[a,b]}(\underline{w}_0)$ to study its cluster algebra $\mathcal{A}^{[a,b]}(\underline{w}_0)$. Denote by $\varphi : K_0(\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}) \to \mathcal{A}^{[a,b]}(\underline{w}_0)$ the isomorphism arising from Theorem 3.5.4 and Proposition 3.5.7. The main result of this section shows how to recover the Λ-invariant in this Λ-monoidal categorification using $\mathcal{H}^{[a,b]}(\underline{w}_0)$.

**Theorem 3.9.2.** *Suppose $\mathfrak{g}$ is untwisted of simply-laced type. Let $(\mathcal{D}, \underline{w}_0)$ be a complete PBW-pair adapted to an orientation of $\Delta$, and let $a, b \in \mathbb{Z}$ with $a \le b$. For reachable simple objects $V, W \in \mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w}_0}$, we have*

$$\Lambda(V, W) = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}},$$

*where $\mathcal{H} = \mathcal{H}^{[a,b]}(\underline{w}_0)$ and $M, N \in \mathcal{H}$ are the corresponding reachable rigid objects such that $\varphi(V) = CC(M)$ and $\varphi(W) = CC(N)$ in $\mathcal{A}^{[a,b]}(\underline{w}_0)$.*

*Proof.* Theorems 3.4.6 and 3.5.4 endow $\mathcal{A}^{[a,b]}(\underline{w}_0)$ with two Λ-cluster algebra structures. If we show that they coincide, then the theorem follows by Proposition 3.4.8 and Theorem 3.5.3, since each side of the equality above computes the same tropical invariant in $\mathcal{A}^{[a,b]}(\underline{w}_0)$. Therefore, it suffices to show that the Poisson coefficient matrices of some Λ-seed are equal.

We assume without loss of generality that $b = 0$. If $\xi$ denotes the height function giving rise to $(\mathcal{D}, \underline{w}_0)$, let $\mathscr{S}$ be the (reachable) monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,0],\mathcal{D},\underline{w}_0}$ that is the restriction of the monoidal seed $\mathscr{S}^{<\xi}$ from Theorem 3.5.12 to $Q^{[a,0]}(\underline{w}_0)$, as explained in Remark 3.5.14. We assume $V, W$ belong to $\mathscr{S}$. Since these objects correspond to the same seed in $\mathcal{A}^{[a,0]}(\underline{w}_0)$, we have $\mathfrak{d}(V, W) = 0$ and $\operatorname{Ext}^1_{\mathcal{H}}(M, N) = 0$. By Proposition 3.5.1, we have

$$\Lambda(V, W) = \sum_{n \ge 1} (-1)^{n-1} \left[ \mathfrak{d}(V, \mathscr{D}^{-n}(W)) - \mathfrak{d}(W, \mathscr{D}^{-n}(V)) \right].$$

On the other hand,

$$[M,N]_{\mathcal{H}} = \sum_{n\geq 0}(-1)^n(\dim \operatorname{Ext}^{-n}_{\mathcal{H}}(M,N) - \dim \operatorname{Ext}^{-n}_{\mathcal{H}}(N,M))$$

by definition. By Proposition 3.9.3 below, the terms of these sums coincide, so we have $\Lambda(V,W) = [M,N]_{\mathcal{H}}$. This shows that the Poisson coefficient matrices of the initial Λ-seeds coincide, concluding the proof. $\surd$

**Proposition 3.9.3.** *Suppose $\mathfrak{g}$ is untwisted of simply-laced type. Let $(\mathcal{D}, \underline{w}_0)$ be a complete PBW-pair adapted to an orientation of $\Delta$, and denote by $\xi : \Delta_0 \to \mathbb{Z}$ the corresponding height function. Take an integer $a \leq 0$ and consider the initial monoidal seed $\mathscr{S}$ of $\mathscr{C}_{\mathfrak{g}}^{[a,0],\mathcal{D},\underline{w}_0}$ obtained by restricting the monoidal seed $\mathscr{S}^{<\xi}$ from Theorem 3.5.12 to $Q^{[a,0]}(\underline{w}_0)$. For simple objects $V$ and $W$ in $\mathscr{S}$, we have*

$$\mathfrak{d}(V, \mathscr{D}^{-n}(W)) = \dim \operatorname{Ext}^{1-n}_{\mathcal{H}}(M,N) \tag{3.18}$$

*for any $n \geq 1$, where $\mathcal{H} = \mathcal{H}^{[a,0]}(\underline{w}_0)$ and $M,N \in \mathcal{H}$ are the corresponding reachable rigid indecomposable objects such that $\varphi(V) = CC(M)$ and $\varphi(W) = CC(N)$ in $\mathcal{A}^{[a,0]}(\underline{w}_0)$.*

*Proof.* The following observation is essential: for fixed $V,W \in \mathscr{C}_{\mathfrak{g}}$, the left-hand side of (3.18) does not depend on $a$ by Remark 3.5.14, and neither does the right-hand side by Proposition 3.3.6 and Theorem 3.8.7 (see also Lemma 3.3.3 and Remark 3.3.4). By taking $a$ such that the length of the interval $[a,0]$ is a multiple of $2l(w_0)$, we can assume every row of $Q^{[a,0]}(\underline{w}_0)$ has exactly $r+1$ vertices for some $r \geq 0$ (see [Béd99, Corollary 2.20]). By Proposition 3.6.9, if we identify $Q^{[a,0]}(\underline{w}_0)$ with a subquiver of $Q_{\mathrm{HL}}^{<\xi}$ via Lemma 3.5.10, we can suppose $r$ is large enough so that $\mathscr{D}^{-n}(W)$ is a reachable simple object of $\mathscr{C}_{\mathfrak{g}}^{[a,0],\mathcal{D},\underline{w}_0}$ that can be reached by applying the sequence of mutations $\mathbf{v}^{\leq r}_{\mathscr{D}^{-n}}$ from Remark 3.6.11 to $\mathscr{S}$. In particular, $\varphi(\mathscr{D}^{-n}(W))$ is a cluster variable of $\mathcal{A}^{[a,0]}(\underline{w}_0)$ and

$$\mathfrak{d}(V, \mathscr{D}^{-n}(W)) = \frac{1}{2}(\varphi(V) \mid\mid \varphi(\mathscr{D}^{-n}(M)))_F$$

by Theorem 3.5.3. Let $\widetilde{N} \in \mathcal{H}$ be the reachable rigid indecomposable object such that $CC(\widetilde{N}) = \varphi(\mathscr{D}^{-n}(W))$. We claim that $\widetilde{N} \cong \Sigma^{-n}N$ (up to increasing $r$), where $\Sigma$ is the suspension functor of the relative cluster category. This is enough to conclude the proof. Indeed, if this is the case, then the number above equals

$$\begin{aligned}\frac{1}{2}(CC(M) \mid\mid CC(\widetilde{N}))_F &= \dim \operatorname{Ext}^1_{\mathcal{H}}(M,\widetilde{N}) \\ &= \dim \operatorname{Ext}^1_{\mathcal{H}}(M,\Sigma^{-n}N) \\ &= \dim \operatorname{Ext}^{1-n}_{\mathcal{H}}(M,N),\end{aligned}$$

where the first equality follows from Proposition 3.4.8. We crucially use here the fact that the $F$-invariant does not depend on the choice of the Poisson coefficient matrices.

To prove our claim, let us first assume that $\Sigma^{-n}N \in \mathcal{H}$. In particular, observe that $N$ is not projective-injective in $\mathcal{H}$ by the definition of the Higgs category. Let $(\overline{Q}, \overline{W})$ be the quiver with potential obtained from $(Q^{[a,0]}(\underline{w}_0), F^{[a,0]}(\underline{w}_0), W^{[a,0]}(\underline{w}_0))$ by deleting the frozen vertices. As explained in Remark 3.6.12, Theorem 3.6.5 implies that the mutation sequence $\mathbf{v}^{\leq r}_{\mathscr{D}-n}$ is a composition of $n$ consecutive maximal green sequences for $\overline{Q}$ and the induced permutation on the vertices of $\overline{Q}$ coincides with the one induced by Proposition 3.6.9. We deduce from Theorem 3.6.4 that $p^*(\widetilde{N}) \cong p^*(\Sigma^{-n}N)$, where $p^* : \mathcal{H} \longrightarrow \mathcal{C}(\overline{Q}, \overline{W})$ is the functor from Section 3.3.3. Since $\widetilde{N}$ and $\Sigma^{-n}N$ are indecomposable objects that are not projective-injective in $\mathcal{H}$, the equivalence (3.4) together with the previous isomorphism imply that $\widetilde{N} \cong \Sigma^{-n}N$, as claimed.

It remains to show that $\Sigma^{-n}N \in \mathcal{H}$ when $r$ is large enough. By the definition of the Higgs category, we must prove that

$$\mathrm{Ext}^p_{\mathcal{C}}(\Sigma^{-n}N, e_s\widehat{\mathbf{\Gamma}}) = \mathrm{Ext}^p_{\mathcal{C}}(e_s\widehat{\mathbf{\Gamma}}, \Sigma^{-n}N) = 0$$

for any $p > 0$ and $s \in F_0^{[a,0]}(\underline{w}_0)$, where $\mathcal{C} = \mathcal{C}(Q^{[a,0]}(\underline{w}_0), F^{[a,0]}(\underline{w}_0), W^{[a,0]}(\underline{w}_0))$ and $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}^{[a,0]}(\underline{w}_0)$. These equalities are equivalent to

$$\mathrm{Ext}^{p+n}_{\mathcal{C}}(N, e_s\widehat{\mathbf{\Gamma}}) = \mathrm{Ext}^{p-n}_{\mathcal{C}}(e_s\widehat{\mathbf{\Gamma}}, N) = 0.$$

Since $N \in \mathcal{H}$ and $n \geq 1$, the vanishing of the extension group on the left is immediate. We similarly deduce that the extension group on the right is zero if $p > n$. For $0 < p \leq n$, since $N = e_t\widehat{\mathbf{\Gamma}}$ for some $t \in [a, 0]$, it suffices to show that

$$\mathrm{Ext}^{p-n}_{\widehat{\mathbf{\Gamma}}}(e_s\widehat{\mathbf{\Gamma}}, e_t\widehat{\mathbf{\Gamma}}) = 0 \tag{3.19}$$

by Proposition 3.3.6. To do so, we identify $Q^{[a,0]}(\underline{w}_0)$ with the triangle product $Q_\xi \boxtimes Q^\circ_{r+1}$. Writing $A = kQ_\xi \otimes A^\circ_{r+1}$ and $B = kQ_\xi \otimes B^\circ_{r+1}$, where $A^\circ_{r+1}$ and $B^\circ_{r+1}$ are as in (3.15), then $\widehat{\mathbf{\Gamma}}$ is quasi-isomorphic to $\mathbf{\Pi}_3(A, B)$ by Lemma 3.3.3 and Section 3.7.4. Therefore,

$$\mathbf{R}\mathrm{Hom}_{\widehat{\mathbf{\Gamma}}}(e_s\widehat{\mathbf{\Gamma}}, e_t\widehat{\mathbf{\Gamma}}) \cong \bigoplus_{q\geq 0} \mathbf{R}\mathrm{Hom}_A(e_sA, F^q(e_tA)) \cong \bigoplus_{q\geq 0} F^q(e_tA)e_s, \tag{3.20}$$

where $F = - \otimes^{\mathbf{L}}_A \Sigma^2\Omega_{A,B}$. But $e_tA \cong P \otimes V_{[r+2-d,r+1]}$, where $P$ is the indecomposable projective $kQ_\xi$-module associated with $i_t \in \Delta_0$ and $d \geq 1$ denotes the integer such that $t$ is the $d$-th vertex of its row in $Q^{[a,0]}(\underline{w}_0)$, counted from right to left. Note that $d$ does not change if we increase $r$. By Corollary 3.8.5, we have

$$F^q(e_tA) \cong \tau^{-q}P \otimes V_{[r+2-d-q,r+1-q]}.$$

Since $s$ is frozen, for $F^q(e_tA)e_s$ not to be zero, we must have $r+2-d-q \leq 1$, that is, $q \geq r+1-d$. If $r$ is large enough, this last inequality ensures that $\tau^{-q}P$ is a complex concentrated in a degree $m$ satisfying $m \leq -n$, so the same holds for $F^q(e_tA)e_s$. Hence, the complex (3.20) is concentrated in degrees lower than $-n$ and (3.19) holds for $0 < p \leq n$. This finishes the proof. $\surd$

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