Université Paris Cité

Master Thesis

On the bounded derived category of a Dynkin quiver

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Abstract

On the bounded derived category of a Dynkin quiver

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This master's thesis is an introduction to some techniques from the representation theory of finite-dimensional associative algebras. We start with the definition of path algebras and how to use quivers to study representations. In this direction, we prove Gabriel's classification of quivers of finite representation type in terms of Dynkin diagrams. Subsequently, we develop the theory of almost split sequences and irreducible maps, due to Auslander and Reiten, and apply its results to better understand the indecomposable modules over the path algebra of a Dynkin quiver. Lastly, we explore what happens in the derived category of these algebras, culminating in a proof of the fact that the bounded derived category of a Dynkin quiver is a fractionally Calabi-Yau triangulated category.

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Introduction

The representation theory of associative algebras is an important and active area of research in mathematics. Since its origins, it has undergone several changes as new ideas have been proposed and explored. This master's thesis serves as an introduction to a few techniques developed during the last fifty years that have greatly influenced the current shape of the subject.

In the first chapter, we start with the concepts of quiver and path algebra. The mathematician Pierre Gabriel and his school extensively studied them in the 1970s and unveiled their relevance for representation theory (see [15] for some results and more references from that time). For example, they showed that, over an algebraically closed field, modules over any algebra can be interpreted in terms of quivers and relations. Gabriel also managed to classify which path algebras are of finite representation type: they are precisely those coming from the Dynkin diagrams of type $\mathbb{A}$, $\mathbb{D}$ and $\mathbb{E}$. We prove this unexpected connection with Lie theory in the second chapter and the first appendix.

Around the same time, Maurice Auslander and Idun Reiten introduced the notion of almost split sequences ([4], [5]). Their work was central in unifying some methods employed by other authors and provided a good framework to study module categories in general. We present the main definitions and theorems of this theory in the third chapter, where we also show how to compute the Auslander-Reiten quiver of some examples, including some of the Dynkin quivers from the previous chapter.

Besides Auslander and Reiten's approach, other techniques from homological algebra began being employed to better understand the representation theory of certain algebras. For instance, in the late 1980s, Dieter Happel methodically studied the behavior of the derived category of a finite-dimensional algebra and obtained significant results ([17], [18]). We will touch on some of his ideas in the fourth chapter, but our main goal will be to prove the following theorem:

Theorem. If K is an algebraically closed field and Q is a Dynkin quiver, then the path algebra KQ is fractionally Calabi-Yau.

As we will explain in detail, this roughly means that the bounded derived category of KQ admits a special autoequivalence, called a Serre functor, which coincides with a shift up to some power. The notion of a (fractionally) Calabi-Yau triangulated category was introduced by Maxim Kontsevich ([24]), who was aware of the result above. We will follow a recent and simple proof given in [12] that also allows us to compute the Calabi-Yau dimension of KQ.

The reader is assumed to be familiar with basic representation theory of algebras in the level of what can be found in [2, Section I.1] or [3, Chapter I]. We will also use some elementary category theory and homological algebra. Knowing about derived categories is essential for the last chapter, but all definitions and results needed are collected in an appendix, where further references are indicated.

Chapter 1

Quiver representations

The goal of this chapter is to set the stage for what is going to be developed afterwards. Quivers and their representations, which will be the main subject of this thesis, are defined. Although it is not essential for what is going to follow, it is important to know the relevance of quivers for the representation theory of associative algebras, notably due to Theorem 1.2.1. Thus, we will usually deal with the slightly more general approach of quivers and relations. This will provide us with more examples to illustrate Auslander-Reiten theory in Chapter 3. Lastly, the final section presents some concepts that do not relate directly to quiver representations but that will show up frequently in subsequent chapters.

The content of this part is mostly based on the initial chapters of [2] and [3]. Some proofs will be incomplete and others will be omitted, but references for more details will be given.

Remark. From now on, all algebras are going to be unital and associative. Modules are *right* modules, unless stated otherwise. In general, they will be of finite dimension over the base field.

1.1 Path algebras

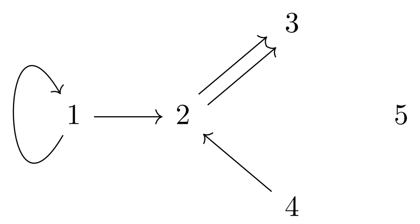
This section introduces the notion of a path algebra of a quiver. The core definitions are given and some initial properties are highlighted.

Definition 1.1.1. A **quiver** is a directed graph. More precisely, a quiver $Q = (Q_0, Q_1, s, t)$ consists of a set of **vertices** Q_0 and a set of **arrows** Q_1, together with two maps $s, t : Q_1 \to Q_0$ which specify the **source** and the **target** of each arrow, respectively.

We use the notation $\alpha : x \to y$ to denote an arrow $\alpha \in Q_1$ whose source is $s(\alpha) = x \in Q_0$ and target is $t(\alpha) = y \in Q_0$. Graphically, we can represent this data by the following picture:

$$x \xrightarrow{\alpha} y$$

We will define quivers by their graphical representations. For example,



represents a quiver with set of vertices $Q_0 = \{1, 2, 3, 4, 5\}$. Unless stated otherwise, a quiver $Q = (Q_0, Q_1)$ is always **finite**, that is, Q_0 and Q_1 are finite sets. In this case, we can suppose that $Q_0 = \{1, 2, \dots, n\}$ for some positive integer n, as in the example.

Remark. As illustrated above, our definition of quiver allows multiple arrows, arrows from a vertex to itself (i.e., **loops**), and vertices with no arrows.

Definition 1.1.2. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A **path** $\alpha_1\alpha_2 \cdots \alpha_l$ of **length** l from $x \in Q_0$ to $y \in Q_0$ is a sequence of l arrows such that $s(\alpha_1) = x$, $t(\alpha_l) = y$ and $t(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i < l$. For each vertex $x \in Q_0$, there is also a **stationary path** ε_x of length zero from x to x. Note that the domain of definition of s and t can be naturally extended to the set of paths of Q.

A path of nonzero length can be represented as

$$x = x_1 \xrightarrow{\alpha_1} x_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_l} x_{l+1} = y.$$

Concatenation defines a partial operation on the set of paths in a quiver. Two paths p and q in Q can be concatenated if $t(p) = s(q)$ and, in this case, we denote their concatenation by[1] pq. For example, a path of the form $\alpha_1\alpha_2 \cdots \alpha_l$ is the successive concatenation of the arrows $\alpha_1, \alpha_2, \dots, \alpha_l$, as the notation suggests.

We can upgrade concatenation to an algebra product:

Definition 1.1.3. Let K be a field and Q be a quiver. The **path algebra** KQ is the K-algebra whose underlying vector space has as a basis the set of paths in Q and such that the product of two basis vectors p and q is given by

$$p \cdot q = \begin{cases} pq & \text{if } t(p) = s(q), \\ 0 & \text{otherwise.} \end{cases}$$

The product is extended to all elements by bilinearity.

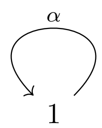
Since concatenation of paths is associative, the path algebra is indeed an associative algebra. Moreover, since we assume Q_0 is finite, we can form the element

$$\sum_{x \in Q_0} \varepsilon_x,$$

which is the identity element for KQ.

Some familiar examples can be realized as path algebras:

Example 1.1.4. Let Q be the following quiver:



A basis for the path algebra KQ is the set

$$\{\varepsilon_1, \alpha, \alpha^2, \alpha^3, \dots\},$$

where α^n is the concatenation of α with itself n times, for $n \geq 1$. With this information, it is easy to see that the homomorphism $K[x] \to KQ$ from the algebra of polynomials to the path algebra that sends x to α is an isomorphism.

[1] If p is a stationary path, the concatenation is just q, and vice versa.

Example 1.1.5. If Q is the quiver

$$1 \xrightarrow{\alpha} 2,$$

the path algebra KQ is three-dimensional with a basis given by the paths $\varepsilon_1, \varepsilon_2$ and α. We have the following identities:

$$\begin{gathered}
\varepsilon_i^2 = \varepsilon_i \quad (i = 1, 2), \\
\varepsilon_1\varepsilon_2 = 0 = \varepsilon_2\varepsilon_1, \\
\varepsilon_1\alpha = \alpha = \alpha\varepsilon_2, \\
\varepsilon_2\alpha = 0 = \alpha\varepsilon_1, \\
\alpha^2 = 0.
\end{gathered}$$

Therefore, if $T_2(K)$ is the algebra of 2×2 upper triangular matrices with entries in K, we have an isomorphism $T_2(K) \to KQ$ sending E_{11} to ε_1, E_{22} to ε_2 and E_{12} to α, where E_{ij} denotes the elementary matrix which has value 1 in the entry (i, j) and 0 elsewhere.

A similar argument shows that the algebra $T_n(K)$ of $n \times n$ upper triangular matrices with entries in K is isomorphic to path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n.$$

In general, if Q has no multiple arrows and its underlying graph is a tree, KQ is isomorphic to some subalgebra of $T_n(K)$, where n is the number of vertices of Q (see [3, Chapter II, Lemma 1.12]).

Note that the first example is infinite-dimensional, while the second is not. This happens due to the existence of cycles only in the first quiver. A **cycle** (or, more precisely, an **oriented cycle**) is a path of positive length which starts and ends at the same vertex. If p is a cycle in Q, then p can be concatenated with itself as many times as we want, showing that there are infinitely many distinct paths in Q or, equivalently, that KQ is infinite-dimensional. On the other hand, if Q is **acyclic**, that is, if Q has no cycles, then there are only finitely many paths in Q since Q is finite.

The second example also illustrates some algebraic properties of the stationary paths ε_x, for $x \in Q_0$. They are **idempotents**: they are equal to their squares. Furthermore, they are **orthogonal**, that is, $\varepsilon_x\varepsilon_y = \varepsilon_y\varepsilon_x = 0$ for distinct $x, y \in Q_0$. It is not hard to show that they cannot be written as a sum of nonzero orthogonal idempotents, so they are also **primitive**. Since the sum of all stationary paths is the identity element, they form a **complete set of primitive orthogonal idempotents** of KQ.

Remark. For a finite-dimensional algebra A, a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ gives rise to a decomposition of the regular right module A_A as a sum of indecomposables:

$$A_A = e_1A \oplus e_2A \oplus \cdots \oplus e_nA.$$

Every such decomposition appears in this way. Therefore, knowing a complete set of idempotents allows us to study indecomposable projective A-modules. We will do this for path algebras in Section 1.3.

Another important fact to know is when path algebras are connected. An algebra A is **connected** if its only central idempotents are 0 and 1. Equivalently, if we write

$A = A_1 \times A_2$ as a direct product of two algebras A_1 and A_2, then $A_1 = 0$ or $A_2 = 0$. A path algebra KQ has this property exactly when Q is **connected**, that is, if we forget the orientation of the arrows in Q, then every two vertices of Q are connected by some undirected path. A proof of this result can be found in [3, Chapter II, Lemma 1.7].

In the finite-dimensional case, that is, when Q is acyclic, it is easy to describe the radical of KQ. Let R_Q be the two-sided ideal of KQ generated by the arrows in Q, the so-called **arrow ideal**. Note that R_Q^m is the ideal generated by all paths of length $m \geq 1$ in Q. Since Q is acyclic, there is a maximal length a path can have, which implies that R_Q is a nilpotent ideal. Moreover, the quotient KQ/R_Q is isomorphic to a direct product of copies of K (one for each vertex of Q) and so is semisimple. These two properties guarantee that $\mathrm{rad}(KQ) = R_Q$.

Since the quotient of KQ by its radical is a direct product of division algebras, it follows that KQ is a **basic** algebra. In terms of modules, this means that, if we have a decomposition

$$KQ = P(1) \oplus P(2) \oplus \cdots \oplus P(n)$$

of the regular module as a sum of (projective) indecomposable modules, then $P(i) \ncong P(j)$ for $i \neq j$. In particular, we have $\varepsilon_x(KQ) \ncong \varepsilon_y(KQ)$ for distinct $x, y \in Q_0$.

We finish this section by collecting all the properties above in the same place.

Proposition 1.1.6. Let K be a field and Q be a quiver.

(1) A complete set of primitive orthogonal idempotents of the path algebra KQ is given by the set

$$\{\varepsilon_x \in KQ \mid x \in Q_0\}$$

of stationary paths.

(2) KQ is a connected algebra if and only if Q is a connected quiver.

(3) KQ is a finite-dimensional algebra if and only if Q is acyclic. In this case, KQ is a basic algebra and its radical is the arrow ideal.

1.2 The Ext-quiver of an algebra

Let A be a finite-dimensional K-algebra. In this section, we will see how to associate a quiver to A which will later help us understand its representation theory.

First of all, since we are only interested in studying the category of A-modules, we can assume A is basic. Indeed, if $P(1), P(2), \ldots, P(n)$ represent all distinct isomorphism classes of indecomposable projective A-modules and if we set

$$P \coloneqq P(1) \oplus P(2) \oplus \cdots \oplus P(n),$$

it is possible to show that the algebra $B = \mathrm{End}_A(P)$ is basic. Since the regular module A_A is a direct sum of the $P(i)$'s (but maybe with multiplicities), A_A is a direct summand of P^n for some $n \geq 1$, showing that P is a progenerator. By the Morita theorem, A is Morita equivalent to B, that is, their module categories are equivalent[1].

Remarkably, over an algebraically closed field, this assumption is enough to guarantee that A is a quotient of a path algebra.

[1] An introduction to Morita theory can be found in [27, Chapter 7], and a more direct proof for the fact that A and B are Morita equivalent is given in [3, Section I.6].

Theorem 1.2.1. Let K be an algebraically closed field. If A is a basic and finite-dimensional K-algebra, then $A \cong KQ/I$ for some finite quiver Q and some admissible ideal I of KQ.

An ideal I of KQ is **admissible** if it satisfies

$$R_Q^m \subseteq I \subseteq R_Q^2$$

for some $m \geq 2$, where R_Q is the arrow ideal. In this case, we call the pair (Q, I) a **bound quiver** and the quotient KQ/I a **bound path algebra**. Since Q is not necessarily acyclic, the first inclusion ensures that paths of length greater than m become zero in KQ/I, so this quotient is of finite dimension. On the other hand, the second inclusion allows us, among other things, to recover Q from KQ/I, as we will see.

A detailed proof of Theorem 1.2.1 can be found in [2, Theorem I.2.13] or in [3, Chapter II, Theorem 3.7]. We will just give a sketch. The first step is to generalize Proposition 1.1.6:

Proposition 1.2.2. Let K be a field and Q be a quiver. Let I be an admissible ideal of the path algebra KQ.

(1) A complete set of primitive orthogonal idempotents of KQ/I is given by the set

$$\{e_x \in KQ/I \mid x \in Q_0\},$$

where $e_x := \varepsilon_x + I$ for $x \in Q_0$.

(2) KQ/I is a connected algebra if and only if Q is a connected quiver.

(3) KQ/I is a basic algebra and its radical is R_Q/I, where R_Q denotes the arrow ideal of KQ.

Proof. This is [3, Chapter II, Corollary 2.12]. □

Let us suppose that $A = KQ/I$ for the moment. The proposition above tells us that there is a bijection between the set of vertices Q_0 and a complete set of primitive orthogonal idempotents of A. Moreover, using that $I \subseteq R_Q^2$, we have an isomorphism of vector spaces:

$$\frac{\operatorname{rad}(A)}{\operatorname{rad}^2(A)} = \frac{R_Q/I}{R_Q^2/I} \cong \frac{R_Q}{R_Q^2}.$$

The residual classes of the arrows in Q form a basis of this space. Thus, a basis of

$$e_x \left(\frac{\operatorname{rad}(A)}{\operatorname{rad}^2(A)} \right) e_y$$

is in bijection with the set of all arrows from $x \in Q_0$ to $y \in Q_0$. Taking the dimension of this vector space and varying x and y, we are able to recover Q.

This motivates the following definition.

Definition 1.2.3. Let A be a basic finite-dimensional algebra and $\{e_1, e_2, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents. The **Ext-quiver** Q_A of A is the quiver whose set of vertices is $\{1, 2, \ldots, n\}$ and such that the number of arrows from i to j is given by the dimension of the K-vector space

$$e_i \left(\frac{\operatorname{rad}(A)}{\operatorname{rad}^2(A)} \right) e_j,$$

for all $i, j \in \{1, 2, \dots, n\}$.

Remark. The reason for the name "Ext-quiver" comes from a different construction of Q_A using the Ext functor which we now describe. Since A is basic, the map $e_i \mapsto e_i A$ defines a bijection between the given complete set of primitive orthogonal idempotents and the set of isomorphism classes of indecomposable projective modules. The latter, in turn, corresponds to the set of isomorphism classes of simple modules via the map

$$e_i A \mapsto \operatorname{top}(e_i A) := \frac{e_i A}{\operatorname{rad}(e_i A)}.$$

One can show (see [3, Chapter III, Lemma 2.12]) that there is a linear isomorphism

$$\operatorname{Ext}^1_A(\operatorname{top}(e_i A), \operatorname{top}(e_j A)) \cong e_i \left(\frac{\operatorname{rad}(A)}{\operatorname{rad}^2(A)} \right) e_j.$$

Therefore, Q_A could have been defined as the quiver whose set of vertices is the set of isomorphism classes of simple modules and such that the number of arrows from S to T is the dimension of $\operatorname{Ext}^1_A(S, T)$ over K. Note that this definition does not depend on the choice of a complete set of primitive orthogonal idempotents.

We can now sketch the proof of Theorem 1.2.1. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A and construct the Ext-quiver Q_A. Given two vertices $x, y \in (Q_A)_0$, let $\alpha_1, \dots, \alpha_m$ be all arrows from x to y. By the definition of Q_A, there are elements $a_{\alpha_1}, \dots, a_{\alpha_m} \in e_x(\operatorname{rad}(A))e_y$ whose residual classes form a basis of

$$e_x \left(\frac{\operatorname{rad}(A)}{\operatorname{rad}^2(A)} \right) e_y.$$

This choice of elements defines a linear map $\varphi : KQ_A \to A$ which takes a stationary path ε_x to e_x, an arrow α to a_α and a path $\alpha_1\alpha_2 \cdots \alpha_l$ to

$$\varphi(\alpha_1)\varphi(\alpha_2) \cdots \varphi(\alpha_l) = a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_l}.$$

Since $e_1, \dots, e_n$ are orthogonal idempotents and since for each arrow $\alpha : x \to y$ we have $a_\alpha = e_x a_\alpha e_y$, it is not hard to see that φ is a homomorphism of algebras.

One can show that φ is surjective. The difficult part is to prove that the products of the elements a_α generate $\operatorname{rad}(A)$. On the other hand, since A is basic, $A/\operatorname{rad}(A)$ is a direct product of division algebras. Recalling that K is algebraically closed, these division algebras are actually copies of K and we deduce that the residual classes of $e_1, \dots, e_n$ generate $A/\operatorname{rad}(A)$. Therefore, the elements e_x for $x \in (Q_A)_0$ and a_α for $\alpha \in (Q_A)_1$ generate A and the surjectivity of φ follows.

From this information, we have $A \cong KQ_A/I$ for $I = \ker \varphi$. By construction, observe that $\varphi(R_{Q_A}) \subseteq \operatorname{rad}(A)$. Since A is finite-dimensional, $\operatorname{rad}(A)$ is a nilpotent ideal and we can find an integer $m \geq 2$ such that $\varphi(R^m_{Q_A}) \subseteq \operatorname{rad}(A)^m = 0$. Thus, $R^m_{Q_A} \subseteq I$. Some further calculations show that $I \subseteq R^2_{Q_A}$ and we conclude that I is an admissible ideal of KQ_A. This completes the proof.

Remark. We used that K is algebraically closed only to guarantee that $A/\operatorname{rad}(A)$ is a direct product of copies of K. An algebra satisfying this property is called **elementary**. We can remove the hypothesis on K with we assume that A is elementary rather than just basic.

It is possible to work with general basic algebras over non-algebraically closed fields. To do so, the main idea is to add the data of the endomorphism rings of the simple modules into the definition of quiver, what leads us to the notion of a "modulated quiver", also known as a "species". However, some hypothesis on K are still necessary for Theorem 1.2.1 to hold. We refer the reader to [8, Section 4.1] for more details.

Example 1.2.4. Let us illustrate the proof above with a concrete example. Inside the algebra of 3×3 matrices over K, consider the subalgebra

$$A = \begin{pmatrix} K & K & K \\ 0 & K & 0 \\ 0 & 0 & K \end{pmatrix}.$$

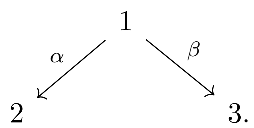
The diagonal elementary matrices E_{11}, E_{22} and E_{33} form a complete set of primitive orthogonal elements of A, because they do the same for the full matrix algebra. Furthermore, we have

$$\operatorname{rad}(A) = \begin{pmatrix} 0 & K & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

since this is a nilpotent ideal of A satisfying $A/\operatorname{rad}(A) \cong K \times K \times K$. This also shows that A is elementary. Note that $\operatorname{rad}^2(A) = 0$, hence, in order to construct the Ext-quiver Q_A of A, we have to find the dimensions of $E_{ii} \operatorname{rad}(A) E_{jj}$ as we vary $i, j \in \{1, 2, 3\}$. Only two of these subspaces are nonzero:

$$E_{11} \operatorname{rad}(A) E_{22} \quad \text{and} \quad E_{11} \operatorname{rad}(A) E_{33},$$

and they are generated by the elementary matrices E_{12} and E_{13}, respectively. We conclude that Q_A is the quiver



We get a surjective homomorphism $\varphi : KQ_A \to A$ sending ε_i to E_{ii} $(i = 1, 2, 3)$, α to E_{12} and β to E_{13}. Since KQ_A and A are both five-dimensional, φ is an isomorphism.

1.3 Modules over bound path algebras

In order to understand the representation theory of a finite-dimensional algebra A over an algebraically closed field K, Theorem 1.2.1 says that we can suppose A is a bound path algebra. It does not simplify our problem very much, but at least gives us a graphical way of writing modules which is helpful to calculate examples. We will see how we can do this in this section.

Definition 1.3.1. Let Q be a quiver and K be a field. A **representation** $M = (M_x, \varphi_\alpha)_{x \in Q_0, \alpha \in Q_1}$ of Q is the data of:

- a K-vector space M_x for each vertex $x \in Q_0$;
- a linear transformation $\varphi_\alpha : M_x \to M_y$ for each arrow $\alpha : x \to y$ in Q_1.

We will write just $M = (M_x, \varphi_\alpha)$ to simplify the notation. If $N = (N_x, \varphi'_\alpha)$ is another representation, a **morphism** $f : M \to N$ is a collection of linear maps $f_x : M_x \to N_x$

for each $x \in Q_0$ such that, for every arrow $\alpha : x \to y$, we have a commutative diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ {\scriptstyle \varphi_\alpha}\downarrow & & \downarrow{\scriptstyle \varphi'_\alpha} \\ M_y & \xrightarrow{f_y} & N_y. \end{array}$$

If we define composition of morphisms vertex-wise, we get the category $\mathrm{Rep}_K(Q)$ of representations of Q. We denote by $\mathrm{rep}_K(Q)$ the full subcategory of representations of finite dimension, that is, representations $M = (M_x, \varphi_\alpha)$ where each M_x is finite-dimensional.

Remark. Note the similarity between a morphism of representations and a natural transformation of functors. This observation can be upgraded to an equivalence of categories. For a quiver Q, we can define its free category CQ: objects are the vertices of Q and the set of morphisms from one vertex to another is the set of paths between them. Composition is given by concatenation. It is easy to check that $\mathrm{Rep}_K(Q)$ is equivalent to the category of functors from CQ to the category of K-vector spaces.

Notice also that $\mathrm{Rep}_K(Q)$ is naturally a K-category, that is, all the Hom-sets are K-vector spaces and composition is bilinear. Moreover, $\mathrm{Rep}_K(Q)$ and $\mathrm{rep}_K(Q)$ are both abelian. These facts can be deduced, for example, from the equivalence above.

Let $M = (M_x, \varphi_\alpha)$ be a representation of Q. For every nonstationary path $w = \alpha_1\alpha_2 \cdots \alpha_l$ from $x \in Q_0$ to $y \in Q_0$, we can define a linear map $\varphi_w : M_x \to M_y$ by setting

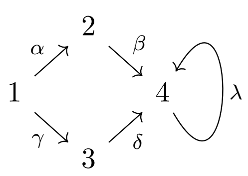
$$\varphi_w = \varphi_{\alpha_l}\varphi_{\alpha_{l-1}} \cdots \varphi_{\alpha_1}.$$

If $p \in KQ$ is a **relation**, that is, a linear combination of paths of length at least two having the same source and the same target, then we can extend linearly the construction above to define φ_p. This allows us to define representations of bound quivers.

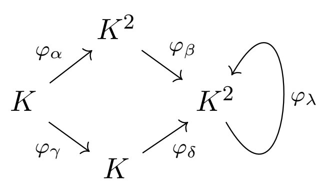
Definition 1.3.2. Let Q be a quiver and I be an admissible ideal of KQ. A representation $M = (M_x, \varphi_\alpha)$ of Q is **bound by** I if $\varphi_p = 0$ for every relation $p \in I$. In this case, we say that M is a representation of the bound quiver (Q, I).

The full subcategory of $\mathrm{Rep}_K(Q)$ of representations bound by I is denoted by $\mathrm{Rep}_K(Q, I)$. We define analogously the full subcategory $\mathrm{rep}_K(Q, I)$ of $\mathrm{rep}_K(Q)$.

Example 1.3.3. Let Q be the quiver



An example of a representation of Q is given by



where the maps are represented by the following matrices in the canonical bases:

$$\varphi_\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \varphi_\gamma = \begin{pmatrix} 1 \end{pmatrix}, \quad \varphi_\delta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \varphi_\lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now, let I be the ideal of KQ generated by the relations $\alpha\beta - \gamma\delta$ and λ^2. It is easy to check that I is an admissible ideal. Since

$$\varphi_\beta\varphi_\alpha - \varphi_\delta\varphi_\gamma = 0 \quad \text{and} \quad \varphi_\lambda^2 = 0,$$

the representation above is bound by I.

The reason for introducing these definitions is that they essentially describe modules over the bound path algebra.

Theorem 1.3.4. Let (Q, I) be a bound quiver and define $A = KQ/I$. There exists a K-linear equivalence of categories

$$F : \operatorname{Mod} A \longrightarrow \operatorname{Rep}_K(Q, I)$$

which restricts to an equivalence between $\operatorname{mod} A$ and $\operatorname{rep}_K(Q, I)$.

We denote by $\operatorname{Mod} A$ the category of *right* A-modules and by $\operatorname{mod} A$ its full subcategory of finite-dimensional modules. If we worked with left modules instead, then we would get an equivalence with $\operatorname{Rep}_K(Q^{\mathrm{op}}, I^{\mathrm{op}})$, where Q^{op} is the quiver obtained from Q by reversing the arrows and I^{op} is obtained from I by reversing the paths.

Proof. By Proposition 1.2.2, there is a canonical complete set of primitive orthogonal idempotents $\{e_x \in A \mid x \in Q_0\}$ of A. We will use them to define the functor F.

Let M be an A-module and let us construct a representation $F(M)$ of (Q, I). For each vertex $x \in Q_0$, we define $F(M)_x \coloneqq Me_x$. For each arrow $\alpha : x \to y$ in Q_1 we associate the linear map

$$\begin{aligned} \varphi_\alpha : Me_x &\longrightarrow Me_y \\ m &\longmapsto m\overline{\alpha}, \end{aligned}$$

where $\overline{\alpha}$ is the residual class of α in $A = KQ/I$. This is well-defined since $\overline{\alpha} = \overline{\alpha}e_y$. In this way, we get a representation of Q and one can check that it is bound by I, since relations in I become zero after taking the quotient.

Now, let us define F on morphisms. Given a homomorphism $f : M \to N$ of A-modules, note that $f(Me_x) \subseteq Ne_x$ for all $x \in Q_0$. Hence, for each $x \in Q_0$, we have a linear map $F(f)_x : F(M)_x \to F(N)_x$ which is simply the restriction of f. Since f commutes with the action of A on M and N, this defines a morphism of representations $F(f) : F(M) \to F(N)$. It is clear that the resulting association F is indeed a K-linear functor from $\operatorname{Mod} A$ to $\operatorname{Rep}_K(Q, I)$.

In order to show that F is an equivalence, we just have to find a quasi-inverse $G : \operatorname{Rep}_K(Q, I) \to \operatorname{Mod} A$. For a representation $M = (M_x, \varphi_\alpha)$ of (Q, I), let $G(M)$ be the K-vector space

$$\bigoplus_{x \in Q_0} M_x.$$

We can endow $G(M)$ with a KQ-module structure as follows: for a path of nonzero length $w \in KQ$ from i to j, remember that we have a linear map $\varphi_w : M_i \to M_j$. We define the action of w on $G(M)$ as the composition

$$\bigoplus_{x\in Q_0} M_x \longrightarrow M_i \xrightarrow{\varphi_w} M_j \longrightarrow \bigoplus_{x\in Q_0} M_x,$$

where the unspecified maps are the natural projection and the natural inclusion, respectively. For a stationary path ε_i, we replace the middle map above by the identity map of M_i. It is not hard to prove that $G(M)$ really becomes a KQ-module in this manner. Notice that I annihilates $G(M)$ because M is bound by I, so $G(M)$ can be seen as an A-module.

Finally, let us define G on morphisms. A morphism of representations $f : M \to N$ is a collection of maps $f_x : M_x \to N_x$ for each $x \in Q_0$, so we can form the linear map

$$G(f) = \bigoplus_{x\in Q_0} f_x : G(M) \to G(N).$$

The compatibility condition between the maps f_x implies that $G(f)$ is a homomorphism of A-modules, as desired.

We omit the verification that F and G are quasi-inverses for each other. For the last statement in the theorem, it is enough to note that $F(M)$ is finite-dimensional if and only if M is, since the direct sums above are finite. □

Remark. When we said that I annihilates $G(M)$ because M is bound by I, we implicitly used that I is generated by relations. This is true because any element $p \in I$ can be written as

$$p = \sum_{x,y\in Q_0} \varepsilon_x p \varepsilon_y$$

and each $\varepsilon_x p \varepsilon_y \in I$ is a relation since $I \subseteq R_Q^2$. This was the only property of I we used. Thus, Theorem 1.3.4 holds more generally. For example, if Q is any quiver, then $\operatorname{Mod} KQ$ is equivalent to $\operatorname{Rep}_K(Q)$, even if Q is not acyclic.

For the rest of the section, we fix $A = KQ/I$ for some bound quiver (Q, I). Using Theorem 1.3.4, we will identify A-modules and representations of (Q, I). This will help us to describe, in particular, the simple, projective and injective A-modules.

For each $x \in Q_0$, we can define a representation $S(x)$ by setting

$$S(x)_y = \begin{cases} K & \text{if } y = x, \\ 0 & \text{otherwise,} \end{cases}$$

and, for each arrow, we associate the zero map. It is clear that $S(x)$ is bound by I.

Lemma 1.3.5. If $x \in Q_0$, then $S(x)$ is a simple A-module isomorphic to the top of the indecomposable projective A-module $e_x A$. In particular, the set

$$\{S(x) \mid x \in Q_0\}$$

is a complete set of representatives of the isomorphism classes of simple A-modules.

Proof. Since $S(x)$ is an one-dimensional A-module, it is simple. Moreover, we have a linear isomorphism

$$\operatorname{Hom}_A(e_x A, S(x)) \cong S(x)e_x = S(x)_x = K.$$

In particular, there is a nonzero map $e_xA \to S(x)$. Since the top of an indecomposable projective module is simple, we conclude that $\mathrm{top}(e_xA) \cong S(x)$.

There is a bijection between the set of indecomposable projective modules and the set of simple modules that sends a projective module to its top. Since $\{e_x \mid x \in Q_0\}$ is a complete set of primitive orthogonal idempotents and A is basic, the set

$$\{e_xA \mid x \in Q_0\}$$

represents all distinct isomorphism classes of indecomposable projective modules. The second statement of the lemma then follows. □

Given $x \in Q_0$, we denote by $P(x)$ the indecomposable projective A-module e_xA. These modules have a simple description too:

Lemma 1.3.6. Let $x \in Q_0$.

(1) If we write $P(x)$ as a representation $(P(x)_y, \varphi_\alpha)$, then $P(x)_y$ is the subspace of A generated by the residual classes of paths from x to y, and, for an arrow $\alpha : y \to z$, the map $\varphi_\alpha : P(x)_y \to P(x)_z$ is given by right multiplication by $\overline{\alpha} := \alpha + I$.

(2) The radical $\mathrm{rad}(P(x))$ has the same description, except that we only consider paths of *nonzero* length from x to y when defining $\mathrm{rad}(P(x))_y$.

Proof. Using the equivalence from Theorem 1.3.4, we have

$$P(x)_y = P(x)e_y = e_xAe_y.$$

Since A is generated by the residual classes of paths in Q, the subspace above is generated by the residual classes of paths from x to y. It also follows from the proof of this theorem that φ_α is right multiplication by $\overline{\alpha}$. In this case, we can interpret it as concatenation with α.

For the second part, recall from Proposition 1.2.2 that the radical of A is given by R_Q/I, which is generated by the elements $\overline{\alpha}$ for $\alpha \in Q_1$. Thus,

$$\mathrm{rad}(P(x)) = P(x) \cdot \mathrm{rad}(A) = \sum_{\alpha \in Q_1} P(x)\overline{\alpha}.$$

In particular, given $y \in Q_0$, $\mathrm{rad}(P(x))_y = \mathrm{rad}(P(x))e_y$ equals

$$\sum_{\alpha : z \to y} P(x)\overline{\alpha} = \sum_{\alpha : z \to y} P(x)(e_z\overline{\alpha}) = \sum_{\alpha : z \to y} P(x)_z\overline{\alpha} = \sum_{\alpha : z \to y} \mathrm{im}\,\varphi_\alpha.$$

If $y \neq x$, then every path from x to y is the concatenation of a path from x to some vertex $z \in Q_0$ with an arrow $\alpha : z \to y$. In other words, $\mathrm{rad}(P(x))_y = P(x)_y$. The same is true for every path from x to x apart from the stationary path, so that $\mathrm{rad}(P(x))_x$ is generated by the residual classes of paths of nonzero length from x to x. This completes the proof. □

Remark. If $M = (M_x, \varphi_\alpha)$ is a representation of (Q, I), the same argument shows that $\mathrm{rad}(M)$ is the representation $N = (N_x, \psi_\alpha)$ where

$$N_x = \sum_{\alpha : y \to x} \mathrm{im}\,\varphi_\alpha$$

and each ψ_α is the restriction of φ_α.

For the study of the indecomposable injective modules, let us recall some facts about duality for finite-dimensional algebras[1]. Given a *right* A-module M, we define the dual vector space

$$DM := \mathrm{Hom}_K(M, K).$$

This space acquires a structure of *left* A-module if we set

$$(a \cdot f)(m) = f(ma)$$

for $f \in DM$, $m \in M$ and $a \in A$. We can naturally extend D to a functor and, in the finite-dimensional case, it becomes a *duality*

$$D : \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, A^{\mathrm{op}},$$

that is, D is a contravariant functor and defines an equivalence between the opposite category of $\mathrm{mod}\, A$ and the category $\mathrm{mod}\, A^{\mathrm{op}}$. The quasi-inverse is also given by taking dual spaces and we will denote it by D too. They are called the **K-duality functors**.

One consequence of this duality, for example, is that an A-module M is injective if and only if DM is projective. Therefore, the indecomposable injective right A-modules are of the form $D(Ae)$ for some primitive idempotent $e \in A$. This module is the injective hull of

$$\mathrm{soc}(D(Ae)) \cong D\,\mathrm{top}(Ae),$$

which is a simple module.

Returning to the case of a bound path algebra $A = KQ/I$, it is not difficult to see that $A^{\mathrm{op}} \cong KQ^{\mathrm{op}}/I^{\mathrm{op}}$. Hence, $\mathrm{mod}\, A^{\mathrm{op}}$ is equivalent to $\mathrm{rep}_K(Q^{\mathrm{op}}, I^{\mathrm{op}})$, as we highlighted after stating Theorem 1.3.4. We already know how to describe the projective modules in $\mathrm{mod}\, A^{\mathrm{op}}$ by Lemma 1.3.6, so we just have to take the dual space to study the injective modules. Observe that, in the context of representations, the dual of $(M_x, \varphi_\alpha)_{x \in Q_0^{\mathrm{op}}, \alpha \in Q_1^{\mathrm{op}}}$ is simply $(DM_x, D\varphi_{\alpha^{\mathrm{op}}})_{x \in Q_0, \alpha \in Q_1}$.

Given $x \in Q_0$, we denote by $I(x)$ the indecomposable injective A-module $D(Ae_x)$. Its socle is isomorphic to $S(x)$. Dualizing Lemma 1.3.6, we get:

Lemma 1.3.7. Let $x \in Q_0$.

(1) If we write $I(x)$ as a representation $(I(x)_y, \varphi_\alpha)$, then $I(x)_y$ is the dual of the subspace of A generated by the residual classes of paths from y to x, and, for an arrow $\alpha : y \to z$, the map $\varphi_\alpha : I(x)_y \to I(x)_z$ is given by the dual of the left multiplication by $\overline{\alpha} := \alpha + I$.

(2) The quotient $I(x)/\mathrm{soc}(I(x))$ has the same description, except that we only consider paths of *nonzero* length from y to x when defining $(I(x)/\mathrm{soc}(I(x)))_y$.

Example 1.3.8. Let Q be the following quiver:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

Let I be the admissible ideal generated by the relation $\alpha\beta\gamma$ and consider the bound path algebra $A = KQ/I$. Let us compute the projective and the injective indecomposable modules.

The projective ones are easily obtained from Lemma 1.3.6:

$$P(1) = \quad K \xrightarrow{\mathrm{id}} K \xrightarrow{\mathrm{id}} K \longrightarrow 0,$$

[1] A more detailed explanation is found in [14, Section 9.1].

$$P(2) = \ 0 \longrightarrow K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K,$$
$$P(3) = \ 0 \longrightarrow 0 \longrightarrow K \xrightarrow{\text{id}} K,$$
$$P(4) = S(4).$$

Notice that the last space in $P(1)$ is zero because the unique path from 1 to 4 is $\alpha\beta\gamma$, which becomes zero in the quotient.

In order to find the injective modules, we repeat the procedure above with the opposite quiver Q^{op} (bound by the "opposite" relation $\gamma^{\text{op}}\beta^{\text{op}}\alpha^{\text{op}}$) and then we apply the duality functor. We get:

$$I(1) = S(1),$$
$$I(2) = \ K \xrightarrow{\text{id}} K \longrightarrow 0 \longrightarrow 0,$$
$$I(3) = \ K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \longrightarrow 0,$$
$$I(4) = \ 0 \longrightarrow K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K.$$

In this case, we have $I(3) \cong P(1)$ and $I(4) \cong P(2)$.

1.4 The Nakayama functor and the Grothendieck group

In the end of the previous section, we saw that there is the same number of projective and injective indecomposable modules over a finite-dimensional K-algebra A, up to isomorphism. It turns out that there is in fact an equivalence

$$\nu : \operatorname{proj} A \longrightarrow \operatorname{inj} A$$

between the full subcategories $\operatorname{proj} A$ and $\operatorname{inj} A$ of $\operatorname{mod} A$ whose objects are the projective and the injective modules, respectively. This section presents some properties of the functor ν above, which will be extremely important in Chapter 4.

The K-duality functor restricts to a duality between projective and injective modules, but it interchanges left and right modules. For example, we have a duality

$$D : \operatorname{proj} A^{\text{op}} \longrightarrow \operatorname{inj} A.$$

In order to construct the functor ν, it is enough to find another duality

$$D' : \operatorname{proj} A \longrightarrow \operatorname{proj} A^{\text{op}},$$

because then the composition DD' will be a covariant functor defining the desired equivalence.

A duality that works in this scenario is given by the **A-duality functor**

$$(-)^t := \operatorname{Hom}_A(-, A_A) : \operatorname{mod} A \longrightarrow \operatorname{mod} A^{\text{op}}.$$

In this case, if M is a right A-module, then $M^t = \operatorname{Hom}_A(M, A_A)$ becomes a left A-module with

$$(a \cdot f)(m) = af(m)$$

for $f \in M^t$, $m \in M$ and $a \in A$. Similarly, there is a functor going in the other direction given by $\operatorname{Hom}_{A^{\text{op}}}(-, {}_AA)$. We will also denote it by $(-)^t$.

Lemma 1.4.1. The A-duality functor $(-)^t$ restricts to a duality between $\operatorname{proj} A$ and $\operatorname{proj} A^{\mathrm{op}}$. A quasi-inverse is also given by $(-)^t$.

Proof. Firstly, let us see that the A-duality functor sends projectives to projectives. Since it is an additive functor, it is enough to check this property for indecomposable projective modules. Indeed, if $e \in A$ is a primitive idempotent, we have an isomorphism of left A-modules

$$\begin{aligned}(eA)^t &\longrightarrow Ae\\ f &\longmapsto f(e),\end{aligned}$$

and Ae is again projective.

Now, for any left (or right) A-module M, we have the evaluation morphism $\varphi_M : M \to M^{tt}$ defined by

$$\varphi_M(m)(f) = f(m)$$

for $m \in M$ and $f \in M^t$. It is a homomorphism of left (or right) A-modules and it is natural in the variable M. Furthermore, using the isomorphism $A^t \cong A$ (as left or right modules), one can check that φ_A is an isomorphism. Since the A-duality functor is additive, it follows that φ_M is an isomorphism whenever M is projective. This shows that applying $(-)^t$ twice, starting either from $\operatorname{proj} A$ or from $\operatorname{proj} A^{\mathrm{op}}$, is naturally isomorphic to the identity functor of the starting category. In other words, we have the duality from the statement. □

As a corollary, we can define our especial functor:

Definition 1.4.2. The **Nakayama functor** $\nu : \operatorname{mod} A \to \operatorname{mod} A$ is defined as the composition of the A-duality and the K-duality functors:

$$\nu := D(-)^t = D\operatorname{Hom}_A(-, A_A).$$

It restricts to an equivalence from $\operatorname{proj} A$ to $\operatorname{inj} A$.

The quasi-inverse for the equivalence above is given by the restriction of the functor $\nu^{-1} : \operatorname{mod} A \longrightarrow \operatorname{mod} A$ defined as the composition in the other order:

$$\nu^{-1} := (D(-))^t = \operatorname{Hom}_{A^{\mathrm{op}}}(D(-), {}_AA).$$

Since D is a duality and D^2 is naturally isomorphic to the identity functor, we can write

$$\operatorname{Hom}_{A^{\mathrm{op}}}(D(-), A) \cong \operatorname{Hom}_A(DA, D^2(-)) \cong \operatorname{Hom}_A(DA, -).$$

We are considering DA as a right A-module above. It also has a structure of left A-module, which turns $\operatorname{Hom}_A(DA, -)$ into a right A-module.

The Nakayama functor also has an alternative description using the bimodule DA.

Proposition 1.4.3. The Nakayama functor ν is naturally isomorphic to the functor $- \otimes_A DA$.

Proof. Since D is a contravariant exact functor and $\operatorname{Hom}_A(-, A)$ is a contravariant left exact functor, it follows that ν is a covariant right exact functor. Moreover, ν is additive and

$$\nu A = D\operatorname{Hom}_A(A, A) \cong DA$$

as bimodules. The result follows from the Eilenberg-Watts theorem (see [1, Chapitre V, Théorème 3.2]). □

Remark. Over the whole module category, ν and ν^{-1} are not quasi-inverses, but they form a pair of adjoint functors by the "tensor-hom" adjunction.

It will be useful to know how ν acts on the Grothendieck group of A. Let us first recall what this is. A more detailed exposition can be found in [2, Section I.1.4] and in [3, Section III.3].

Definition 1.4.4. The **Grothendieck group** $K_0(A)$ of A (more precisely, of $\operatorname{mod} A$) is the quotient of the free abelian group on the set of isomorphism classes $[M]$ of modules M in $\operatorname{mod} A$ by the subgroup generated by all expressions of the form

$$[L] - [M] + [N]$$

whenever there is a short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

The element in $K_0(A)$ corresponding to a module M will also be denoted by $[M]$.

If $\{S(1), \dots, S(n)\}$ is a complete set of representatives of isomorphism classes of simple A-modules, then we have an isomorphism of abelian groups

$$\begin{aligned} \underline{\dim} : K_0(A) &\longrightarrow \mathbb{Z}^n \\ [M] &\longmapsto (\mu_1(M), \dots, \mu_n(M)), \end{aligned}$$

where $\mu_i(M)$ denotes the multiplicity of $S(i)$ as a composition factor of M, for all $1 \leq i \leq n$. In particular, the elements $[S(1)], \dots, [S(n)]$ form a basis of the free abelian group $K_0(A)$. The vector $\underline{\dim}([M])$, or simply $\underline{\dim}\, M$, is called the **dimension vector** of M.

Remark. If $A = KQ/I$ for some bound quiver (Q, I) with $Q_0 = \{1, \dots, n\}$, we could have defined $\underline{\dim}$ as the function given by

$$[(M_i, \varphi_\alpha)] \longmapsto (\dim_K(M_1), \dots, \dim_K(M_n)).$$

It follows easily from Lemma 1.3.5 that this definition agrees with the original one, justifying the name "dimension vector".

When A is an algebra of finite global dimension, that is, when every finite-dimensional module has a finite projective (or, equivalently, injective) resolution, $K_0(A)$ has also two other noteworthy bases. For the following, $\{P(1), \dots, P(n)\}$ and $\{I(1), \dots, I(n)\}$ will denote complete sets of representatives of indecomposable projective and indecomposable injective modules, respectively, and we enumerate them so that $S(i) \cong \operatorname{top}(P(i)) \cong \operatorname{soc}(I(i))$.

Lemma 1.4.5. If $\operatorname{gldim} A < \infty$, then the sets

$$\{[P(1)], \dots, [P(n)]\} \quad \text{and} \quad \{[I(1)], \dots, [I(n)]\}$$

are bases for the Grothendieck group $K_0(A)$.

Proof. We will prove the result for the projective modules. The other case is similar.

If $1 \leq i \leq n$, then $S(i)$ has a finite projective resolution by hypothesis, that is, we have an exact sequence

$$0 \longrightarrow P_m \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow S(i) \longrightarrow 0$$

where $P_0, \ldots, P_m$ are finite-dimensional projective modules. In the Grothendieck group, this gives us an equality:

$$[S(i)] = \sum_{j=0}^{m} (-1)^j [P_j].$$

Each P_j is a direct sum of copies of the modules $P(1), \ldots, P(n)$, so we can write $[S(i)]$ as a linear combination of the first set in the statement. This holds for all i. Since $\{[S(1)], \ldots, [S(n)]\}$ is a basis for $K_0(A)$ and has the same cardinality of $\{[P(1)], \ldots, [P(n)]\}$, this latter set must be a basis too. □

Remark. We can consider the dimension vector of the modules $P(1), \ldots, P(n)$ as the columns of a $n \times n$ matrix. This is the **Cartan matrix** C_A of A. The rows of C_A are the dimension vectors of the modules $I(1), \ldots, I(n)$ (see [3, Chapter III, Proposition 3.8]). When $\operatorname{gldim} A < \infty$, the argument above shows that C_A has a right inverse, thus it is invertible since it is a square matrix.

This result allows us to define an important transformation.

Definition 1.4.6. Suppose $\operatorname{gldim} A < \infty$. The **Coxeter transformation** of A is the map

$$\begin{aligned} c_A : K_0(A) &\longrightarrow K_0(A) \\ [P(i)] &\longmapsto -[I(i)]. \end{aligned}$$

By Lemma 1.4.5, this is a well-defined invertible map.

Remark. Using the basis of projective modules for the domain and the basis of injective modules for the target, c_A is represented by the opposite of the identity matrix. Changing back to the canonical basis of simple modules, we get that c_A is represented by the matrix $C_A^t \cdot (-I_n) \cdot C_A^{-1} = -C_A^t C_A^{-1}$. This is the **Coxeter matrix** of A.

If $P(i) \cong eA$ for some primitive idempotent $e \in A$, then

$$\nu P(i) \cong D \operatorname{Hom}_A(eA, A) \cong D(Ae) \cong I(i).$$

Therefore, for any projective module P we have the equality

$$c_A([P]) = -[\nu P].$$

It does not hold for arbitrary modules because ν is not exact, so it does not induce a well-defined map on the Grothendieck group. However, in Section 4.3, we will consider the derived functor of ν and the equality above will make sense more generally.

There are two reasons for putting a minus sign in the definition of c_A. Firstly, in some cases, c_A coincides with a Coxeter element coming from the theory of root systems (see Section A.3). Secondly, it also coincides with the action of the Auslander-Reiten translation in the Grothendieck group of the bounded derived category of $\operatorname{mod} A$ (see Lemma 4.3.4). We explore these facts in Chapter 4.

Chapter 2

Gabriel's theorem

One of the main objectives of representation theory is to classify all representations of a given object, up to isomorphism, and them understand the morphisms between them. However, this task is in general unfeasible and there is no hope of finding a complete classification theorem. In this sense, it is important to know if we are dealing with these "wild" cases or not. The most promising case is that of finite representation type.

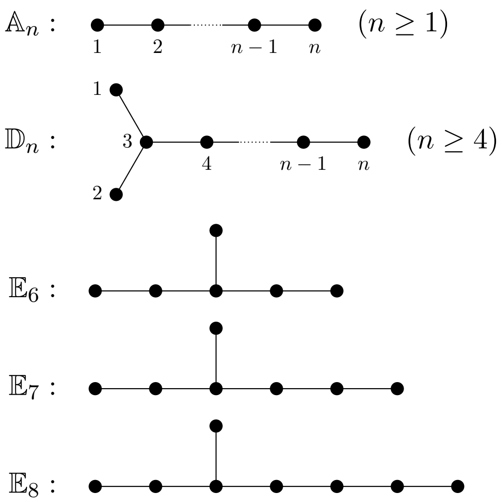
Definition 2.0.1. Let A be an algebra over a field K. We say that A is **of finite representation type** if there are only finitely many isomorphism classes of finite-dimensional indecomposable A-modules.

In this chapter, we will restrict ourselves to path algebras of quivers and give a proof (following [11] and [13]) of a famous result due to Pierre Gabriel:

Theorem 2.0.2 (Gabriel)**.** *Let Q be a finite quiver and K an algebraically closed field. The path algebra KQ is of finite representation type if, and only if, the underlying graph of Q is a disjoint union of Dynkin diagrams of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$.*

The first remarkable aspect of the theorem above is that the orientation of the quiver is irrelevant. The representation type depends only on the underlying graph, which is obtained by forgetting the orientation of the arrows and considering them as normal edges in a graph.

The second striking fact is the appearance of the Dynkin diagrams, which come from the seemingly unrelated theory of root systems. Those that we are concerned are divided into the two infinite families $\mathbb{A}$ and $\mathbb{D}$, and the finite family $\mathbb{E}$. They are the following:



The subscript indicates the number of vertices in the corresponding graph. For what follows, we will say that the quiver Q is a **Dynkin quiver** if its underlying graph is one of the Dynkin diagrams above.

The connection with root systems does not stop here. We will see that, for these quivers, the map

$$M \longmapsto \underline{\dim}\, M$$

defines a bijection between the set of isomorphism classes of indecomposable modules in $\operatorname{mod} KQ$ and the set of positive roots of the root system associated to Q. This is the fact that will prove the finiteness property in the theorem.

Remark. Note that the statement does not require that Q is acyclic. Thus, Gabriel's theorem implies that, if KQ is not of finite dimension, then it cannot be of finite representation type. Nevertheless, we will stick to finite-dimensional algebras and focus on the case when Q is indeed acyclic.

We can also assume that Q is connected because, otherwise, KQ decomposes as a direct product of smaller path algebras and $\operatorname{mod} KQ$ is equivalent to the product of their module categories.

From now on, K will denote an algebraically closed field and all modules will be of finite dimension over K.

2.1 Hereditary algebras

One important property of path algebras is that they are hereditary. Let us first recall what this means.

Definition 2.1.1. A finite dimensional K-algebra A is **hereditary** if the following equivalent conditions hold:

(1) Every submodule of a projective A-module is projective.

(2) The radical of any indecomposable projective A-module is projective.

(3) The global dimension of A is at most one.

(4) For every A-module M and $n \geq 2$, we have

$$\operatorname{Ext}^n_A(M, -) = 0 = \operatorname{Ext}^n_A(-, M).$$

For a proof of the equivalence of the conditions above and for some further properties of hereditary algebras, see [1, Chapitre XII, Section 1] or [3, Section VII.1].

Proposition 2.1.2. If Q is acyclic, then KQ is a hereditary algebra.

Proof. Let us check that the radical of an indecomposable projective module is again projective. Let $x \in Q_0$ and consider the module $P(x)$, as defined in Section 1.3. By Lemma 1.3.6, if we see $\operatorname{rad}(P(x))$ as a representation (M_y, φ_α), then M_y is isomorphic to the subspace of KQ generated by paths of length at least one from x to y. Adapting the proof of this lemma, we see that $\operatorname{rad}^2(P(x))$ has the same description, except that we have to consider only paths of length at least two. Therefore, after constructing the quotient, we are essentially left with paths of length exactly one and we get

$$\operatorname{top}(\operatorname{rad}(P(x))) \cong S(y_1)^{n_1} \oplus \cdots \oplus S(y_r)^{n_r},$$

where $y_1, \dots, y_r$ are the successors of x in Q and n_i is the number of arrows from x to y_i, for each $1 \le i \le r$. Since $P(y)$ is the projective cover of $S(y)$ for all $y \in Q_0$, we have a surjective homomorphism

$$P(y_1)^{n_1} \oplus \cdots \oplus P(y_r)^{n_r} \longrightarrow \operatorname{rad}(P(x)).$$

Again by Lemma 1.3.6, the dimension of each $P(y_i)$ is the number of paths in Q starting at y_i. On the other hand, the dimension of $\operatorname{rad}(P(x))$ is the number of paths of length at least one starting at x, and each of these paths is the concatenation of some path starting at some y_i and some arrow from x to y_i. We conclude that the two modules above have the same dimension, hence this surjective homomorphism is an isomorphism. In particular, $\operatorname{rad}(P(x))$ is projective, as desired. □

Remark. There is a sort of converse to the result above. If A is a basic finite-dimensional algebra, we know from Theorem 1.2.1 that $A \cong KQ/I$ for some bound quiver (Q, I). If A is also hereditary, one can show that Q is acyclic and $I = 0$ (see [2, Proposition I.2.28]).

We also remark that KQ is hereditary even if Q is not acyclic, but the proof above does not work in this case. The idea for the general case is to explicitly construct a "standard" projective resolution of length one for every module (see [13, p. 7]).

If A is an algebra of finite global dimension, then there is an interesting bilinear form $\langle -, - \rangle_A : K_0(A) \times K_0(A) \to \mathbb{Z}$ called the **Euler form** of A. It is defined by the formula

$$\langle [M], [N] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}^i_A(M, N)$$

for A-modules M and N. The sum above is always finite and goes up to $i = \operatorname{gldim} A$. It is well-defined over $K_0(A)$ due to the long exact sequence of Ext associated to a short exact sequence. If $A = KQ$, then we have a hereditary algebra and the formula simplifies to

$$\langle [M], [N] \rangle_A = \dim_K \operatorname{Hom}_A(M, N) - \dim_K \operatorname{Ext}^1_A(M, N).$$

In this case, we can write the Euler form in terms of the data of Q. For this, we will suppose $Q_0 = \{1, \dots, n\}$ and identify $K_0(KQ)$ with $\mathbb{Z}^n$ via the dimension vector map. We will also denote the Euler form just by $\langle -, - \rangle_Q$, as the independence of K will be apparent.

Lemma 2.1.3. If $v, w \in \mathbb{Z}^n$, then

$$\langle v, w \rangle_Q = \sum_{i=1}^{n} v_i w_i - \sum_{\alpha \in Q_1} v_{s(\alpha)} w_{t(\alpha)}.$$

Proof. We can write v and w as a linear combination of the dimension vectors of the simple modules $S(1), \dots, S(n)$, the coefficients being the v_i's and the w_j's. By bilinearity of the Euler form, we get

$$\begin{aligned} \langle v, w \rangle_Q &= \sum_{i,j=1}^{n} v_i w_j \cdot \langle [S(i)], [S(j)] \rangle_Q \\ &= \sum_{i,j=1}^{n} v_i w_j \cdot (\dim_K \operatorname{Hom}_{KQ}(S(i), S(j)) - \dim_K \operatorname{Ext}^1_{KQ}(S(i), S(j))). \end{aligned}$$

By the description of the simple modules, it is immediate that the first dimension above equals δ_{ij}, resulting in the first sum in the formula of the statement. On the other hand, the other dimension is the number of arrows between the vertices corresponding to $S(i)$ and $S(j)$ in the Ext-quiver of KQ, which is Q itself (see the remark after Definition 1.2.3). This accounts for the second sum above, finishing the proof. □

The **Tits form** $q_Q : \mathbb{Z}^n \to \mathbb{Z}$ of KQ is the quadratic form associated to the Euler form. Using the previous result, it is given by

$$q_Q(v) = \langle v, v\rangle_Q = \sum_{i=1}^{n} v_i^2 - \sum_{\alpha\in Q_1} v_{s(\alpha)}v_{t(\alpha)}$$

for $v \in \mathbb{Z}^n$. Notice that it is independent of the orientation of the arrows in Q.

We will see that KQ has finite representation type exactly when q_Q is **positive definite**, that is, $q_Q(v) > 0$ for all nonzero $v \in \mathbb{Z}^n$. This will prove Gabriel's theorem due to the following result:

Theorem 2.1.4. If Q is connected, then q_Q is positive definite if and only if Q is a Dynkin quiver.

The proof is completely combinatorial and it is done in Section A.1 of the appendix.

2.2 A tour through algebraic geometry

In this section, we will use some results from algebraic geometry to deduce properties about quiver representations with the same dimension vector.

Write $Q_0 = \{1, \dots, n\}$ and fix a nonzero vector $d \in \mathbb{Z}^n$ with nonnegative entries. If $M = (M_i, \varphi_\alpha)$ is a representation of Q with dimension vector d, we can assume that $M_i = K^{d_i}$ for all $1 \le i \le n$. What characterizes M among representations of dimension vector d are the linear maps φ_α for $\alpha \in Q_1$. This leads us to the following definition.

Definition 2.2.1. The **representation variety** associated to the dimension vector d is the set

$$V(d) := \prod_{\alpha\in Q_1} \mathrm{Hom}_K(K^{d_{s(\alpha)}}, K^{d_{t(\alpha)}}).$$

For $x \in V(d)$, we have a representation $M(x) = (M(x)_i, \varphi(x)_\alpha)$ defined by $M(x)_i = K^{d_i}$ and $\varphi(x)_\alpha = x_\alpha$ for all $1 \le i \le n$ and $\alpha \in Q_1$. Note that $\underline{\dim}\, M(x) = d$ and that every representation of Q with dimension vector d is isomorphic to $M(x)$ for some $x \in V(d)$.

Let us consider the group

$$\mathrm{GL}(d) := \mathrm{GL}(d_1) \times \cdots \times \mathrm{GL}(d_n),$$

where $\mathrm{GL}(d_i)$ denotes the group of linear automorphisms of K^{d_i} for $1 \le i \le n$. It acts on $V(d)$ by conjugation: for $g \in \mathrm{GL}(d)$ and $x \in V(d)$, we have

$$(g \cdot x)_\alpha = g_t x_\alpha g_s^{-1}$$

for every arrow $\alpha : s \to t$ in Q_1.

If $x, y \in V(d)$ and there is $g \in \mathrm{GL}(d)$ with $y = g \cdot x$, then the maps defining g assemble into an isomorphism of representations $M(x) \to M(y)$. Every isomorphism from $M(x)$ to $M(y)$ appears in this way. Therefore, the orbits of this action are in bijection with the isomorphism classes of representations of Q with dimension vector d. If M is such a representation, we denote by $\mathcal{O}_M$ the associated orbit in $V(d)$.

We will use some results from algebraic geometry to study this action. The representation variety $V(d)$ is naturally an algebraic variety isomorphic to the affine space of dimension

$$\sum_{\alpha \in Q_1} d_{s(\alpha)} d_{t(\alpha)}.$$

Furthermore, $\mathrm{GL}(d)$ is an affine algebraic group of dimension

$$\sum_{i=1}^{n} d_i^2$$

which acts algebraically on $V(d)$. Note that $q_Q(d)$ is the difference between these two dimensions.

We will admit the following properties of actions of algebraic groups. A proof can be found in [11, Proposition 2.1.7].

Lemma 2.2.2. Let X be an affine variety equipped with an algebraic action of an affine algebraic group G. Let $x \in X$ and denote by $\mathcal{O}$ its orbit.

(1) $\mathcal{O}$ is locally closed, that is, $\mathcal{O}$ is open in its closure $\overline{\mathcal{O}}$.

(2) $\overline{\mathcal{O}} \setminus \mathcal{O}$ is a union of orbits of dimension strictly smaller than $\dim \mathcal{O}$.

(3) If G_x denotes the stabilizer of x, then G_x is a closed subgroup of G and

$$\dim \mathcal{O} = \dim G - \dim G_x.$$

Returning to our particular action, we obtain some corollaries.

Corollary 2.2.3. If M is a representation of Q with dimension vector d, then

$$\dim V(d) - \dim \mathcal{O}_M = \dim_K \mathrm{End}_{KQ}(M) - q_Q(d) = \dim_K \mathrm{Ext}^1_{KQ}(M, M).$$

Proof. The second equality follows from the definition of $q_Q(d)$ as $\langle d, d \rangle_Q$. For the first equality, we remarked above that

$$q_Q(d) = \dim \mathrm{GL}(d) - \dim V(d),$$

so we have to prove that

$$\dim \mathcal{O}_M = \dim \mathrm{GL}(d) - \dim_K \mathrm{End}_{KQ}(M).$$

The vector space $\mathrm{End}_{KQ}(M)$ can be seen as an algebraic variety isomorphic to an affine space. In the Zariski topology, $\mathrm{Aut}_{KQ}(M)$ is an open subset of the irreducible space $\mathrm{End}_{KQ}(M)$, hence, it is dense and its dimension is $\dim_K \mathrm{End}_{KQ}(M)$. Finally, by the description of the action of $\mathrm{GL}(d)$ on $V(d)$, note that $\mathrm{Aut}_{KQ}(M)$ is isomorphic to the stabilizer of a point in $\mathcal{O}_M$, so the desired equality follows from Lemma 2.2.2. □

Corollary 2.2.4. If $q_Q(d) \leq 0$, then there are infinitely many orbits in $V(d)$. In particular, if KQ is of finite representation type, then q_Q is positive definite.

Proof. If M is a representation with dimension vector d, then $\operatorname{End}_{KQ}(M)$ is nonzero. By Corollary 2.2.3 and the hypothesis,

$$\dim V(d) - \dim \mathcal{O}_M = \dim_K \operatorname{End}_{KQ}(M) - q_Q(d) > -q_Q(d) \geq 0.$$

This shows that every orbit in $V(d)$ has dimension strictly smaller than the dimension of $V(d)$. Since the dimension of a finite union of subsets in $V(d)$ equals the maximum of the dimensions of these subsets, we conclude that $V(d)$ must contain infinitely many orbits.

For the second part, assume KQ is of finite representation type and take a nonzero $v \in \mathbb{Z}^n$. If v has nonnegative coordinates, we can consider it as a dimension vector and it follows that $V(v)$ has finitely many orbits under the action of $\mathrm{GL}(v)$, so the first part implies that $q_Q(v) > 0$. Now, if v has some negative coordinate, let $v' \in \mathbb{Z}^n$ be the vector whose coordinates are the absolute values of the coordinates of v. It is not hard to see from the definition of q_Q that $q_Q(v) \geq q_Q(v')$, and this last number is positive by the same argument as before. We conclude that q_Q is positive definite. □

In view of Theorem 2.1.4, we have just proved one of the implications in Gabriel's theorem. We will need some more technical results in order to show the other implication. For what follows, we will say that a KQ-module M **has no self-extensions** if $\operatorname{Ext}^1_{KQ}(M, M) = 0$.

Corollary 2.2.5. If M is a representation of Q with dimension vector d, then $\mathcal{O}_M$ is open in $V(d)$ if and only if M has no self-extensions. In particular, there exists at most one representation M of dimension vector d without self-extensions, up to isomorphism.

Proof. By Corollary 2.2.3, the representation M has no self-extensions if and only if $\dim V(d) = \dim \mathcal{O}_M$. The dimension of $\mathcal{O}_M$ is the same as the dimension of its closure $\overline{\mathcal{O}_M}$ and, since $V(d)$ is irreducible, proper closed subsets of $V(d)$ are of a strictly lower dimension. It follows that the desired condition is equivalent to $\overline{\mathcal{O}_M} = V(d)$. If this happens, then $\mathcal{O}_M$ is open in $V(d)$ since it is locally closed by Lemma 2.2.2. Conversely, if $\mathcal{O}_M$ is an open subset of the irreducible space $V(d)$, then $\mathcal{O}_M$ is dense and we have $\overline{\mathcal{O}_M} = V(d)$. This proves the first statement of the corollary.

For the second part, take two representations M and N as in the statement. By the first part, $\mathcal{O}_M$ and $\mathcal{O}_N$ are open in $V(d)$, hence dense. Therefore, these two orbits intersect and so they must coincide, proving that $M \cong N$. □

Lemma 2.2.6. If there is a nonsplit short exact sequence of KQ-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

then $\mathcal{O}_{X \oplus Z} \subseteq \overline{\mathcal{O}_Y} \setminus \mathcal{O}_Y$.

Proof. Write $X = (X_i, \varphi^X_\alpha)$, $Y = (Y_i, \varphi^Y_\alpha)$ and $Z = (Z_i, \varphi^Z_\alpha)$. For every $1 \leq i \leq n$, we can identify X_i as a subspace of Y_i. Extending a basis of each X_i to a basis of Y_i and using these bases to represent the maps φ^Y_α as matrices, we can find $x \in V(\underline{\dim}\, X)$, $y \in V(\underline{\dim}\, Y)$ and $z \in V(\underline{\dim}\, Z)$ such that $X \cong M(x)$, $Y \cong M(y)$ and $Z \cong M(z)$, and such that y_α is of the form

$$y_\alpha = \begin{pmatrix} x_\alpha & w_\alpha \\ 0 & z_\alpha \end{pmatrix}$$

for $\alpha \in Q_1$. For a nonzero $\lambda \in K$, consider the element $g_\lambda \in \mathrm{GL}(\underline{\dim}\, Y)$ represented by

$$(g_\lambda)_i = \begin{pmatrix} \lambda \cdot I_{\dim X_i} & 0 \\ 0 & I_{\dim Z_i} \end{pmatrix}$$

for $1 \le i \le n$, where I_k denotes the identity matrix of size k. We have

$$(g_\lambda \cdot y)_\alpha = g_t y_\alpha g_s^{-1} = \begin{pmatrix} x_\alpha & \lambda w_\alpha \\ 0 & z_\alpha \end{pmatrix}$$

for every arrow $\alpha : s \to t$ in Q_1. This constructs a line inside the orbit of y, except that the point corresponding to $\lambda = 0$ is missing. Thus, in the closure of this orbit, we must have the element $y' \in V(\underline{\dim}\, Y)$ given by

$$y'_\alpha = \begin{pmatrix} x_\alpha & 0 \\ 0 & z_\alpha \end{pmatrix}$$

for $\alpha \in Q_1$. It is immediate that $X \oplus Z \cong M(y')$ and, since $\overline{\mathcal{O}_Y}$ is a union of orbits by Lemma 2.2.2, it follows that $\mathcal{O}_{X\oplus Z} \subseteq \overline{\mathcal{O}_Y}$.

It remains to prove that $Y \ncong X \oplus Z$. Applying the functor $\mathrm{Hom}_{KQ}(Z, -)$ to our short exact sequence, we get the sequence

$$0 \longrightarrow \mathrm{Hom}_{KQ}(Z, X) \longrightarrow \mathrm{Hom}_{KQ}(Z, Y) \xrightarrow{f} \mathrm{Hom}_{KQ}(Z, Z),$$

which is exact. Since the initial sequence does not split, the map $Y \to Z$ does not admit a section and the map f above cannot be surjective. We obtain

$$\begin{aligned} \dim_K \mathrm{Hom}_{KQ}(Z, Y) &= \dim_K \mathrm{Hom}_{KQ}(Z, X) + \dim_K \operatorname{im} f \\ &\neq \dim_K \mathrm{Hom}_{KQ}(Z, X) + \dim_K \mathrm{Hom}_{KQ}(Z, Z). \end{aligned}$$

On the other hand,

$$\mathrm{Hom}_{KQ}(Z, X \oplus Z) \cong \mathrm{Hom}_{KQ}(Z, X) \oplus \mathrm{Hom}_{KQ}(Z, Z),$$

so we must have $Y \ncong X \oplus Z$, as needed. □

Corollary 2.2.7. If $\mathcal{O}_M$ is an orbit in $V(d)$ of maximal dimension and $M = X \oplus Z$, then

$$\mathrm{Ext}^1_{KQ}(X, Z) = \mathrm{Ext}^1_{KQ}(Z, X) = 0.$$

Proof. If $\mathrm{Ext}^1_{KQ}(Z, X) \neq 0$, then there exists a nonsplit short exact sequence of KQ-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

By Lemma 2.2.6, $\mathcal{O}_M$ is contained in $\overline{\mathcal{O}_Y} \setminus \mathcal{O}_Y$. However, by Lemma 2.2.2, this last set is a union of orbits of dimension strictly smaller than $\dim \mathcal{O}_Y$, contradicting the hypothesis on $\mathcal{O}_M$. Thus, we must have $\mathrm{Ext}^1_{KQ}(Z, X) = 0$. A similar argument proves that the other extension group is zero. □

2.3 From indecomposable modules to positive roots

Suppose that Q is a Dynkin quiver. By Theorem 2.1.4, q_Q is positive definite. The goal of this section is to relate indecomposable KQ-modules and positive roots of q_Q.

Definition 2.3.1. A nonzero vector $v \in \mathbb{Z}^n$ is a **root** of q_Q if $q_Q(v) = 1$. It is **positive** if its entries are all nonnegative.

In Section A.2, we discuss the relationship between this definition of root and the one coming from the theory of root systems. The only property that we are going to use is that the set of roots is *finite*.

We will prove that the dimension vector of an indecomposable module is a positive root. In fact, we will obtain something slightly stronger: any indecomposable has no self-extensions and is a **brick**, that is, its endomorphism algebra is one-dimensional. If M is an indecomposable KQ-module with these properties, then indeed we have

$$q_Q(\underline{\dim}\, M) = \dim_K \operatorname{End}_{KQ}(M) - \dim_K \operatorname{Ext}^1_{KQ}(M, M) = 1.$$

We start with a technical lemma.

Lemma 2.3.2. If M is an indecomposable KQ-module that is not a brick, then M has a submodule which is a brick *with* self-extensions.

Proof. It suffices to show that M has an indecomposable proper submodule N with self-extensions. Having proved this property, if N is not a brick, then we can iterate the proof until we reach a submodule that is a brick.

Note that $\operatorname{End}_{KQ}(M)$ has nonzero noninvertible elements. If this were not the case, then $\operatorname{End}_{KQ}(M)$ would be a finite-dimensional division algebra over K and, since K is algebraically closed, it would be isomorphic to K, contradicting that M is not a brick. Therefore, if $\theta \in \operatorname{End}_{KQ}(M)$ is a nonzero endomorphism whose image has minimal dimension, then θ is noninvertible. Since M is indecomposable, $\operatorname{End}_{KQ}(M)$ is in fact a local algebra and it follows that θ is nilpotent. In this case, notice that the dimension of the image of θ^2 is lower than that of θ, so we must have $\theta^2 = 0$ and $\operatorname{im}\theta \subseteq \ker\theta$. Given a decomposition

$$\ker\theta = X_1 \oplus X_2 \oplus \cdots \oplus X_r$$

of $\ker\theta$ as a sum of indecomposable submodules, there exists $1 \le j \le r$ such that the canonical projection $\pi_j : \ker\theta \to X_j$ does not annihilate $\operatorname{im}\theta$. We will now prove that X_j has self-extensions and so we can set $N = X_j$ to complete the proof.

Let α be the restriction of π_j to $\operatorname{im}\theta$. The composition

$$M \xrightarrow{\ \theta\ } \operatorname{im}\theta \xrightarrow{\ \alpha\ } X_j \hookrightarrow M$$

is an endomorphism of M with image $\operatorname{im}\alpha$. Since α is nonzero and $\dim_K \operatorname{im}\alpha \le \dim_K \operatorname{im}\theta$, the minimality condition on θ forces α to be injective. Hence, α fits into a short exact sequence:

$$0 \longrightarrow \operatorname{im}\theta \xrightarrow{\ \alpha\ } X_j \longrightarrow Y \longrightarrow 0.$$

Applying the functor $\operatorname{Hom}_{KQ}(-, X_j)$ and using the long exact sequence with the Ext functors, we get in particular a sequence

$$\operatorname{Ext}^1_{KQ}(X_j, X_j) \xrightarrow{\ \alpha^*\ } \operatorname{Ext}^1_{KQ}(\operatorname{im}\theta, X_j) \longrightarrow \operatorname{Ext}^2_{KQ}(Y, X_j)$$

which is exact. But KQ is hereditary, so $\operatorname{Ext}^2_{KQ}(Y, X_j) = 0$ and α^* must be surjective. Thus, in order to show that X_j has self-extensions, it is enough to show that

$\mathrm{Ext}^1_{KQ}(\mathrm{im}\,\theta, X_j) \neq 0$. For this purpose, consider the following diagram:

$$\begin{array}{ccc} \ker\theta & \hookrightarrow & M \\ {\scriptstyle \pi_j}\downarrow & & \\ X_j & & \end{array}$$

After taking its pushout, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\theta & \longrightarrow & M & \xrightarrow{\theta} & \mathrm{im}\,\theta & \longrightarrow & 0 \\ & & {\scriptstyle \pi_j}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \longrightarrow & X_j & \xrightarrow{g} & M' & \longrightarrow & \mathrm{im}\,\theta & \longrightarrow & 0 \end{array}$$

whose rows are exact[1]. If $\mathrm{Ext}^1_{KQ}(\mathrm{im}\,\theta, X_j)$ vanished, the bottom row would split and we would find $h : M' \to X_j$ such that $hg = \mathrm{id}_{X_j}$, but then hf would be a retraction for the inclusion $X_j \hookrightarrow M$ and X_j would be a direct summand of M, contradicting the indecomposability of M. Therefore, $\mathrm{Ext}^1_{KQ}(\mathrm{im}\,\theta, X_j) \neq 0$ and the proof is finished. □

Corollary 2.3.3. If q_Q is positive definite and M is an indecomposable KQ-module, then M is a brick without self-extensions. In particular, $\underline{\dim}\, M$ is a positive root.

Proof. If M is not a brick, then Lemma 2.3.2 gives us an indecomposable module N which is a brick with self-extensions. It follows that

$$q(\underline{\dim}\, N) = \dim_K \mathrm{End}_{KQ}(N) - \dim_K \mathrm{Ext}^1_{KQ}(N, N) \leq 0,$$

contradicting that q_Q is positive definite. Therefore, M must be a brick. Again by the positive definiteness of q_Q, we have

$$\dim_K \mathrm{Ext}^1_{KQ}(M, M) = \dim_K \mathrm{End}_K(M) - q_Q(\underline{\dim}\, M) = 1 - q_Q(\underline{\dim}\, M) \leq 0$$

and we conclude that M has no self-extensions. □

We can now finish the proof of Gabriel's theorem.

Theorem 2.3.4. Suppose Q is a Dynkin quiver. The map

$$M \longmapsto \underline{\dim}\, M$$

defines a bijection between the set of isomorphism classes of indecomposable KQ-modules and the set of positive roots of q_Q. Consequently, KQ is of finite representation type.

Proof. By Corollary 2.3.3, the dimension vector of an indecomposable module is indeed a positive root and the map above is well-defined.

If M and N are indecomposable modules, Corollary 2.3.3 says that M and N have no self-extensions. If $\underline{\dim}\, M = \underline{\dim}\, N$, it follows from Corollary 2.2.5 that $M \cong N$. This proves that the map of the statement is injective.

In order to prove the surjectivity, let $d \in \mathbb{Z}^n$ be a positive root of q_Q. Choose an orbit $\mathcal{O}_M$ of maximal dimension in the representation variety $V(d)$. If $M = X \oplus Z$

[1]See [1, Chapitre III, Théorème 5.9] for a proof that the cokernel is preserved under the pushout.

with $X, Z \neq 0$, Corollary 2.2.7 gives us $\operatorname{Ext}^1_{KQ}(X, Z) = \operatorname{Ext}^1_{KQ}(Z, X) = 0$, so

$$\begin{aligned} q_Q(d) &= \langle \underline{\dim}\, M, \underline{\dim}\, M \rangle_Q \\ &= \langle \underline{\dim}\, X + \underline{\dim}\, Z, \underline{\dim}\, X + \underline{\dim}\, Z \rangle_Q \\ &= q_Q(\underline{\dim}\, X) + q_Q(\underline{\dim}\, Z) + \dim_K \operatorname{Hom}_{KQ}(X, Z) + \dim_K \operatorname{Hom}_{KQ}(Z, X) \\ &\geq 1 + 1 + 0 + 0 = 2, \end{aligned}$$

but this contradicts the fact that d is a root. Hence, M must be indecomposable and it is mapped to d by the map from the statement. □

Chapter 3

Auslander-Reiten theory

In the previous chapter, Gabriel's theorem gave us the exact condition for a path algebra KQ to be of finite representation type: it has to come from a Dynkin quiver. Moreover, Theorem 2.3.4 showed that the dimension vectors of the indecomposable modules can be obtained combinatorially from the theory of root systems. Yet, even with these strong results, we have only a blurred picture of the module category of KQ. For example, we do not know exactly how the indecomposable modules look like or how to find them, and we have no clue about the behavior of the morphisms between them.

Our next step is to address these problems. For this purpose, we will develop a bit of the so-called Auslander-Reiten theory, following mainly [2], [3, Chapter IV], [6] and [8, Sections 4.12 and 4.13]. It is convenient to work in the more general setting of an arbitrary finite-dimensional K-algebra, and we specialize to the case of path algebras in most examples of Section 3.4 and in Section 3.5. At the end, we will see an algorithm to find the Auslander-Reiten quiver of KQ when Q is a Dynkin quiver. It allows us, in particular, to explicitly find the indecomposable modules, as we portray in Example 3.4.1 for the case $Q = \mathbb{A}_3$.

Remark. Throughout the chapter, A denotes a K-algebra of finite dimension and all modules are in $\operatorname{mod} A$. For the last three sections, we assume that K is algebraically closed.

3.1 Irreducible morphisms and almost split sequences

In order to understand $\operatorname{mod} A$, it is not enough to find all indecomposable modules. We also need to study the morphisms between them. In this section, we will define and study a class of morphisms analogous to that of indecomposable objects. They will not be the building blocks of all morphisms in $\operatorname{mod} A$ in general, but they will give us a good picture of this category.

Let $\varphi : M \to N$ be a homomorphism of A-modules. Given another module L and maps $\psi : M \to L$ and $\psi' : L \to N$, we always have two factorizations of φ:

$$M \xrightarrow{\begin{pmatrix}\varphi\\ \psi\end{pmatrix}} N \oplus L \twoheadrightarrow N \quad \text{and} \quad M \hookrightarrow M \oplus L \xrightarrow{\begin{pmatrix}\varphi & \psi'\end{pmatrix}} N.$$

The unspecified maps are the canonical projection and inclusion. These are **trivial factorizations**. Recall that a surjective homomorphism can be seen as the canonical projection from a direct sum if and only if it is a **retraction**, that is, it has a right inverse. Dually, an injective homomorphism can be seen as the canonical inclusion into a direct sum if and only if it is a **section**, that is, it has a left inverse. With this nomenclature, a factorization $\varphi = g \circ f$ is trivial if either f is a section or g is a retraction.

Definition 3.1.1. A homomorphism of A-modules $\varphi : M \to N$ is **irreducible** if it is neither a section nor a retraction, and every factorization of φ is trivial.

Remark. Every homomorphism $\varphi : M \to N$ has a factorization

$$M \twoheadrightarrow \operatorname{im} \varphi \hookrightarrow N.$$

If φ is irreducible, one of the two maps above must be an isomorphism. This shows that irreducible morphisms are always injective or surjective (but not both).

Example 3.1.2. Let P be a nonsimple indecomposable projective module. We claim that the inclusion $\varphi : \operatorname{rad}(P) \hookrightarrow P$ is irreducible. First of all, φ is not a section because $\operatorname{rad}(P) \neq 0$ is not a direct summand of P. Now, if we write $\varphi = gf$ with $f : \operatorname{rad}(P) \to M$ and $g : M \to P$, we have two cases. If g is surjective, then it is a retraction because P is projective. Otherwise, the image of g is contained in $\operatorname{rad}(P)$ since this is the unique maximal submodule of P. In this case, if g' is the restriction of g as a map from M to $\operatorname{rad}(P)$, we have $g'f = \operatorname{id}_{\operatorname{rad}(P)}$, proving that f is a section. Therefore, every factorization of φ is trivial.

Dually, if I is a nonsimple indecomposable injective module, the quotient map $I \twoheadrightarrow I/\operatorname{soc}(I)$ is also irreducible.

There is an interesting characterization of irreducible morphisms which will motivate the appearance of almost split morphisms. It will also start relating irreducible morphisms and indecomposable modules.

Lemma 3.1.3. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a nonsplit short exact sequence of A-modules.

(1) f is irreducible if, and only if, for every morphism $v : V \to N$, there is $v_1 : V \to M$ such that $v = gv_1$ or $v_2 : M \to V$ such that $g = vv_2$.

(2) g is irreducible if, an only if, for every morphism $u : L \to U$, there is $u_1 : M \to U$ such that $u = u_1 f$ or $u_2 : U \to M$ such that $f = u_2 u$.

Proof. We will just prove (1). The proof for (2) is dual.

($\Longrightarrow$) Suppose f is irreducible. Given a morphism $v : V \to N$, we can take its pullback along g to get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f'} & E & \xrightarrow{g'} & V & \longrightarrow & 0 \\ & & \| & & \downarrow u & & \downarrow v & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows (see [1, Chapitre III, Théorème 5.5]). The square on the left gives a factorization of f, which must be trivial by hypothesis. If f' is a section, then the first row splits and g' is a retraction. In this case, we can take $v_1 = uh_1$ with $h_1 : V \to E$ a right inverse for g'. If u is a retraction, we can take $v_2 = g'h_2$ with $h_2 : M \to E$ a right inverse for u.

($\Longleftarrow$) Note that f is neither a section nor a retraction because the sequence is nonsplit by hypothesis. Now, let $f = f_2 f_1$ be a factorization with $f_1 : L \to E$ and $f_2 : E \to M$. Assuming the condition stated above, let us show this factorization is trivial. By taking the cokernel $g' : E \to V$ of the injective map f_1, we arrive at a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f_1} & E & \xrightarrow{g'} & V & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle v} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

Since the rows are exact, the square on the right is a pullback square (by the same reference as before). By our assumption, there is $v_1 : V \to M$ with $v = gv_1$ or $v_2 : M \to V$ with $g = vv_2$. In the first case, applying the universal property of the pullback with $v_1 : V \to M$ and $\mathrm{id}_V : V \to V$, we get a right inverse for g', so it is a retraction and f_1 is a section. In the second case, a similar argument proves that f_2 is a retraction. $\square$

Corollary 3.1.4. The cokernel of an injective irreducible morphism is indecomposable. Dually, the kernel of a surjective irreducible morphism is indecomposable.

Proof. Let $f : L \to M$ be an injective irreducible morphism. Let N be its cokernel and let $g : M \to N$ be the canonical map. These two maps fit into a nonsplit short exact sequence because f is not a section. Write $N = N_1 \oplus N_2$ and denote by $\mu_i : N_i \to N$ $(i = 1, 2)$ the canonical inclusions. If there is $v_i : M \to N_i$ with $g = \mu_i v_i$, then μ_i is surjective and $N_i = N$, so the direct sum decomposition is trivial. If v_1 and v_2 do not exist, then Lemma 3.1.3 guarantees the existence of $v_i' : N_i \to M$ with $\mu_i = gv_i'$ for $i = 1, 2$. If $\pi_i : N \to N_i$ denotes the canonical projection, we have

$$g(v_1'\pi_1 + v_2'\pi_2) = \mu_1\pi_1 + \mu_2\pi_2 = \mathrm{id}_N \implies g \text{ is a retraction} \implies f \text{ is a section},$$

a contradiction. This proves that N is indecomposable. The second statement of the corollary is proved similarly. $\square$

The property appearing in Lemma 3.1.3 is close to the one satisfied by almost split morphisms, which we now define.

Definition 3.1.5. Let L, M and N be A-modules.

(1) A homomorphism $f : L \to M$ is called **left almost split** if it is not a section and, for every homomorphism of A-modules $u : L \to U$ that is not a section, there exists $u' : M \to U$ such that $u'f = u$, that is, the following triangle is commutative:

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle u'} & \\ U & & \end{array}$$

(2) A homomorphism $g : M \to N$ is called **right almost split** if it is not a retraction and, for every homomorphism of A-modules $v : V \to N$ that is not a retraction, there exists $v' : V \to M$ such that $gv' = v$, that is, the following triangle is commutative:

$$\begin{array}{ccc} & & V \\ & \swarrow{\scriptstyle v'} & \downarrow{\scriptstyle v} \\ M & \xrightarrow{g} & N \end{array}$$

Remark. Let $f : L \to M$ be a morphism in $\operatorname{mod} A$. If f is a section with left inverse h, then any $u : L \to U$ admits a lift $u' : M \to U$ as above: we just take $u' = uh$. On the other hand, if we can find a section $u : L \to U$ (with left inverse h') for which there exists a lift $u' : M \to U$, then the composition $h'u'$ is a left inverse for f and f is a section. This explains the nomenclature above: if f is left almost split, then it is not a split monomorphism, but it still satisfies the lifting property for all morphisms which can possibly admit a lift. A similar remark applies for right almost split morphisms.

Lemma 3.1.6. If $f : L \to M$ is a left almost split morphism in $\operatorname{mod} A$, then L is indecomposable. Dually, if $g : M \to N$ is a right almost split morphism in $\operatorname{mod} A$, then N is indecomposable.

Proof. It is similar to the proof of Corollary 3.1.4 (see [3, Chapter IV, Lemma 1.3]). □

Let $f : L \to M$ and $f' : L \to M'$ be two left almost split morphisms in $\operatorname{mod} A$ with the same domain. Both of them are not sections, so we can find morphisms $\varphi : M \to M'$ and $\psi : M' \to M$ commuting

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle f'}\downarrow & \swarrow_{\varphi} & \\ M' & & \end{array} \quad \text{and} \quad \begin{array}{ccc} L & \xrightarrow{f'} & M' \\ {\scriptstyle f}\downarrow & \swarrow_{\psi} & \\ M. & & \end{array}$$

Hence, we have $f = (\psi\varphi)f$ and $f' = (\varphi\psi)f'$. If this implied that $\psi\varphi$ and $\varphi\psi$ are automorphisms of M and M', respectively, then φ and ψ would be isomorphisms. This leads us to another definition.

Definition 3.1.7. Let $f : L \to M$ and $g : M \to N$ be homomorphisms of A-modules.

(1) We say that f is **left minimal** if every endomorphism $h : M \to M$ with $hf = f$ is an automorphism. Dually, g is **right minimal** if every endomorphism $h : M \to M$ with $gh = g$ is an automorphism.

(2) If f is both left almost split and left minimal, we call it **left minimal almost split**. Analogously, if g is both right almost split and right minimal, we call it **right minimal almost split**.

We get a uniqueness result for minimal almost split morphisms.

Lemma 3.1.8. If $f : L \to M$ and $f' : L \to M'$ are left minimal almost split morphisms in $\operatorname{mod} A$ with the same domain, then there exists an isomorphism $\varphi : M \to M'$ such that $f' = \varphi f$. Dually, if $g : M \to N$ and $g' : M' \to N$ are right minimal almost split morphisms in $\operatorname{mod} A$ with the same target, then there exists an isomorphism $\psi : M \to M'$ such that $g = g'\psi$.

We can now relate these concepts with irreducible morphisms.

Lemma 3.1.9. Every nonzero left or right minimal almost split morphism in $\operatorname{mod} A$ is irreducible.

Proof. Let us prove the statement for a nonzero left minimal almost split morphism $f : L \to M$, the other case being dual.

By definition, f is not a section. Since L is indecomposable by Lemma 3.1.6, f is not a retraction either, because otherwise $M \neq 0$ would be a direct summand of L and f would have to be an isomorphism. Now, assume that $f = f_2 f_1$ with $f_1 : L \to U$ and

$f_2 : U \to M$. Suppose that f_1 is not a section and let us show that f_2 is a retraction. By the definition of almost split morphism, we have a lift

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle f_1}\downarrow & \swarrow {\scriptstyle u} & \\ U. & & \end{array}$$

Thus, $f = f_2 f_1 = f_2 u f$ and, by left minimality, $f_2 u$ is an automorphism. Consequently, f_2 is a retraction, as claimed. □

Theorem 3.1.10. The following assertions hold:

(1) Let $f : L \to M$ be a left minimal almost split morphism in $\operatorname{mod} A$. Then, a morphism $f' : L \to M'$ is irreducible if, and only if, $M' \neq 0$ and there is a morphism $f'' : L \to M''$ such that $M \cong M' \oplus M''$ and

$$\begin{pmatrix} f' \\ f'' \end{pmatrix} : L \longrightarrow M' \oplus M''$$

is left minimal almost split.

(2) Let $g : M \to N$ be a right minimal almost split morphism in $\operatorname{mod} A$. Then, a morphism $g' : M' \to N$ is irreducible if, and only if, $M' \neq 0$ and there is a morphism $g'' : M'' \to N$ such that $M \cong M' \oplus M''$ and

$$\begin{pmatrix} g' & g'' \end{pmatrix} : M' \oplus M'' \longrightarrow N$$

is right minimal almost split.

Proof. As before, we will just prove (1). Let $f' : L \to M'$ be a homomorphism of A-modules.

($\Longrightarrow$) Suppose f' is irreducible. Since f' is not a retraction, $M' \neq 0$. As f' is not a section either, we can use that f is left almost split to get $h : M \to M'$ such that $f' = hf$. It follows that h is a retraction because f' is irreducible and f is not a section. Therefore, $M'' := \ker h$ is a direct summand of M and, if $\pi : M \to M''$ is the canonical projection associated to some direct sum decomposition, the map

$$\begin{pmatrix} h \\ \pi \end{pmatrix} : M \longrightarrow M' \oplus M''$$

is an isomorphism. The composition of this isomorphism with f is again left minimal almost split and, since $f' = hf$, it has the form as in the statement.

($\Longleftarrow$) Let $f'' : L \to M''$ be as in the statement and let us prove that f' is irreducible. Firstly, f' cannot be a section because the existence of a left inverse h would give us

$$\begin{pmatrix} h & 0 \end{pmatrix} \begin{pmatrix} f' \\ f'' \end{pmatrix} = \mathrm{id}_L,$$

contradicting that the map from the statement is not a section. Since $M' \neq 0$ and L is indecomposable (Lemma 3.1.6), f' cannot be a retraction either.

Assume now that f' factorizes as $f' = f_2 f_1$ with $f_1 : L \to U$ and $f_2 : U \to M'$. If f_1 is not a section, there is $\begin{pmatrix} h' & h'' \end{pmatrix} : M' \oplus M'' \to U$ such that

$$\begin{pmatrix} h' & h'' \end{pmatrix} \begin{pmatrix} f' \\ f'' \end{pmatrix} = f_1$$

because this last matrix represents a left almost split morphism by hypothesis. We have

$$\begin{pmatrix} f_2h' & f_2h'' \\ 0 & \mathrm{id}_{M''} \end{pmatrix} \begin{pmatrix} f' \\ f'' \end{pmatrix} = \begin{pmatrix} f_2 \circ (h'f' + h''f'') \\ f'' \end{pmatrix} = \begin{pmatrix} f_2 f_1 \\ f'' \end{pmatrix} = \begin{pmatrix} f' \\ f'' \end{pmatrix},$$

so left minimality implies that the square matrix above is invertible. Thus, f_2h' has to be an automorphism and f_2 is a retraction. This proves that f' is irreducible. □

Let L be an indecomposable module. We will see in Section 3.2 that there exists a left minimal almost split morphism $L \to M$, which is unique up to some isomorphism by Lemma 3.1.8. By Theorem 3.1.10, any nontrivial direct sum decomposition of M yields an irreducible morphism going out of L, and every such morphism appears in this way. We have a similar characterization of irreducible morphisms arriving at a given indecomposable module. Hence, if we know how to find minimal almost split morphisms, we know how to compute irreducible morphisms between indecomposable modules.

It turns out that nearly all minimal almost split morphisms come from a special type of short exact sequence characterized by the following result:

Theorem 3.1.11. Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence of A-modules. The following assertions are equivalent:

(1) f and g are irreducible.

(2) f is left minimal almost split.

(3) g is right minimal almost split.

(4) f is left almost split and N is indecomposable.

(5) g is right almost split and L is indecomposable.

Definition 3.1.12. A short exact sequence $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an **almost split sequence** if one (and hence all) of the assertions of Theorem 3.1.11 is satisfied.

In order to prove the equivalence above, we need another lemma.

Lemma 3.1.13. Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

be a commutative diagram in $\mathrm{mod}\, A$ whose rows are nonsplit short exact sequences.

(1) If L is indecomposable and w is an automorphism, then u and hence v are automorphisms.

(2) If N is indecomposable and u is an automorphism, then w and hence v are automorphisms.

Proof. We will only prove (1). Since L is indecomposable, $\mathrm{End}_A(L)$ is a local algebra. Thus, if u is not invertible, it is nilpotent. Let $m \geq 1$ be such that $u^m = 0$. We have $v^m f = f u^m = 0$, so v^m factors through the cokernel of f. Hence, there is

$h : N \to M$ such that $v^m = hg$. We have $ghg = gv^m = w^m g$ and, because g is an epimorphism, $gh = w^m$. But w is an automorphism, so this implies that g is a retraction, contradicting that the rows are not split. We conclude that u must be an automorphism and, by the five lemma, v as well. □

Proof of Theorem 3.1.11. (1) $\implies$ (2). If $h : M \to M$ satisfies $hf = f$, the irreducibility of f implies that h is a retraction and, in particular, surjective. Since M is finite-dimensional, this implies that h is an isomorphism. This shows that f is left minimal.

Now, let $u : L \to U$ be a map that is not a section. Let us find a lift for u along f as in the definition of left almost split morphisms. Note that the composition of u with the projection to some direct summand is not a section either, so we may suppose that U is indecomposable. Since g is irreducible, Lemma 3.1.3 gives $u_1 : M \to U$ with $u = u_1 f$ or $u_2 : U \to M$ with $f = u_2 u$. In the first case, u_1 is the desired lift. In the second case, by the irreducibility of f and by the fact that u is not a section, we have that u_2 is a retraction. Since U is indecomposable, u_2 is in fact an isomorphism and we can take u_2^{-1} as the lift. This proves that f is left almost split.

(2) $\implies$ (3). Let $h : M \to M$ be a morphism with $gh = g$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & {\scriptstyle h'}\downarrow & & {\scriptstyle h}\downarrow & & \| & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

The morphism h' comes from the functoriality of taking kernels. Since f is left almost split, L is indecomposable by Lemma 3.1.6, so Lemma 3.1.13 implies that h is an automorphism. Therefore, g is right minimal.

Let us check that g is also right almost split. Notice that g is not a retraction because f is not a section. Let $v : V \to N$ be a map that is not a retraction. Taking the pullback of v along g, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{u} & U & \xrightarrow{k} & V & \longrightarrow & 0 \\ & & \| & & {\scriptstyle h}\downarrow & & {\scriptstyle v}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with exact rows. If k is a retraction, the composition of some right inverse of k and h gives the desired lift of v along g. We will prove that this is the case by contradiction. Suppose that k is not a retraction. Thus, the first row above is not split and u is not a section. Since f is left almost split, there is $u' : M \to U$ such that $u = u'f$. This fits into a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \| & & {\scriptstyle u'}\downarrow & & {\scriptstyle v'}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{u} & U & \xrightarrow{k} & V & \longrightarrow & 0 \\ & & \| & & {\scriptstyle h}\downarrow & & {\scriptstyle v}\downarrow & & \\ 0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

The morphism v' comes from the functoriality of taking cokernels. Now, because f is left minimal almost split, it is irreducible by Lemma 3.1.9 and, thus, N is indecomposable by Corollary 3.1.4. It follows from Lemma 3.1.13 that vv' is an automorphism and so v is a retraction, contradicting the initial hypothesis on v.

A dual argument proves that (3) implies (2), hence these two assertions are equivalent. Lemma 3.1.9 implies that (2) and (3) together imply (1). We deduce that (1), (2) and (3) are equivalent. Observe that, while proving that (2) implies (3), we only used that f was left minimal to conclude that N was indecomposable, so the same proof shows that (4) implies (3) and, similarly, (5) implies (2). Finally, Lemma 3.1.6 gives that (2) and (3) together imply (4) and (5). This completes the proof. □

3.2 Existence of almost split morphisms

In this section, we show how to find minimal almost split morphisms going in or out of a given indecomposable module.

Let N be an indecomposable A-module. If there exists an almost split sequence

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0,$$

then g is a right minimal almost split morphism arriving at N. However, such sequence is necessarily nonsplit, so N cannot be projective. Dually, we can only use almost split sequences to find a left minimal almost split morphism going out of an indecomposable module L if L is not injective. Fortunately, the projective and the injective cases are much easier to deal with.

Proposition 3.2.1. If P is an indecomposable projective A-module, then the right minimal almost split morphism with target P is the inclusion $\mathrm{rad}(P) \hookrightarrow P$. Dually, if I is an indecomposable injective A-module, then the left minimal almost split morphism with domain I is the quotient map $I \twoheadrightarrow I/\operatorname{soc}(I)$.

By Lemma 3.1.8, left and right minimal almost split morphisms are determined by their domain and their target, respectively. Thus, it makes sense to say "the" right minimal almost split morphism with target P and "the" left minimal almost split morphism with target I.

Proof. Let $g : \mathrm{rad}(P) \to P$ be the inclusion. Since g is a monomorphism, it is right minimal. Let us prove that it is right almost split. Firstly, g cannot be a retraction as it is not surjective. Now, if $v : V \to P$ is not a retraction, then it cannot be surjective because P is projective. Therefore, the image of v is contained in $\mathrm{rad}(P)$, which is the unique maximal submodule of P, so v factors through g, as needed.

The second statement is proved dually. □

Let N be a nonprojective indecomposable A-module. Our goal is to find an almost split sequence in which N is the last term. By the uniqueness of minimal almost split morphisms, the first term L of this sequence should be a noninjective indecomposable module depending only on N. If we can do this for every N, we get a bijection between the sets of isomorphism classes of nonprojective indecomposables and noninjective indecomposables. With this in mind, our approach will be to first establish this correspondence, and we will do this functorially as we did for the Nakayama functor in Section 1.4.

We introduce two new categories. For two A-modules M and N, denote by $\mathcal{P}(M, N)$ the subset of $\mathrm{Hom}_A(M, N)$ consisting of all morphisms that factor through a projective module. It is not difficult to see that the assignment $(M, N) \mapsto \mathcal{P}(M, N)$

defines an **ideal** $\mathcal{P}$ of $\operatorname{mod} A$, that is, each $\mathcal{P}(M,N)$ is a subspace of $\operatorname{Hom}_A(M,N)$ and, for morphisms $g : L \to M$ and $h : N \to P$, we have

$$fg \in \mathcal{P}(L,N) \quad \text{and} \quad hf \in \mathcal{P}(M,P)$$

for all $f \in \mathcal{P}(M,N)$. The **projectively stable category** $\underline{\operatorname{mod}}\, A$ is the quotient of $\operatorname{mod} A$ by the ideal $\mathcal{P}$. In other words, the objects of $\underline{\operatorname{mod}}\, A$ and $\operatorname{mod} A$ coincide, but the space of morphisms between two modules M and N in $\underline{\operatorname{mod}}\, A$ is given by the quotient space

$$\underline{\operatorname{Hom}}_A(M,N) := \frac{\operatorname{Hom}_A(M,N)}{\mathcal{P}(M,N)}.$$

Similarly, we define the subset $\mathcal{I}(M,N)$ of $\operatorname{Hom}_A(M,N)$ consisting of all morphisms that factor through an injective module. We get an ideal $\mathcal{I}$ and the quotient $\overline{\operatorname{mod}}\, A := (\operatorname{mod} A)/\mathcal{I}$ is the **injectively stable category**. The space of morphisms between two modules M and N in $\overline{\operatorname{mod}}\, A$ is

$$\overline{\operatorname{Hom}}_A(M,N) := \frac{\operatorname{Hom}_A(M,N)}{\mathcal{I}(M,N)}.$$

For more details on these constructions, see [2, Section III.1.1] and [6, Chapter 4, Section 1]. The idea to keep in mind is that we are forcing projective or injective modules to be isomorphic to zero, so the stable categories encode information about nonprojective and noninjective modules.

Proposition 3.2.2. There exists an equivalence $\tau : \underline{\operatorname{mod}}\, A \to \overline{\operatorname{mod}}\, A$.

Proof. We sketch a proof based on [2, Section III.1.2 and Corollary III.1.6].

Let $\operatorname{mp} A$ be the category of morphisms of $\operatorname{proj} A$, that is, the objects in $\operatorname{mp} A$ are morphisms in $\operatorname{proj} A$ and the morphisms in $\operatorname{mp} A$ are commutative squares in $\operatorname{proj} A$. By taking cokernels and applying the quotient functor $\operatorname{mod} A \to \underline{\operatorname{mod}}\, A$, we get a functor $F : \operatorname{mp} A \to \underline{\operatorname{mod}}\, A$. Since every A-module has a projective presentation, F is essentially surjective, and it is not hard to see that it is also full. Therefore, if $\mathcal{K}_F$ denotes the kernel of F, we get an induced equivalence $(\operatorname{mp} A)/\mathcal{K}_F \to \underline{\operatorname{mod}}\, A$. In the same fashion, we define the category $\operatorname{mi} A$ of morphisms of $\operatorname{inj} A$. By taking kernels, we get a functor $G : \operatorname{mi} A \to \overline{\operatorname{mod}}\, A$ which induces an equivalence $(\operatorname{mi} A)/\mathcal{K}_G \to \overline{\operatorname{mod}}\, A$.

Recall from Section 1.4 that the Nakayama functor ν is an equivalence from $\operatorname{proj} A$ to $\operatorname{inj} A$. It upgrades to an equivalence between the categories of morphisms $\operatorname{mp} A$ and $\operatorname{mi} A$. One can show that this equivalence takes the ideal $\mathcal{K}_F$ to the ideal $\mathcal{K}_G$, so it descends to an equivalence between the corresponding quotients. In this way, we get the equivalence τ as the composition

$$\underline{\operatorname{mod}}\, A \xrightarrow{\sim} (\operatorname{mp} A)/\mathcal{K}_F \xrightarrow{\nu} (\operatorname{mi} A)/\mathcal{K}_G \xrightarrow{\sim} \overline{\operatorname{mod}}\, A.$$

The first functor is a quasi-inverse of the first equivalence in the previous paragraph, and it can be defined by choosing a projective presentation for every A-module. Thus, if M is an A-module, then τM is obtained by considering the chosen projective presentation of M, applying ν and taking the kernel. Analogously, we can define a quasi-inverse τ^{-1} for τ such that, for a given A-module, $\tau^{-1} M$ is obtained by choosing an injective presentation of M, applying ν^{-1} and taking the cokernel. □

Definition 3.2.3. The equivalences $\tau : \underline{\operatorname{mod}}\, A \to \overline{\operatorname{mod}}\, A$ and $\tau^{-1} : \overline{\operatorname{mod}}\, A \to \underline{\operatorname{mod}}\, A$ from Proposition 3.2.2 and its proof are called the **Auslander-Reiten translations**.

Remark. In order to compute τM for some module M, we will always choose a *minimal* projective presentation (see [3, Section I.5]). By the uniqueness of such presentation, τM is a module defined up to isomorphism in $\operatorname{mod} A$. However, we cannot see τ as a functor from $\operatorname{mod} A$ to itself because there is no canonical way of lifting morphisms from the stable category. A similar remark applies to τ^{-1}.

Remark. As we did with the Nakayama functor, we can write τ and τ^{-1} as the composition of two dualities. If D denotes the K-duality functor, it interchanges projective and injective modules, inducing a duality between $\overline{\operatorname{mod}}\, A$ and $\underline{\operatorname{mod}}\, A^{\mathrm{op}}$. Hence, we have a duality

$$\mathrm{Tr} := D\tau : \underline{\operatorname{mod}}\, A \longrightarrow \underline{\operatorname{mod}}\, A^{\mathrm{op}}$$

called the **transposition functor**. Since the functor D already appears in the definition of ν, it "cancels" after we apply D again, thus, we can see Tr in the following way: for an A-module M, $\mathrm{Tr}\, M$ is calculated by choosing a minimal projective presentation of M, applying the A-duality functor $(-)^t$ and taking the cokernel. A quasi-inverse of Tr is computed in the same way and we also denote it by Tr. With these conventions, we have

$$\tau = D\,\mathrm{Tr} \quad \text{and} \quad \tau^{-1} = \mathrm{Tr}\, D.$$

We prove some important properties of the Auslander-Reiten translations.

Lemma 3.2.4. Let M be an indecomposable A-module and choose a minimal projective presentation $P_1 \xrightarrow{p} P_0 \xrightarrow{q} M \longrightarrow 0$.

(1) We have an exact sequence

$$0 \longrightarrow \tau M \longrightarrow \nu P_1 \xrightarrow{\nu p} \nu P_0 \xrightarrow{\nu q} \nu M \longrightarrow 0.$$

Moreover, if M is not projective, then the first three terms form a minimal injective presentation of τM.

(2) $\tau M = 0$ if and only if M is projective.

(3) τM has no nonzero injective summands.

(4) If M is nonprojective, then τM is an indecomposable noninjective module and $\tau^{-1}\tau M \cong M$.

(5) τ induces a bijection between the isomorphism classes of nonprojective and noninjective indecomposable modules.

A dual statement holds for τ^{-1}.

Proof. (1). Since the Nakayama functor is right exact, the sequence of the statement is exact on νP_0 and νM. In turn, τM is by definition the kernel of the middle map, so the sequence also becomes exact on τM and νP_1 if the map $\tau M \to \nu P_1$ is the inclusion.

For the second part, take a minimal injective presentation

$$0 \longrightarrow \tau M \longrightarrow I_1 \xrightarrow{d} I_0.$$

By [1, Chapitre X, Lemme 1.11], there are morphisms f_1 and f_0 making the following diagram commute:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tau M & \longrightarrow & I_1 & \xrightarrow{d} & I_0 \\ & & \| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ 0 & \longrightarrow & \tau M & \longrightarrow & \nu P_1 & \xrightarrow{\nu p} & \nu P_0. \end{array}$$

Furthermore, f_1 and f_0 are sections and we can find left inverses f_1' and f_0', respectively, such that $d \circ f_1' = f_0' \circ \nu p$. Our goal is to prove that f_1 and f_0 are isomorphisms or, equivalently, that $\ker f_1'$ and $\ker f_0'$ are zero. By the commutativity relations, note that νp sends $\operatorname{im} f_1$ into $\operatorname{im} f_0$ and $\ker f_1'$ into $\ker f_0'$. But $\nu P_i = \operatorname{im} f_i \oplus \ker f_i'$ for $i = 0, 1$, so we can write $\nu p = \varphi \oplus \psi$ for certain morphisms

$$\varphi : \operatorname{im} f_1 \longrightarrow \operatorname{im} f_0 \quad \text{and} \quad \psi : \ker f_1' \longrightarrow \ker f_0'.$$

Applying the inverse Nakayama functor ν^{-1} to νp and identifying $\nu^{-1}\nu P_i$ with P_i for $i = 0, 1$, we can write $p = \nu^{-1}\varphi \oplus \nu^{-1}\psi$. Since M is indecomposable and is the cokernel of p, it follows that M is isomorphic to the cokernel of $\nu^{-1}\varphi$ or of $\nu^{-1}\psi$. This gives us a new projective presentation of M whose terms have dimension at most the dimension of the terms of our original presentation. By minimality, these two presentations must be isomorphic. This can only happen if $\operatorname{im} f_1 = \operatorname{im} f_0 = 0$ or $\ker f_1' = \ker f_2' = 0$. Now, if we have the first case, then $I_0 = I_1 = 0$ and $\tau M = 0$, contradicting (2) and that M is not projective. Therefore, assuming (2), this concludes the proof of (1).

(2). If M is projective, then $P_0 \cong M$ and $P_1 = 0$, so we get $\tau M = 0$ by the exact sequence in (1). Conversely, if $\tau M = 0$, then νp is injective. Since νP_1 is an injective module, νp must be a section. Applying ν^{-1}, it follows that p is a section, so q is a retraction and M is a direct summand of P_0. Thus, M is projective.

(3). We may suppose that M is not projective by (2), so we can consider the minimal injective presentation of τM given in (1). If τM has an injective summand I, then I is also a direct summand of νP_1. By exactness of the sequence in (1), νp sends I to zero. Applying ν^{-1}, we find a direct summand of P_1 isomorphic to $\nu^{-1} I$ which is sent to zero by p. For the projective presentation in the statement to be minimal, we must have $\nu^{-1} I = 0$ and, consequently, $I = 0$.

(4). Suppose M is nonprojective. By (2) and (3), τM is nonzero and noninjective. Using the minimal injective resolution from (1), we see that $\tau^{-1}\tau M$ is isomorphic to the cokernel $\nu^{-1}(\nu p)$. Hence, $\tau^{-1}\tau M$ is isomorphic to the cokernel of p, which is M.

To conclude, let us prove that τM is indecomposable. If not, then $\tau M = N_1 \oplus N_2$ for nonzero submodules N_1 and N_2. By (3), N_1 and N_2 are not injective. Thus, the dual of (2) implies that $\tau^{-1} N_1$ and $\tau^{-1} N_2$ are also nonzero. But we have $M \cong \tau^{-1}\tau M \cong \tau^{-1} N_1 \oplus \tau^{-1} N_2$, contradicting that M is indecomposable.

(5). It follows from (4) and its dual version. □

The most important result about the Auslander-Reiten translations that we are going to use are the **Auslander-Reiten formulas** given by the theorem below.

Theorem 3.2.5. If M and N are A-modules, then there are isomorphisms

$$\operatorname{Ext}^1_A(M, \tau N) \cong D\, \underline{\operatorname{Hom}}_A(N, M) \quad \text{and} \quad \operatorname{Ext}^1_A(\tau^{-1} M, N) \cong D\, \overline{\operatorname{Hom}}_A(N, M)$$

which are functorial in both variables.

Remark. Since $\operatorname{Ext}^1_A(P,-) = 0 = \operatorname{Ext}^1_A(-,I)$ for P projective and I injective, we can view Ext^1_A as a functor from

$$(\underline{\operatorname{mod}}\, A)^{\mathrm{op}} \times \overline{\operatorname{mod}}\, A$$

to the category of K-vector spaces. In this way, it makes sense to say that the isomorphisms above are "functorial".

Proof. See [2, Theorem III.2.4]. We also refer the reader to [25] for a proof where it could be easier to check the functoriality. □

We are now ready to construct almost split sequences. Let N be a nonprojective indecomposable module. We will search for an almost split sequence of the form

$$0 \longrightarrow \tau N \longrightarrow M \longrightarrow N \longrightarrow 0.$$

This short exact sequence should correspond to an element in $\operatorname{Ext}^1_A(N, \tau N)$, so it will be helpful to characterize almost split morphisms in terms of the Ext functor.

Lemma 3.2.6. Let $0 \longrightarrow L \longrightarrow M \xrightarrow{\ g\ } N \longrightarrow 0$ be a nonsplit short exact sequence of A-modules. Then g is right almost split if, and only if, for every $v : V \to N$ which is not a retraction, the element $\xi \in \operatorname{Ext}^1_A(N, L)$ corresponding to this short exact sequence is in the kernel of the morphism

$$\operatorname{Ext}^1_A(v, L) : \operatorname{Ext}^1_A(N, L) \longrightarrow \operatorname{Ext}^1_A(V, L).$$

Proof. It suffices to check that ξ is in the kernel of the map above if and only if v factors through g. Using the naturality of the long exact sequence of Ext obtained after applying $\operatorname{Hom}_A(N,-)$ and $\operatorname{Hom}_A(V,-)$ to the short exact sequence above, we have in particular a commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}_A(N,M) & \longrightarrow & \operatorname{Hom}_A(N,N) & \xrightarrow{\delta_1} & \operatorname{Ext}^1_A(N,L) \\ \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \operatorname{Ext}^1_A(v,L)} \\ \operatorname{Hom}_A(V,M) & \xrightarrow{\psi} & \operatorname{Hom}_A(V,N) & \xrightarrow{\delta_2} & \operatorname{Ext}^1_A(V,L) \end{array}$$

with exact rows. The maps δ_1 and φ take id_N to ξ and v, respectively[1]. Hence, $\operatorname{Ext}^1_A(v, L)$ sends ξ to zero if and only if δ_2 sends v to zero. By exactness, this happens if and only if v is in the image of ψ or, in other words, if v factors through g, as needed. □

Now we can use the Auslander-Reiten formulas to identify $\operatorname{Ext}^1_A(N, \tau N)$ with $D\,\underline{\operatorname{Hom}}_A(N,N) = D\,\underline{\operatorname{End}}_A(N)$. Continuing with the assumption that N is a nonprojective indecomposable module, this vector space is nonzero. Moreover, it has a natural structure of bimodule over the algebra $\operatorname{End}_A(N)$ induced by composition.

Lemma 3.2.7. With the hypotheses above, $\operatorname{Ext}^1_A(N, \tau N)$ has a simple socle when regarded as a left or a right $\operatorname{End}_A(N)$-module.

Proof. By taking the dual space, it is equivalent to show that the top of $\underline{\operatorname{End}}_A(N)$ as a left or a right $\operatorname{End}_A(N)$-module is simple. In order to do so, let us first prove that $\mathcal{P}(N,N)$ is contained in the radical of the algebra $\operatorname{End}_A(N)$. Since N is indecomposable, $\operatorname{End}_A(N)$ is a local algebra, so it is enough to check that an invertible map

[1] It is essentially by definition that δ_1 takes id_N to ξ (see [1, Chapitre IX, Section 5]).

$N \to N$ cannot factor through a projective module. Indeed, if it factored through a projective module P, we would find a retraction $P \to N$, thus N would be a direct summand of P, contradicting the fact that N is nonprojective.

The $\mathrm{End}_A(N)$-module $\underline{\mathrm{End}}_A(M)$ is the quotient of $\mathrm{End}_A(N)$ by its submodule $\mathcal{P}(N,N)$. By the previous paragraph, its radical is isomorphic to the quotient

$$\frac{\mathrm{End}_A(N)}{\mathrm{rad}(\mathrm{End}_A(N))},$$

which is a division algebra because $\mathrm{End}_A(N)$ is local. Any nonzero element is invertible and generates such quotient, hence it is simple if regarded either as a left or a right $\mathrm{End}_A(N)$-module. □

We finish this section by proving our previous claims.

Proposition 3.2.8. (1) If N is a nonprojective indecomposable A-module, then there exists an almost split sequence of the form

$$0 \longrightarrow \tau N \longrightarrow M \longrightarrow N \longrightarrow 0.$$

(2) If L is a noninjective indecomposable A-module, then there exists an almost split sequence of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow \tau^{-1} L \longrightarrow 0.$$

Proof. If L is noninjective indecomposable, then $\tau^{-1}L$ is nonprojective indecomposable and $\tau(\tau^{-1}L) \cong L$ by the dual of Lemma 3.2.4. Hence, (2) will follow after we prove (1).

Take a nonprojective indecomposable module N. According to Lemma 3.2.7, we can choose a nonzero element ξ of the socle of the left $\mathrm{End}_A(N)$-module $\mathrm{Ext}^1_A(N,\tau N)$. It corresponds to a nonsplit short exact sequence

$$0 \longrightarrow \tau N \longrightarrow M \xrightarrow{\ g\ } N \longrightarrow 0.$$

Since τN is indecomposable by Lemma 3.2.4, Theorem 3.1.11 tells us that we have to check that g is right almost split to get an almost split sequence. To prove this, we will use Lemma 3.2.6.

Let $v : V \to N$ be a morphism of A-modules which is not a retraction. We claim that the residual class of id_N is not in the image of the map

$$\underline{\mathrm{Hom}}_A(N,v) : \underline{\mathrm{Hom}}_A(N,V) \longrightarrow \underline{\mathrm{Hom}}_A(N,N) = \underline{\mathrm{End}}_A(N).$$

If it were, then we would have $vu - \mathrm{id}_N \in \mathcal{P}(N,N)$ for some map $u : N \to V$. Using the proof of Lemma 3.2.7, $vu - \mathrm{id}_N$ would be in the radical of $\mathrm{End}_A(N)$. Since id_N is not in the radical, vu could not be either, hence it would have to be invertible and v would be a retraction, a contradiction. Therefore, the map above is not surjective. By taking its dual and applying the Auslander-Reiten formula, we get that the map

$$\mathrm{Ext}^1_A(v,\tau N) : \mathrm{Ext}^1_A(N,\tau N) \longrightarrow \mathrm{Ext}^1_A(V,\tau N)$$

is not injective. It is not hard to see that $\underline{\mathrm{Hom}}_A(N,v)$ is a homomorphism of right $\mathrm{End}_A(N)$-modules, so this last map is a homomorphism of left $\mathrm{End}_A(N)$-modules. Because the socle of $\mathrm{Ext}^1_A(N,\tau N)$ is simple by Lemma 3.2.7, it must be sent to zero. In particular, $\mathrm{Ext}^1_A(v,\tau N)$ sends ξ to zero, as desired. □

3.3 The Auslander-Reiten quiver

Before giving examples of almost split sequences in concrete cases, we will first introduce the Auslander-Reiten quiver of A. It succinctly describes the irreducible morphisms between indecomposable A-modules.

The only ingredient missing for this definition is a way of measuring the "number" of irreducible morphisms between two given modules. This will be achieved by constructing a vector space which encodes such information. In order to do so, we need to know what is the radical of $\operatorname{mod} A$.

Definition 3.3.1. Let M and N be A-modules. The **space of radical morphisms** $\operatorname{rad}_A(M, N)$ between M and N is the subset of $\operatorname{Hom}_A(M, N)$ of homomorphisms $f : M \to N$ such that, for every indecomposable summand M' of M and every indecomposable summand N' of N, the composition

$$M' \hookrightarrow M \xrightarrow{f} N \twoheadrightarrow N'$$

is not an isomorphism, where the unspecified maps are the canonical inclusion and projection. This defines an ideal rad_A of $\operatorname{mod} A$ called the **radical** of $\operatorname{mod} A$.

For some motivation and for more details on the radical, see [2, Section II.1].

Remark. If M is indecomposable, note that a homomorphism $M \to N$ is radical if and only if it is not a section. Similarly, if N is indecomposable, a map $M \to N$ is radical if and only if it is not a retraction. If both M and N are indecomposable, $\operatorname{rad}_A(M, N)$ consists of the nonisomorphisms from M to N.

For any ideal in $\operatorname{mod} A$, we can consider its powers. In particular, for A-modules M and N, we define $\operatorname{rad}^2_A(M, N)$ as the set of morphisms $M \to N$ which can be written as the composition of two radical morphisms. This gives us a new characterization of irreducible morphisms.

Lemma 3.3.2. Let $f : M \to N$ be a homomorphism between two indecomposable A-modules. Then f is irreducible if, and only if,

$$f \in \operatorname{rad}_A(M, N) \setminus \operatorname{rad}^2_A(M, N).$$

Proof. Since M and N are indecomposable, f is radical if and only if it is a nonisomorphism, which in this case is equivalent to saying that f is neither a section nor a retraction.

On the other hand, $f \in \operatorname{rad}^2_A(M, N)$ if and only if there is a module L and radical morphisms $g : M \to L$ and $h : L \to N$ such that $f = hg$. Because of the indecomposability of M and N, saying that g and h are radical is the same as saying that g is not a section and h is not a retraction. Thus, $f \in \operatorname{rad}^2_A(M, N)$ if and only if f admits a nontrivial factorization.

The lemma then follows from the definition of an irreducible morphism. □

Definition 3.3.3. Let M and N be indecomposable A-modules. The **space of irreducible morphisms** between M and N is the quotient space

$$\operatorname{Irr}_A(M, N) := \frac{\operatorname{rad}_A(M, N)}{\operatorname{rad}^2_A(M, N)}.$$

By the previous lemma, the dimension of this space quantifies in some sense the "number" of irreducible maps from M to N. We can now give the main definition of this section.

Definition 3.3.4. Let A be a finite-dimensional K-algebra. Its **Auslander-Reiten quiver** (or **AR-quiver** for short) is the quiver $\Gamma(\operatorname{mod} A)$ defined as follows:

- The vertices of $\Gamma(\operatorname{mod} A)$ are the isomorphism classes of indecomposable A-modules.
- The number of arrows from M to N is the dimension of $\operatorname{Irr}_A(M, N)$ over K.

Remark. All the notions appearing above depend only on $\operatorname{mod} A$. That is the reason for using the notation $\Gamma(\operatorname{mod} A)$. A similar definition can be used to define the AR-quiver of any K-linear Krull-Schmidt category. For example, in Chapter 4, we will implicitly deal with the AR-quiver of a bounded derived category.

Just as almost split morphisms help us find irreducible morphisms, they also can be used to compute the dimension of $\operatorname{Irr}_A(M, N)$.

Theorem 3.3.5. Suppose K is an algebraically closed field. Let L and N be indecomposable A-modules.

(1) Let $f : L \to M$ be the left minimal almost split morphism with domain L. If $M = \bigoplus_{i=1}^t M_i^{n_i}$ with the M_i indecomposable and pairwise nonisomorphic, then

$$\dim_K \operatorname{Irr}_A(L, M_i) = n_i$$

for all $1 \leq i \leq t$, and $\operatorname{Irr}_A(L, M') \neq 0$ for some indecomposable M' if and only if $M' \cong M_i$ for some i.

(2) Let $g : M \to N$ be the right minimal almost split morphism with target N. If $M = \bigoplus_{i=1}^t M_i^{n_i}$ with the M_i indecomposable and pairwise nonisomorphic, then

$$\dim_K \operatorname{Irr}_A(M_i, N) = n_i$$

for all $1 \leq i \leq t$, and $\operatorname{Irr}_A(M', N) \neq 0$ for some indecomposable M' if and only if $M' \cong M_i$ for some i.

Proof. We will only prove (1). The other item is analogous.

Note that $\operatorname{Irr}_A(L, M') \neq 0$ if and only if there is an irreducible morphism from L to M. Thus, the second part of the statement follows from Theorem 3.1.10.

For the first part, denote by $M_{j1}, \dots, M_{jn_j}$ the copies of M_j appearing in the decomposition of M. Let $f_{jk} : L \to M_{jk}$ be the components of f, with $1 \leq j \leq t$ and $1 \leq k \leq n_j$. Each of these morphisms is irreducible by Theorem 3.1.10 and hence radical by Lemma 3.3.2. For $1 \leq i \leq t$, we will show that the residual classes of $f_{i1}, \dots, f_{in_i}$ form a basis of the quotient $\operatorname{Irr}_A(L, M_i) = \operatorname{rad}_A(L, M_i)/\operatorname{rad}_A^2(L, M_i)$. This will finish the proof.

Initially, let us check that these elements are linearly independent. Suppose that there are $\lambda_1, \dots, \lambda_{n_i} \in K$ such that

$$v := \sum_{k=1}^{n_i} \lambda_k f_{ik} \in \operatorname{rad}_A^2(L, M_i).$$

This map is the composition of the morphisms $f_i : L \to M_i^{n_i}$ and $u : M_i^{n_i} \to M_i$ given by the matrices

$$f_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{in_i} \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} \lambda_1 \cdot \operatorname{id}_{M_i} & \cdots & \lambda_{n_i} \cdot \operatorname{id}_{M_i} \end{pmatrix}.$$

Observe that f_i is irreducible by Theorem 3.1.10. If some $\lambda_k \neq 0$, it is easy to see that u is a retraction. As $v = uf_i$, we can use the direct sum decomposition in $M_i^{n_i}$ induced by the retraction u to see v as a component of f_i (up to composing with an isomorphism). Hence, Theorem 3.1.10 implies that v is irreducible, but this contradicts Lemma 3.3.2 since $v \in \operatorname{rad}_A^2(L, M_i)$. Therefore, $\lambda_k = 0$ for all $1 \leq k \leq n_i$, as desired.

Now, let us prove that the residual classes $\overline{f_{i1}}, \dots, \overline{f_{in_i}}$ generate $\operatorname{Irr}_A(L, M_i)$. Let $h \in \operatorname{rad}_A(L, M_i)$. Since h is not a section and f is left almost split, there exists $u : M \to M_i$ such that $h = uf$. Let $u_{jk} : M_{jk} \to M_i$ denote the components of u, with $1 \leq j \leq t$ and $1 \leq k \leq n_j$. If $j \neq i$, then $M_{jk} \ncong M_i$ and u_{jk} cannot be an isomorphism, so it is radical. In this case, $u_{jk} f_{jk} \in \operatorname{rad}_A^2(L, M_i)$ because f_{jk} is also radical. Hence, in the quotient $\operatorname{Irr}_A(L, M_i)$, we get

$$\overline{h} = \overline{uf} = \sum_{j=1}^{t} \sum_{k=1}^{n_j} \overline{u_{jk} f_{jk}} = \sum_{k=1}^{n_i} \overline{u_{ik} f_{ik}}.$$

For $1 \leq k \leq n_i$, u_{ik} is an element of the local algebra $\operatorname{End}_A(M_i)$. Since K is algebraically closed, the quotient of $\operatorname{End}_A(M_i)$ by its radical is one-dimensional, so we can write $u_{ik} = \lambda_k \cdot \operatorname{id}_{M_i} + u'_{ik}$ for some $\lambda_k \in K$ and some noninvertible map $u'_{ik} \in \operatorname{rad}_A(M_i, M_i)$. It follows that

$$\overline{h} = \sum_{k=1}^{n_i} \overline{u_{ik} f_{ik}} = \sum_{k=1}^{n_i} \lambda_k \cdot \overline{f_{ik}},$$

concluding the proof. □

Remark. It is still possible to work when K is not algebraically closed. In this case, we see $\operatorname{Irr}_A(M, N)$ as a bimodule over the top of the local algebras $\operatorname{End}_A(M)^{\mathrm{op}}$ and $\operatorname{End}_A(N)$. These are division algebras and we can compute the dimension of $\operatorname{Irr}_A(M, N)$ over them. However, we now have two dimensions which might be different and it is convenient to use another definition for the AR-quiver of A. The reader is referred to [6, Chapter VII, Section 1] for more details.

Remark. From now on, K is **always algebraically closed** so that we can apply Theorem 3.3.5.

Corollary 3.3.6. Suppose

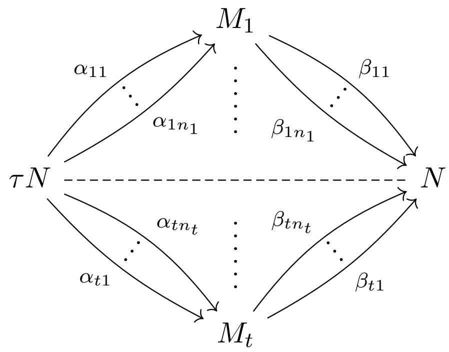
$$0 \longrightarrow L \longrightarrow \bigoplus_{i=1}^{t} M_i^{n_i} \longrightarrow N \longrightarrow 0$$

is an almost split sequence, where the M_i are indecomposable and pairwise nonisomorphic. Then, for each $1 \leq i \leq n$, we have

$$\dim_K \operatorname{Irr}_A(L, M_i) = n_i = \dim_K \operatorname{Irr}_A(M_i, N).$$

Proof. This follows immediately from Theorem 3.3.5. □

One consequence of these results is that the AR-quiver of A has a very particular shape. If N is a nonprojective indecomposable module, we have the following **mesh** inside $\Gamma(\mathrm{mod}\,A)$:



The modules $M_1, \ldots, M_t$ are the indecomposable modules appearing in the middle term of the almost split sequence with τN and N. The multiplicity of M_i is n_i. We have listed above all arrows leaving τN and all arrows arriving at N. On the other hand, there might be other arrows in $\Gamma(\mathrm{mod}\,A)$ going in or out of the modules M_i. The dashed line is not part of $\Gamma(\mathrm{mod}\,A)$ but it is there to represent that τN and N are Auslander-Reiten translates.

If N is an indecomposable projective module, we still have a "half-mesh" consisting of the right part of the picture. In this case, we have $\mathrm{rad}(N) = \bigoplus_{i=1}^{t} M_i^{n_i}$ since the inclusion $\mathrm{rad}(N) \hookrightarrow N$ is the right minimal almost split morphism with target N. Dually, we also get a "half-mesh" starting from an indecomposable injective module.

We conclude the section with a result describing the "corners" of $\Gamma(\mathrm{mod}\,A)$. It will provide the starting point of an algorithm to calculate the AR-quiver in some particular cases.

Proposition 3.3.7. Let S be an indecomposable A-module.

(1) S has no predecessors in $\Gamma(\mathrm{mod}\,A)$ if, and only if, S is simple and projective.

(2) If S is simple, projective and noninjective, then the middle terms of the mesh defined by S and $\tau^{-1}S$ are all projective. Moreover, they are exactly the indecomposable projective modules P such that S is a direct summand of $\mathrm{rad}(P)$.

(3) In the previous item, the number of arrows from S to P in $\Gamma(\mathrm{mod}\,A)$ is the multiplicity of S as a direct summand of $\mathrm{rad}(P)$.

The dual statement also holds.

Proof. (1). If S is projective, then the predecessors of S in $\Gamma(\mathrm{mod}\,A)$ are the direct summands of $\mathrm{rad}(S)$ (Proposition 3.2.1 and Theorem 3.3.5). Thus, in this case, S has no predecessors if and only if $\mathrm{rad}(S) = 0$, which is equivalent to saying that S is simple. If S is not projective, then the mesh defined by τS and S provides S with at least one predecessor.

(2) and (3). Let P be a middle term of the mesh defined by S and $\tau^{-1}S$. In particular, there is an irreducible map from S to P. If P were not projective, we could form a mesh with extremes τP and P. One of the middle terms would be S and so τP would be a predecessor of S, contradicting (1). Therefore, P is projective. Since the inclusion $\mathrm{rad}(P) \hookrightarrow P$ is the right minimal almost split morphism with target P, S must be a direct summand of $\mathrm{rad}(P)$ by Theorem 3.3.5. Furthermore, the same theorem says that $\dim_K \mathrm{Irr}_A(S, P)$ is the multiplicity of S as a direct summand of $\mathrm{rad}(P)$. Conversely, a similar argument shows that, if P' is an indecomposable projective module such that S is a direct summand of $\mathrm{rad}(P')$, then P' is a successor of S in $\Gamma(\mathrm{mod}\,A)$. □

3.4 Examples

We are finally ready to compute some examples! As we did in the previous section, we assume that the base field K is algebraically closed.

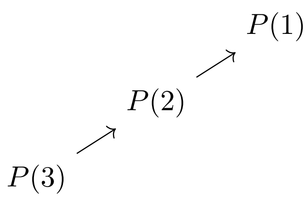
Example 3.4.1. Suppose $A = KQ$, where Q is the Dynkin diagram $\mathbb{A}_3$ with the following orientation:

$$1 \longrightarrow 2 \longrightarrow 3.$$

From Lemmas 1.3.6 and 1.3.7, we can find the projective and the injective indecomposable modules:

$$\begin{aligned} P(1) &= \ K \xrightarrow{\text{id}} K \xrightarrow{\text{id}} K \ = I(3), \\ P(2) &= \ 0 \longrightarrow K \xrightarrow{\text{id}} K, \\ P(3) &= S(3), \\ I(1) &= S(1), \\ I(2) &= \ K \xrightarrow{\text{id}} K \longrightarrow 0. \end{aligned}$$

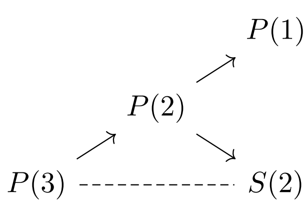
Note that $P(3)$ is the radical of $P(2)$, which in turn is the radical of $P(1)$. Thus, the full subquiver of $\Gamma(\operatorname{mod} A)$ with projective vertices is



The arrow from $P(3)$ to $P(2)$ represents the inclusion. Since $P(3)$ is simple and projective, Proposition 3.3.7 says that $P(2)$ is the only successor of $P(3)$ in $\Gamma(\operatorname{mod} A)$. Hence, by Theorems 3.1.10 and 3.3.5, the irreducible morphism $P(3) \to P(2)$ must be the left minimal almost split morphism with source $P(3)$. Taking the cokernel, we get an almost split sequence:

$$0 \longrightarrow P(3) \longrightarrow P(2) \xrightarrow{\pi} S(2) \longrightarrow 0.$$

It follows that $\tau^{-1}P(3) \cong S(2)$ and we get a mesh in the AR-quiver:

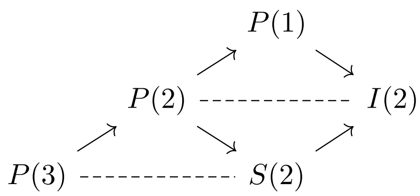


Let us compute the mesh defined by $P(2)$ and $\tau^{-1}P(2)$. We claim that $P(1)$ and $S(2)$ are the only successors of $P(2)$. Indeed, let M be a successor of $P(2)$ in $\Gamma(\operatorname{mod} A)$. If M is projective, then we already verified that M has to be $P(1)$. If M is not projective, then $P(2)$ is a middle term in the mesh defined by τM and M. But the only predecessor of $P(2)$ is $P(3)$ because the inclusion $P(3) \to P(2)$ is the right minimal almost split morphism with target $P(2)$. Thus, $\tau M \cong P(3)$ and $M \cong \tau^{-1}P(3) \cong S(2)$, as claimed.

We already know that there is only one arrow from $P(2)$ to $P(1)$ and from $P(2)$ to $S(2)$. This means that the left minimal almost split morphism with domain $P(2)$ is of the form $P(2) \to P(1) \oplus S(2)$. The component $P(2) \to P(1)$ is an irreducible map but, since $\mathrm{Irr}_A(P(2), P(1))$ is one-dimensional, there is only one irreducible map from $P(2)$ to $P(1)$ (up to multiplication by a nonzero scalar), which is the inclusion. Similarly, the component $P(2) \to S(2)$ must be the projection π from the previous almost split sequence. In this way, we have found the morphism $P(2) \to P(1) \oplus S(2)$ and, calculating the cokernel, we get another almost split sequence:

$$0 \longrightarrow P(2) \longrightarrow P(1) \oplus S(2) \longrightarrow I(2) \longrightarrow 0.$$

Hence, $\tau^{-1}P(2) \cong I(2)$ and we update our picture of $\Gamma(\mathrm{mod}\, A)$:

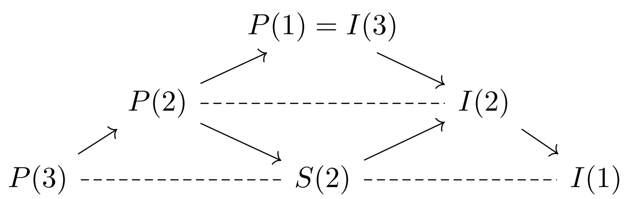


Since $S(2)$ is not injective, there is still a mesh left to be found. We affirm that $I(2)$ is the only successor of $S(2)$. Indeed, let M be a successor of $S(2)$. It cannot be projective because we already know the predecessors of every projective module and $S(2)$ is not one of them. Thus, $S(2)$ is a middle term in the mesh defined by τM and M and, in particular, τM is a predecessor of $S(2)$. But we have already found the mesh ending at $S(2)$, so we know that the only predecessor of $S(2)$ is $P(2)$. This proves that $\tau M \cong P(2)$ and $M \cong \tau^{-1}P(2) \cong I(2)$, as desired.

Arguing as before, the inclusion $S(2) \to I(2)$ (which appears in the previous almost split sequence) is the left minimal almost split morphism with source $S(2)$. Taking the cokernel, we arrive at the almost split sequence

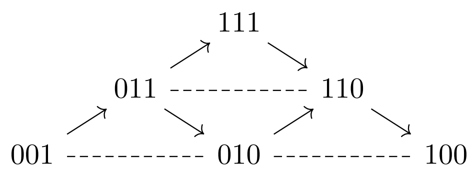
$$0 \longrightarrow S(2) \longrightarrow I(2) \longrightarrow I(1) \longrightarrow 0,$$

and we get the third mesh in $\Gamma(\mathrm{mod}\, A)$:

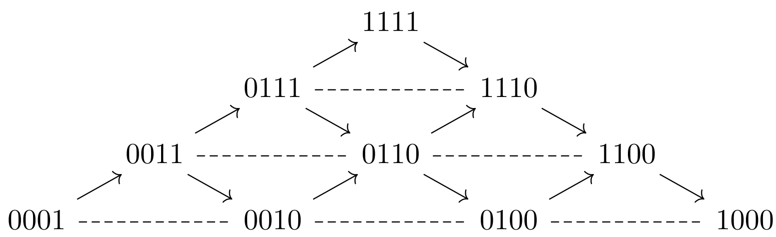


To conclude, observe that the successors of the injective modules $I(3)$ and $I(2)$ are all listed above since $I(2)$ is the quotient of $I(3)$ by its socle and $I(1)$ is the quotient of $I(2)$ by its socle (see Proposition 3.2.1). The simple module $I(1)$ has no successors by the dual of Proposition 3.3.7. Therefore, we have constructed a connected component of $\Gamma(\mathrm{mod}\, A)$.

Actually, by Theorem 2.3.4 and Section A.2, A has exactly six indecomposable modules, so the last quiver above is the AR-quiver of A. Writing the dimension vector in place of each indecomposable module, we can also represent $\Gamma(\operatorname{mod} A)$ by



We remark that the AR-quiver of the path algebra of $\mathbb{A}_n$ (with arrows pointing in the same direction) has also a triangular shape, where each side has as vertices the indecomposable projective modules, the indecomposable injective modules and the simple modules, respectively. For example, for $n = 4$, it is



This can be proved in a similar way as we did above.

This example illustrates an algorithm to compute the AR-quiver of A. Its steps can be (not so much precisely) described as follows:

(1) Find the indecomposable projective modules;

(2) Find the indecomposable summands of their radicals and draw the full subquiver of $\Gamma(\operatorname{mod} A)$ with projective vertices;

(3) Use the information above to compute the almost split sequences that start with a simple projective module;

(4) Repeat step (3) with the successors of the simple projectives in $\Gamma(\operatorname{mod} A)$, and then with their successors, and so on;

(5) Stop if you reach the indecomposable injective modules.

This is the **knitting algorithm**. It does not work in all cases but, for path algebras, it always constructs the so-called "postprojective" component of $\Gamma(\operatorname{mod} A)$ (see Proposition 3.5.3). For some more details and examples where the knitting algorithm can be applied, see [2, Section IV.1.3] and [7, Chapter 7].

For the next examples, we will just calculate the dimension vectors of the indecomposable modules, as we did at the end of Example 3.4.1. This is enough to distinguish indecomposable modules over path algebras of Dynkin quivers due to Theorem 2.3.4. It also makes it easier to apply the knitting algorithm: if we have already found a non-injective indecomposable module L and its successors $M_1, \ldots, M_t$, with multiplicities $n_1, \ldots, n_t$, then the dimension vector of $\tau^{-1}L$ can be calculated as

$$\underline{\dim}\, \tau^{-1}L = \sum_{i=1}^{t} n_i \cdot \underline{\dim}\, M_i - \underline{\dim}\, L$$

because we have an almost split sequence

$$0 \longrightarrow L \longrightarrow \bigoplus_{i=1}^{t} M_i^{n_i} \longrightarrow \tau^{-1}L \longrightarrow 0.$$

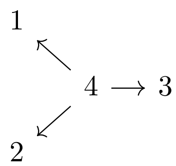
We remark that, with some more work, we can do better and calculate explicitly the indecomposable modules, as we did in Example 3.4.1.

In Example 3.4.1, we did not know immediately that the quiver we found was the AR-quiver of A, but only a connected component. We had to count the isomorphism classes of indecomposable modules to check that we indeed had found $\Gamma(\operatorname{mod} A)$. Thanks to the next theorem, this was not necessary.

Theorem 3.4.2 (Auslander)**.** Suppose A is a connected finite-dimensional K-algebra. If $\Gamma(\operatorname{mod} A)$ has a connected component Γ whose modules have bounded composition length, then A is of finite representation type and $\Gamma(\operatorname{mod} A) = \Gamma$. In particular, this is the case if $\Gamma(\operatorname{mod} A)$ admits a finite connected component Γ.

A proof can be found in [2, Theorem VI.1.7] and [3, Chapter IV, Theorem 5.4].

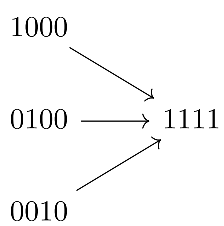
Example 3.4.3. Suppose $A = KQ$, where Q is the Dynkin diagram $\mathbb{D}_4$ with the following orientation:



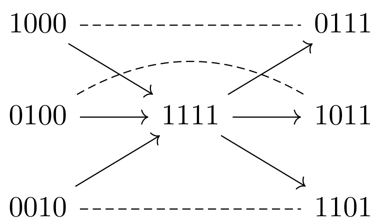
By Lemmas 1.3.6 and 1.3.7, the dimension vectors of the projective and the injective indecomposable modules are:

$$\begin{aligned} P(1) &= 1000 = S(1), \\ P(2) &= 0100 = S(2), \\ P(3) &= 0010 = S(3), \\ P(4) &= 1111, \\ I(1) &= 1001, \\ I(2) &= 0101, \\ I(3) &= 0011, \\ I(4) &= 0001 = S(4). \end{aligned}$$

To simplify the notation, these four juxtaposed numbers represent the dimension at each vertex of the representation, and the i-th number corresponds to the vertex labeled by i. Lemma 1.3.6 also shows that $P(1)$, $P(2)$ and $P(3)$ are the direct summands of the radical of $P(4)$, so the full subquiver of $\Gamma(\operatorname{mod} A)$ with projective vertices is



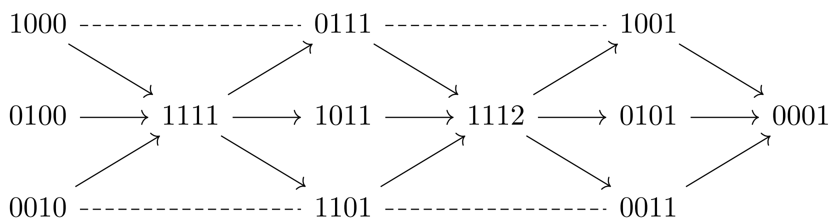
Now, Proposition 3.3.7 says that the simple projective modules $P(1), P(2)$ and $P(3)$ have just $P(4)$ as a successor in the AR-quiver. Hence, we can compute the meshes that start with them. In order to do so, we simply subtract $\underline{\dim}\, P(i)$ from $\underline{\dim}\, P(4)$ for $1 \leq i \leq 3$. We get:



As we argued in Example 3.4.1, we can show that these three new indecomposable modules are all the successors of $P(4)$, so we can form a new mesh. In terms of dimension vectors, we have

$$\underline{\dim}\, \tau^{-1} P(4) = (0111 + 1011 + 1101) - 1111 = 1112.$$

We continue in this way until we arrive at[1]:



The four rightmost dimension vectors are exactly the dimension vectors of the indecomposable injective modules. By Theorem 2.3.4, they indeed represent such modules. One can check that all their successors are again injective and are drawn in the picture. We have thus constructed a connected component of $\Gamma(\operatorname{mod} A)$ and, by Auslander's theorem, this is the whole AR-quiver.

Note that there are twelve isomorphism classes of indecomposable modules, as expected from the table in Section A.2.

Example 3.4.4. Take $A = KQ/I$, where Q is the quiver

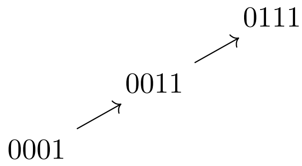
$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\gamma} 4$$

and I is the admissible ideal generated by the relation $\alpha\beta\gamma$. Let us find the AR-quiver of A. As we will see, the knitting algorithm will only produce indecomposable modules whose dimension vectors are made of zeros and ones. If we fix a vector like this, it is not hard to check that, up to isomorphism, there exists at most one indecomposable A-module with such dimension vector. Hence, we will again describe $\Gamma(\operatorname{mod} A)$ just with dimension vectors.

We described the projective and the injective indecomposable modules in Example 1.3.8. One can check that the simple projective $P(4)$ is the radical of $P(3)$, which in

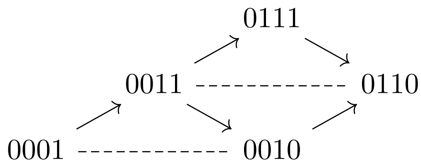
[1] In the middle row, every module is the Auslander-Reiten translate of the module that comes two positions to the right, but the dashed lines were omitted for clarity.

turn is the radical of $P(2)$. The full subquiver of $\Gamma(\text{mod}\, A)$ with these vertices is thus

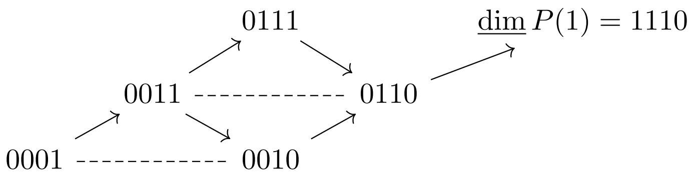


The projective module $P(1)$ is missing. The problem is that its radical is not projective, so it is not directly connected to the other other projective vertices in the AR-quiver. For the moment, we ignore this problem and continue applying the knitting algorithm.

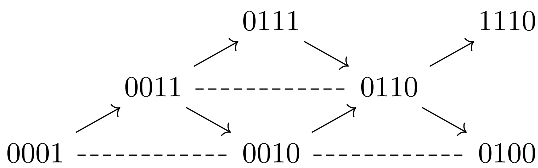
Calculating the mesh of the simple projective with dimension vector 0001 and then the mesh of the indecomposable projective with dimension vector 0011, we get:



The indecomposable module with dimension vector 0110 is the radical of $P(1)$, hence we have to add an arrow:

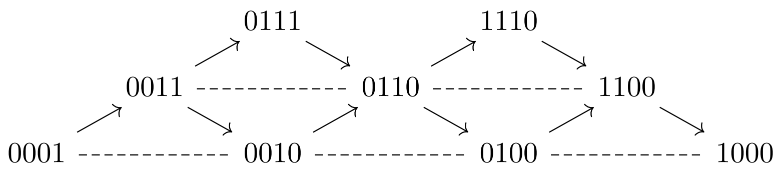


There is no mesh starting or ending at the indecomposable modules represented by 0111 and 1110 because they are both projective and injective at the same time. We continue with the algorithm an add the mesh starting at 0010:



Now, we claim that the module X represented by 0110 has all its successors in the diagram above, even though we had to add the projective $P(1)$ in the middle of the algorithm. Indeed, let M be a successor of X. If M is projective, then M has to be $P(1)$, which is depicted above. If M is not projective, then τM is a predecessor of X. Since we computed the mesh ending at X, we must have $\underline{\dim}\, \tau M = 0111$ or $\underline{\dim}\, \tau M = 0010$. The first case cannot happen because τM is not injective by Lemma 3.2.4. Therefore, τM is the module with dimension vector 0010 and $M \cong \tau^{-1}(\tau M)$ has dimension vector 0100, as desired. In this way, we can continue applying the knitting algorithm.

After drawing two more meshes, we arrive at:



We have found all projective and injective indecomposable modules. Moreover, every noninjective indecomposable above is the left term of some mesh and every nonprojective indecomposable is the right term of some mesh. Hence, this is a connected component of $\Gamma(\operatorname{mod} A)$. By Auslander's theorem, A has finite representation type and the quiver above is its AR-quiver.

We remark once more that the modules with dimension vectors 0111 and 1110 are not Auslander-Reiten translates.

In all previous examples, the AR-quivers did not have multiple arrows between two vertices. The following result explains this coincidence.

Proposition 3.4.5. If $\Gamma(\operatorname{mod} A)$ has multiple arrows, then A has infinite representation type.

We refer the reader to [3, Chapter IV, Proposition 4.9] for a proof. The idea is that, if there are two arrows from a vertex to another, then the Auslander-Reiten translates of these vertices are of higher dimension and still connected by a multiple arrow, so we can repeat the argument and get indecomposable modules of arbitrarily high dimension. This is illustrated in the next example.

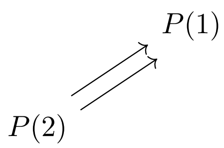
Example 3.4.6. Take $A = KQ$, where Q is the quiver

$$1 \rightrightarrows 2.$$

The indecomposable projective module $P(2)$ is simple and $P(1)$ is given by the following representation:

$$K \underset{\begin{pmatrix}0\\1\end{pmatrix}}{\overset{\begin{pmatrix}1\\0\end{pmatrix}}{\rightrightarrows}} K^2,$$

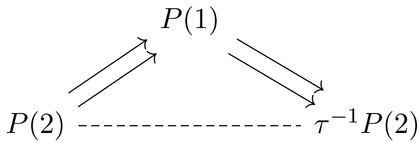
where the maps are represented by their matrices in the canonical bases. We have $\operatorname{rad}(P(1)) = P(2) \oplus P(2)$, so the knitting algorithm first gives us



The dimension vector of $\tau^{-1}P(2)$ is then

$$\underline{\dim}\,\tau^{-1}P(2) = 2 \cdot \underline{\dim}\,P(1) - \underline{\dim}\,P(2) = (2,3).$$

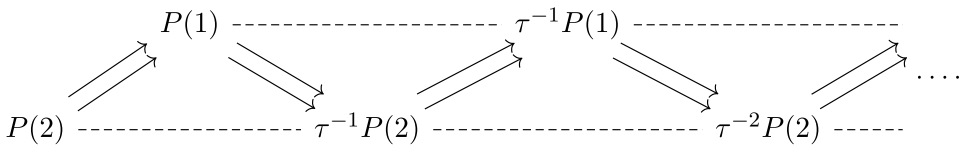
Drawing the mesh, we get:



Continuing in the same way, we get that the dimension vector of $\tau^{-1}P(1)$ is $(3,4)$. Note that the dimension of $\tau^{-1}P(1)$ and $\tau^{-1}P(2)$ exceed the dimension of A, so these modules are not injective and we can continue with the algorithm. The dimension keeps increasing and one can show that

$$\underline{\dim}\,\tau^{-n}P(i) = \begin{cases} (1+2n, 2+2n) & \text{if } i=1, \\ (2n, 1+2n) & \text{if } i=2, \end{cases}$$

for $n \geq 0$. In the end, we get an infinite connected component of $\Gamma(\operatorname{mod} A)$:



This is not the whole AR-quiver of A. For example, the indecomposable injective modules do not appear above, and we can actually find them with a "dual" knitting algorithm which produces another infinite connected component of $\Gamma(\operatorname{mod} A)$ similar to the one above. There are still connected components left to be found. See [2, Section IV.4] for a full description.

3.5 The hereditary case

Suppose Q is an acyclic and connected quiver. In this section, we prove some particular properties of the Auslander-Reiten quiver of the path algebra KQ, where K is an algebraically closed field. All the results will essentially be a consequence of the fact that KQ is a hereditary algebra (see Section 2.1).

Lemma 3.5.1. Every predecessor of a projective module in $\Gamma(\operatorname{mod} KQ)$ is again projective. Moreover, the full subquiver of $\Gamma(\operatorname{mod} KQ)$ consisting of the indecomposable projective modules is connected and isomorphic to the opposite quiver Q^{op}. The dual result for injective modules also holds.

Proof. By Proposition 3.2.1 and Theorem 3.3.5, the predecessors of an indecomposable projective module P are the indecomposable direct summands of $\operatorname{rad}(P)$. Since KQ is hereditary, they are all projective.

Let $x \in Q_0$. From the proof of Proposition 2.1.2, we know that

$$\operatorname{rad}(P(x)) \cong P(y_1)^{n_1} \oplus \cdots \oplus P(y_r)^{n_r},$$

where $y_1, \ldots, y_r$ are the *successors* of x in Q and n_i is the number of arrows from x to y_i, for each $1 \leq i \leq r$. Therefore, $P(y_1), \ldots, P(y_r)$ are the *predecessors* of $P(x)$ in $\Gamma(\operatorname{mod} KQ)$ and, by Theorem 3.3.5, the number of arrows from $P(y_i)$ to $P(x)$ is n_i. The second statement of the lemma follows. □

This lemma allows us to find some special connected components of $\Gamma(\operatorname{mod} KQ)$.

Definition 3.5.2. Let A be an arbitrary finite-dimensional K-algebra and let Γ be a connected component of $\Gamma(\operatorname{mod} A)$. We say that Γ is **postprojective** if every indecomposable module in Γ is isomorphic to $\tau^{-t}P$ for some indecomposable projective module P and some $t \geq 0$. Dually, we say that Γ is **preinjective** if every indecomposable module in Γ is isomorphic to $\tau^{t}I$ for some indecomposable injective module I and some $t \geq 0$.

In the first three examples of Section 3.4, $\Gamma(\operatorname{mod} A)$ is connected and it is both postprojective and preinjective. The connected component described in Example 3.4.6 is postprojective.

Proposition 3.5.3. If Q is a connected and acyclic quiver, then $\Gamma(\operatorname{mod} KQ)$ admits a unique postprojective component (which contains all indecomposable projective modules) and a unique preinjective component (which contains all indecomposable injective modules). They are both acyclic.

Proof. We will consider just the case of the postprojective component, the other case being analogous. By Lemma 3.5.1, all indecomposable projective modules lie in the same connected component Γ of $\Gamma(\operatorname{mod} KQ)$. We will prove by contradiction that Γ is postprojective.

Suppose there is an indecomposable module M in Γ which is not of the form $\tau^{-t}P$ for some projective P and $t \geq 0$. Since Γ is connected, there is an undirected path between M and some indecomposable projective. In particular, there exists an indecomposable module L of the form $\tau^{-t}P$ and an irreducible morphism $L \to M$ or $M \to L$. We may change M and L so that t is minimal. We have two cases.

Suppose first that there is an irreducible morphism $M \to L$. If L is projective, then M is also projective by Lemma 3.5.1, a contradiction. If L is not projective, then $t \geq 1$ and, due to the mesh defined by τL and L, there is an irreducible morphism $\tau L \to M$. But $\tau L \cong \tau^{-(t-1)}P$, contradicting the minimality of t.

Now, suppose there is an irreducible morphism $L \to M$. By hypothesis, M is not projective and so there is an irreducible morphism $\tau M \to L$. If $t \geq 1$, then $L \cong \tau^{-t}P$ is not projective either and now we get an irreducible morphism $\tau L \to \tau M$. Actually, $\tau^{s}M$ is not projective for all $s \geq 0$, so we can repeat this argument t times to find an irreducible morphism $\tau^{t}L \cong P \to \tau^{t}M$. Finally, we have an irreducible morphism $\tau^{t+1}M \to P$ and $\tau^{t+1}M$ is projective by Lemma 3.5.1, a contradiction.

This concludes the proof that Γ is postprojective. For the uniqueness, note that, for a nonprojective indecomposable M, both τM and M are in the same connected component due to the mesh they define. Hence, any postprojective component of $\Gamma(\operatorname{mod} KQ)$ contains a projective module and must be equal to Γ.

Let us prove that Γ is acyclic. If not, then we have a cycle of irreducible morphisms

$$M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_{m-1} \longrightarrow M_m = M_0$$

in Γ. Since Γ is postprojective, for each $1 \leq i \leq m$, the module M_i is isomorphic to $\tau^{t_i}P_i$ for some indecomposable projective P_i and some $t_i \geq 0$. If $t = \min\{t_i \mid 1 \leq i \leq m\}$, a similar argument as given before shows that we can apply the Auslander-Reiten translation t times and still arrive at a cycle of irreducible morphisms:

$$\tau^{t}M_0 \longrightarrow \tau^{t}M_1 \longrightarrow \cdots \longrightarrow \tau^{t}M_{m-1} \longrightarrow \tau^{t}M_0.$$

By definition of t, one of the modules above is projective, hence all of them are projective by Lemma 3.5.1. Therefore, there is a cycle in the full subquiver of $\Gamma(\operatorname{mod} KQ)$

consisting of the projective vertices, but this subquiver is isomorphic to the acyclic quiver Q^{op} by the same lemma, a contradiction. □

We finish the section by highlighting some properties of the orbits of the Auslander-Reiten translation in the case of finite representation type.

Definition 3.5.4. Let M be an indecomposable A-module. The **τ-orbit** of M is the set of isomorphism classes of indecomposable modules of the form $\tau^t M$ for $t \in \mathbb{Z}$. It is **periodic** if $\tau^t M \cong M$ for some $t \geq 1$.

Corollary 3.5.5. Suppose Q is a Dynkin quiver. Then, the τ-orbit of any indecomposable KQ-module contains a unique projective module and a unique injective module. In particular, the number of τ-orbits is the cardinality of Q_0 and they are all nonperiodic.

Proof. By Gabriel's theorem, $\Gamma(\operatorname{mod} KQ)$ is a finite quiver, hence connected by Auslander's theorem. Proposition 3.5.3 then implies that $\Gamma(\operatorname{mod} KQ)$ is both postprojective and preinjective. In particular, every τ-orbit contains a projective module and an injective module.

For the uniqueness, if P and P' are indecomposable projective modules in the same τ-orbit, then we may assume there is $t \geq 0$ such that $P \cong \tau^t P'$. If $t \geq 1$, then $\tau^t P' = 0$ by Lemma 3.2.4, which is not the case, so $t = 0$ and $P \cong P'$. A similar argument holds for the injective case.

It is easy to see that different τ-orbits are disjoint. Thus, the number of τ-orbits is the number of isomorphism classes of projective (or injective) indecomposable modules, which is the number of vertices of Q.

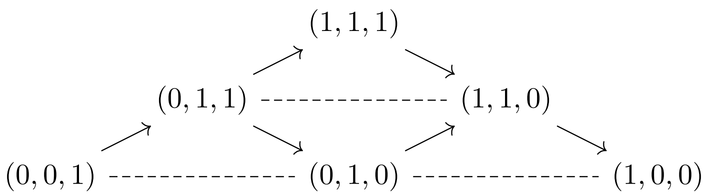
Lastly, if a τ-orbit is periodic, one can check that there is $t \geq 1$ such that $\tau^t M \cong M$ for every M in it. Taking M to be projective, we get a contradiction. □

Chapter 4

Fractionally Calabi-Yau algebras

Let Q be a Dynkin quiver. We saw in Section 3.4 how to construct the AR-quiver of the path algebra KQ using the knitting algorithm. By Theorem 2.3.4, it suffices to work with dimension vectors and, in this the case, the meshes drawn by the algorithm can be computed combinatorially. If we did not know the dimension vectors of the indecomposable injective modules in advance, we could theoretically continue applying the algorithm indefinitely. Let us see what would happen.

Take Q to be the Dynkin diagram $\mathbb{A}_3$ with the orientation given in Example 3.4.1. This example showed that $\Gamma(\operatorname{mod} KQ)$ is



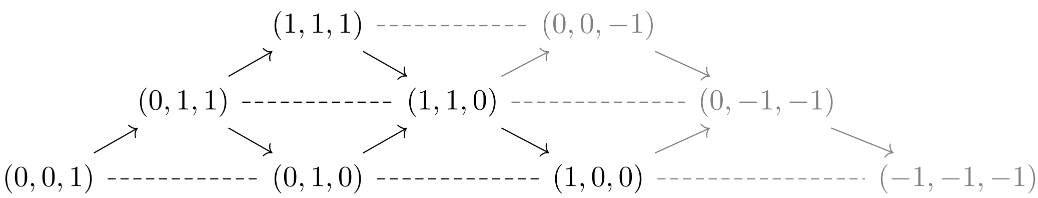
where each indecomposable module is represented by its dimension vector. Computing the "theoretical" mesh starting at the indecomposable injective with dimension vector $(1, 1, 1)$, we get a "module" with "dimension vector"

$$(1, 1, 0) - (1, 1, 1) = (0, 0, -1).$$

This is nonsense, but let us continue anyway. Now, the "dimension vector" of the "Auslander-Reiten translate" of $(1, 1, 0)$ is

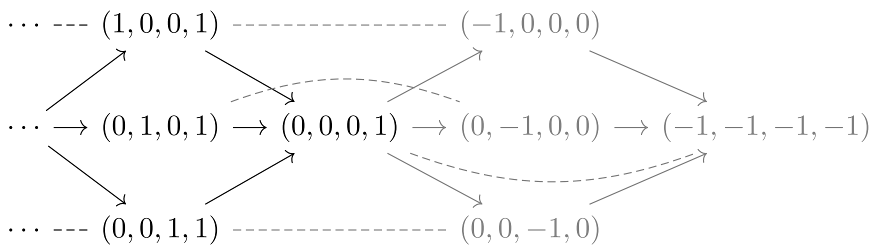
$$(0, 0, -1) + (1, 0, 0) - (1, 1, 0) = (0, -1, -1).$$

Drawing these two meshes and then the one starting with $(1, 0, 0)$, we get the following picture:



Remarkably, the full subquiver consisting of these three new vectors is the same as the full subquiver consisting of the projective vertices, but all the signs are changed. Therefore, if we continue with the knitting algorithm, we get again the AR-quiver of KQ, but now it is flipped upside down and with opposite signs. Computing even more meshes, we recover $\Gamma(\operatorname{mod} KQ)$ and the pattern repeats.

This behavior occurs for any Dynkin quiver. For example, if we do the same for $Q = \mathbb{D}_4$ with the orientation given in Example 3.4.3, the new vectors we get are the following:



They are again the opposite of the dimension vectors of the indecomposable projective modules. Thus, if we carry on with the algorithm forever, we alternately find "positive" and "negative" copies of $\Gamma(\operatorname{mod} KQ)$.

The first goal of this chapter is to explain the idea behind this phenomenon. The right framework for our study will be the bounded derived category of $\operatorname{mod} KQ$, and it is its AR-quiver that this extended knitting algorithm is drawing. However, instead of developing Auslander-Reiten theory for derived categories, we will take another approach: our main objective is to prove that KQ is a fractionally Calabi-Yau algebra. We need some new concepts in order to define what this means but, during the process, we will see how to upgrade the Auslander-Reiten translation to the derived case and understand the mystery depicted above.

The first three sections introduce the definitions and properties that we need, following mostly [26]. A proof of the main result, which is based on the appendix of [12], is given in the last section. Although we assume the reader has some familiarity with derived categories, a quick review and references can be found in Appendix B.

Remark. Throughout the chapter, A denotes a finite-dimensional K-algebra. At some points, we will assume it to be hereditary or of finite global dimension. All A-modules are of finite dimension over K, which is an algebraically closed field.

4.1 The derived category of a hereditary algebra

In this section, we will study some properties of the bounded derived category $\mathcal{D}^b(A)$ of $\operatorname{mod} A$ when A is hereditary.

As detailed in Appendix B, we will use a homological notation instead of a cohomological one, that is, an object $X \in \mathcal{D}^b(A)$ is a bounded *chain* complex

$$X = \quad \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

of A-modules, where the indices are *decreasing*. Moreover, by taking *homology* in the n-th degree, we get a functor $H_n : \mathcal{D}^b(A) \to \operatorname{mod} A$. For $X \in \mathcal{D}^b(A)$, the shift $X[1]$ is the chain complex with

$$X[1]_n \coloneqq X_{n-1} \quad \text{and} \quad d_n^{X[1]} \coloneqq -d_{n-1}^X$$

for all $n \in \mathbb{Z}$. This is the translation functor from the triangulated structure of $\mathcal{D}^b(A)$.

The main result of this section is the following.

Proposition 4.1.1. If A is a hereditary algebra, then every chain complex $X \in \mathcal{D}^b(A)$ is isomorphic in $\mathcal{D}^b(A)$ to its homology.

The homology of X as a complex is the chain complex

$$\cdots \longrightarrow H_{n+1}(X) \xrightarrow{0} H_n(X) \xrightarrow{0} H_{n-1}(X) \longrightarrow \cdots.$$

If we denote by $M[n]$ the *stalk complex* whose only nonzero entry is the A-module M in the n-th degree, then the homology of X is the direct sum

$$\bigoplus_{n\in\mathbb{Z}} H_n(X)[n]$$

in $\mathcal{D}^b(A)$. Observe that this is a finite direct sum because X is bounded.

Proof. It suffices to find, for each $n \in \mathbb{Z}$, a morphism $H_n(X)[n] \to X$ in $\mathcal{D}^b(A)$ which induces an isomorphism after taking homology in degree n. Indeed, with these morphisms, we get a quasi-isomorphism (in $\mathcal{D}^b(A)$) from the homology of X to X due to the direct sum decomposition above.

Fix $n \in \mathbb{Z}$. In $\operatorname{mod} A$, if we consider the long exact sequence of Ext obtained after applying the functor $\operatorname{Hom}_A(H_n(X), -)$ to the short exact sequence

$$0 \longrightarrow \ker d_{n+1} \longrightarrow X_{n+1} \longrightarrow \operatorname{im} d_{n+1} \longrightarrow 0,$$

we get in particular an exact sequence

$$\operatorname{Ext}^1_A(H_n(X), X_{n+1}) \xrightarrow{\alpha} \operatorname{Ext}^1_A(H_n(X), \operatorname{im} d_{n+1}) \longrightarrow \operatorname{Ext}^2_A(H_n(X), \ker d_{n+1}).$$

The last term vanishes because A is hereditary, hence the map α is surjective. If we view the short exact sequence

$$0 \longrightarrow \operatorname{im} d_{n+1} \longrightarrow \ker d_n \longrightarrow H_n(X) \longrightarrow 0$$

as an element of $\operatorname{Ext}^1_A(H_n(X), \operatorname{im} d_{n+1})$, the surjectivity of α gives us a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_{n+1} & \xrightarrow{f} & E_n & \xrightarrow{g} & H_n(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle h} & & \| & & \\
0 & \longrightarrow & \operatorname{im} d_{n+1} & \longrightarrow & \ker d_n & \longrightarrow & H_n(X) & \longrightarrow & 0
\end{array} \qquad (*)$$

with exact rows[1]. Thus, if $i : \ker d_n \to X_n$ denotes the inclusion, we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H_n(X) & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle g} & & \uparrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & X_{n+1} & \xrightarrow{f} & E_n & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle ih} & & \downarrow & & \\
\cdots & \longrightarrow & X_{n+2} & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots
\end{array}$$

Denote the complex in the middle row by E. The upper part of the diagram above represents a map of complexes $\varphi : E \to H_n(X)[n]$ which is a quasi-isomorphism since

[1] If we interpret Ext^1_A as a group of extensions, this comes from the definition of the map $\alpha = \operatorname{Ext}^1_A(H_n(X), d_{n+1})$ (see [1, Chapitre IX, Section 5]).

the first row in $(*)$ is exact. Finally, the bottom part represents a map $\psi : E \to X$ and $(*)$ implies that $H_n(\psi)$ is an isomorphism. Therefore, the morphism $\psi\varphi^{-1}$ in $\mathcal{D}^b(A)$ is the one we were searching for. □

Recall that, for every $n \in \mathbb{Z}$, there is a copy of $\operatorname{mod} A$ inside $\mathcal{D}^b(A)$ consisting of complexes concentrated in degree n. The result above is saying that, if A is hereditary, any object of $\mathcal{D}^b(A)$ is a finite direct sum of objects in these subcategories. In addition, it is easy to tell what the morphisms between two stalk complexes are:

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(M[m], N[n]) \cong \begin{cases} 0 & \text{if } n < m \text{ or } n > m+1 \\ \operatorname{Hom}_A(M, N) & \text{if } n = m \\ \operatorname{Ext}^1_A(M, N) & \text{if } n = m+1 \end{cases}$$

for $m, n \in \mathbb{Z}$ and A-modules M and N. This comes from Proposition B.3.2 and from the fact that A is hereditary.

Corollary 4.1.2. Suppose A is a hereditary algebra. Then the indecomposable objects of $\mathcal{D}^b(A)$ are the stalk complexes $M[n]$ where M is an indecomposable A-module and $n \in \mathbb{Z}$. Furthermore, $\mathcal{D}^b(A)$ is a Krull-Schmidt category.

An additive category is called **Krull-Schmidt** if every object can be written as a finite direct sum of indecomposable objects having local endomorphism rings. In such categories, we can adapt the proof of the Krull-Schmidt theorem for modules and get the uniqueness result for direct sum decompositions with indecomposable objects.

Proof. Let $X \in \mathcal{D}^b(A)$ be an indecomposable object. By Proposition 4.1.1, we have an isomorphism

$$X \cong \bigoplus_{n \in \mathbb{Z}} H_n(X)[n] \qquad (*)$$

in $\mathcal{D}^b(A)$, and this direct sum is finite because we are in the bounded derived category. Since X is indecomposable, there is exactly one $n \in \mathbb{Z}$ with $H_n(X) \neq 0$ and X is the stalk complex $H_n(X)[n]$. Using the embedding of $\operatorname{mod} A$ in $\mathcal{D}^b(A)$ in degree n, it is easy to see that $H_n(X)$ has to be indecomposable.

Conversely, let M be an indecomposable A-module and $n \in \mathbb{Z}$. If $M[n] \cong X \oplus Y$ for two complexes $X, Y \in \mathcal{D}^b(A)$, then, by taking homology, it follows that $H_m(X) = H_m(Y) = 0$ for $m \neq n$ and $M \cong H_n(X) \oplus H_n(Y)$. By the indecomposability of M, we get that X or Y is exact and hence quasi-isomorphic to the zero complex. This implies that direct sum decomposition is trivial, proving that $M[n]$ is an indecomposable object of $\mathcal{D}^b(A)$.

For the second part, any object $X \in \mathcal{D}^b(A)$ can be written as in $(*)$ by Proposition 4.1.1. Decomposing each $H_n(X)$ as a finite direct sum of indecomposable A-modules, we see that X is isomorphic to a finite direct sum of indecomposable objects in $\mathcal{D}^b(A)$. Moreover, for an indecomposable stalk complex $M[n]$, note that

$$\operatorname{End}_{\mathcal{D}^b(A)}(M[n]) \cong \operatorname{End}_A(M),$$

which is a local algebra because M is an indecomposable module. □

4.2 Revisiting the Grothendieck group

We can adapt the definition of Grothendieck group for $\mathcal{D}^b(A)$ if we replace short exact sequences by distinguished triangles.

Definition 4.2.1. The **Grothendieck group** $K_0(\mathcal{D}^b(A))$ of $\mathcal{D}^b(A)$ is the quotient of the free abelian group on the set of isomorphism classes $[X]$ of complexes X in $\mathcal{D}^b(A)$ by the subgroup generated by all expressions of the form

$$[X] - [Y] + [Z]$$

whenever there is a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1].$$

The element in $K_0(\mathcal{D}^b(A))$ corresponding to a complex X will also be denoted by $[X]$.

A similar definition works for any essentially small triangulated category.

Lemma 4.2.2. For $X \in \mathcal{D}^b(A)$, we have $[X[1]] = -[X]$ in $K_0(\mathcal{D}^b(A))$.

Proof. By the axiom (TR1) of triangulated categories (see Section B.2),

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$

is a distinguished triangle. By (TR2), the triangle

$$X \longrightarrow 0 \longrightarrow X[1] \xrightarrow{-\text{id}} X[1]$$

is distinguished too. Therefore, $[X] + [X[1]] = [0]$ in the Grothendieck group and it is enough to check that $[0]$ is the zero element of $K_0(\mathcal{D}^b(A))$. Indeed, (TR1) gives the distinguished triangle

$$0 \xrightarrow{\text{id}} 0 \longrightarrow 0 \longrightarrow 0[1],$$

hence $[0] = [0] - [0] + [0] = 0$ in $K_0(\mathcal{D}^b(A))$. □

Interestingly, this new definition is closely related to $K_0(A)$. Remember that any short exact sequence of A-modules

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

induces a distinguished triangle of stalk complexes

$$L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow L[1].$$

Thus, the inclusion $\operatorname{mod} A \hookrightarrow \mathcal{D}^b(A)$ induces a homomorphism of abelian groups $\varphi : K_0(A) \to K_0(\mathcal{D}^b(A))$ which sends $[M]$ to the class of the stalk complex M concentrated in degree zero.

Proposition 4.2.3. The map φ is an isomorphism.

Proof. Let us first find a left inverse for φ. Any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

induces a long exact sequence in homology:

$$\cdots \longrightarrow H_{n+1}(Z) \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots .$$

This sequence is bounded because we started in the bounded derived category. Hence, in the Grothendieck group of A we get the following relation:

$$\sum_{n\in\mathbb{Z}}(-1)^n[H_n(X)] - \sum_{n\in\mathbb{Z}}(-1)^n[H_n(Y)] + \sum_{n\in\mathbb{Z}}(-1)^n[H_n(Z)] = 0.$$

This proves that the map

$$\begin{aligned}\psi : K_0(\mathcal{D}^b(A)) &\longrightarrow K_0(A)\\ [X] &\longmapsto \sum_{n\in\mathbb{Z}}(-1)^n[H_n(X)]\end{aligned}$$

is well-defined. It is immediate that ψ is a left inverse for φ. In particular, φ is injective.

Now, let X be a bounded chain complex of A-modules. Suppose $m \in \mathbb{Z}$ is such that $n > m$ implies $X_n = 0$. We have the following morphism of complexes

$$\begin{array}{ccccccccc}\cdots & \longrightarrow & 0 & \longrightarrow & X_m & \longrightarrow & 0 & \longrightarrow & \cdots\\ & & \downarrow & & \downarrow{\scriptstyle d_m} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & X_{m-1} & \xrightarrow{d_{m-1}} & X_{m-2} & \xrightarrow{d_{m-2}} & \cdots\end{array}$$

where the top row is concentrated in degree $m-1$ and the bottom row is a (stupid) truncation X' of X. From the definition, it is not hard to see that the cone of this morphism is X, so we have

$$[X] = [X'] - [X_m[m-1]] = [X'] + (-1)^m[X_m]$$

in $K_0(\mathcal{D}^b(A))$, where we view X_m as a stalk complex concentrated in degree zero. Repeating this procedure with X' and its subsequent truncations, we get

$$[X] = \sum_{n\in\mathbb{Z}}(-1)^n[X_n] = \varphi\left(\sum_{n\in\mathbb{Z}}(-1)^n[X_n]\right).$$

This shows that φ is surjective, concluding the proof. Observe that it also gives us another way of writing the inverse ψ. □

This isomorphism allows us to interpret the dimension vectors with negative coordinates from the introduction of the chapter. Instead of looking at dimension vectors while drawing the AR-quiver of A, we can equivalently look at the class of each indecomposable A-module in $K_0(A)$. By Proposition 4.2.3, these classes are elements of $K_0(\mathcal{D}^b(A))$. If $A = KQ$ with Q Dynkin, we will see in the next section that the weird dimension vectors we found in the beginning correspond to the shifts of the indecomposable projective modules, which explains the negative signs.

We end this section with a result in this direction.

Lemma 4.2.4. Suppose Q is a Dynkin quiver. If X and Y are indecomposable objects in $\mathcal{D}^b(KQ)$ with the same class in the Grothendieck group, then $X \cong Y[2a]$ for some $a \in \mathbb{Z}$. Similarly, if $[X] = -[Y]$ in $K_0(\mathcal{D}^b(A))$, then $X \cong Y[2b+1]$ for some $b \in \mathbb{Z}$.

Proof. Since KQ is hereditary, we can assume by Corollary 4.1.2 that $X = M[m]$ and $Y = N[n]$ for $m, n \in \mathbb{Z}$ and indecomposable KQ-modules M and N. Thus, in

$K_0(\mathcal{D}^b(A))$, we have

$$[M] = (-1)^m[X] = (-1)^m[Y] = (-1)^{m+n}[N].$$

On the other hand, in $K_0(A)$, Theorem 2.3.4 says that $[M]$ and $[N]$ correspond to *positive* roots of Q. Hence, using the identification from Proposition 4.2.3, we must have $(-1)^{m+n} = 1$ and, consequently, $m - n$ is even. Knowing this, the two previous results also imply that $M \cong N$, so

$$Y[m-n] = N[m] \cong M[m] = X,$$

as desired. The second statement follows from the first after shifting Y. □

4.3 Serre functors and the derived Nakayama functor

For the sake of defining what is a fractionally Calabi-Yau algebra, we first need to introduce Serre functors and see how they appear in our study of $\mathcal{D}^b(A)$. As a result, we will develop the tools to explain what we discussed in the introduction of the chapter.

Definition 4.3.1. Let $\mathcal{C}$ be a K-linear category with finite-dimensional Hom-sets. A **Serre functor** is an equivalence $F : \mathcal{C} \to \mathcal{C}$ together with natural isomorphisms

$$\eta_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X,Y) \longrightarrow D\operatorname{Hom}_{\mathcal{C}}(Y, FX)$$

for all $X, Y \in \mathcal{C}$. Here, recall that $D\operatorname{Hom}_{\mathcal{C}}(Y, FX)$ is the dual space of $\operatorname{Hom}_{\mathcal{C}}(Y, FX)$.

Remark. We will not use other properties of Serre functors besides the definition. Nevertheless, it is important to know that they are essentially unique. Indeed, if F and F' are two Serre functors with natural isomorphisms $\eta_{X,Y}$ and $\eta'_{X,Y}$, then we have a natural isomorphism of functors

$$\operatorname{Hom}_{\mathcal{C}}(-, FX) \xrightarrow{D\eta_{X,-}} D\operatorname{Hom}_{\mathcal{C}}(X,-) \xrightarrow{\left(D\eta'_{X,-}\right)^{-1}} \operatorname{Hom}_{\mathcal{C}}(-, F'X)$$

for every $X \in \mathcal{C}$. We get from the Yoneda lemma a natural isomorphism from F to F' which is compatible with $\eta_{X,Y}$ and $\eta'_{X,Y}$. In this sense, having or not a Serre functor is an intrinsic property of the category $\mathcal{C}$ and not extra information.

For more details on Serre functors, see [9] and [10].

Suppose that A has finite global dimension. We will prove that the left derived functor of the Nakayama functor is a Serre functor on $\mathcal{D}^b(A)$. Let us make this sentence more precise. In Section 1.4, we saw that the Nakayama functor induces an equivalence $\nu : \operatorname{proj} A \to \operatorname{inj} A$. Since it is an additive functor, it is not difficult to see that we can extended it to an equivalence

$$\nu : \mathcal{K}^b(\operatorname{proj} A) \longrightarrow \mathcal{K}^b(\operatorname{inj} A),$$

where $\mathcal{K}^b(\operatorname{proj} A)$ (resp., $\mathcal{K}^b(\operatorname{inj} A)$) denotes the full subcategory of the bounded homotopy category $\mathcal{K}^b(A)$ of $\operatorname{mod} A$ whose objects are bounded chain complexes of projective (resp., injective) A-modules[1]. By the hypothesis on A, the canonical functor

[1] See Appendix B for more details on this and on what follows.

$\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{D}^b(A)$ is an equivalence and we can take the left derived functor $\mathbb{L}\nu$ of ν. If G denotes a quasi-inverse of this equivalence, $\mathbb{L}\nu$ is defined as the composition

$$\mathcal{D}^b(A) \xrightarrow{G} \mathcal{K}^b(\operatorname{proj} A) \xrightarrow{\nu} \mathcal{K}^b(\operatorname{inj} A) \longrightarrow \mathcal{D}^b(A).$$

The fact that A is of global dimension also implies that the last functor above is an equivalence, so we deduce that $\mathbb{L}\nu$ is an autoequivalence[1] of $\mathcal{D}^b(A)$. For $\mathbb{L}\nu$ to be a Serre functor, it remains to find the natural isomorphism in Definition 4.3.1.

Lemma 4.3.2. Let X and Y be bounded complexes of A-modules. If X_n is projective for all $n \in \mathbb{Z}$, then there is a natural isomorphism

$$D \operatorname{Hom}_{\mathcal{D}^b(A)}(X, Y) \cong \operatorname{Hom}_{\mathcal{D}^b(A)}(Y, \nu X).$$

Proof. Since X is a complex of projective modules and νX is a complex of injective modules, it is equivalent to find a natural isomorphism

$$D \operatorname{Hom}_{\mathcal{K}^b(A)}(X, Y) \cong \operatorname{Hom}_{\mathcal{K}^b(A)}(Y, \nu X)$$

by Lemma B.3.4. We will see how to do this in a particular case and then we explain the idea behind the general case.

Suppose X and Y are concentrated in degree zero. For simplicity, we also denote X_0 and Y_0 by X and Y, respectively. We have

$$D \operatorname{Hom}_{\mathcal{K}^b(A)}(X, Y) \cong D \operatorname{Hom}_A(X, Y) = \operatorname{Hom}_K(\operatorname{Hom}_A(X, Y), K).$$

Now, observe that there is a natural homomorphism

$$\begin{aligned} Y \otimes_A X^t &\longrightarrow \operatorname{Hom}_A(X, Y) \\ y \otimes f &\longmapsto (x \mapsto y f(x)), \end{aligned}$$

where we recall from Section 1.4 that X^t is the left A-module $\operatorname{Hom}_A(X, A_A)$. If X is the regular module A_A, it is not hard to see that this map is an isomorphism. Hence, it is also an isomorphism for a general projective module X because all operations used in the definition of the map above are linear on the variable X. With this isomorphism in hand, we obtain

$$\operatorname{Hom}_K(\operatorname{Hom}_A(X, Y), K) \cong \operatorname{Hom}_K(Y \otimes_A X^t, K) \cong \operatorname{Hom}_A(Y, \operatorname{Hom}_K(X^t, K))$$

where we used the “tensor-hom” adjunction. But $\operatorname{Hom}_K(X^t, K) = DX^t = \nu X$, so the last vector space above is

$$\operatorname{Hom}_A(Y, \nu X) \cong \operatorname{Hom}_{\mathcal{K}^b(A)}(Y, \nu X),$$

as desired.

For general X and Y, the proof is essentially the same, but we need to upgrade the dual, the tensor product and the functor Hom to complexes. The definitions can be found in [8, Section 2.7]. They satisfy the same properties that we used above (see Lemmas 4.3.13 and 6.4.4 of [26]), so we get a natural isomorphism

$$D \operatorname{Hom}_A(X, Y) \cong \operatorname{Hom}_A(Y, \nu X)$$

[1] A quasi-inverse is given by the right derived functor $\mathbb{R}\nu^{-1}$ of the inverse Nakayama functor ν^{-1}.

of complexes of K-vector spaces. By [26, Lemma 4.3.14], we arrive at

$$D\operatorname{Hom}_{\mathcal{K}^b(A)}(X,Y) \cong \operatorname{Hom}_{\mathcal{K}^b(A)}(Y,\nu X)$$

after taking homology in degree zero. □

Proposition 4.3.3. If A has finite global dimension, then $\mathbb{L}\nu : \mathcal{D}^b(A) \to \mathcal{D}^b(A)$ is a Serre functor.

Proof. We already saw that $\mathbb{L}\nu$ is an autoequivalence. If G denotes a quasi-inverse of the canonical functor $i : \mathcal{K}^b(\operatorname{proj} A) \to \mathcal{D}^b(A)$, then there is a natural isomorphism[1] $GX \to X$ in $\mathcal{D}^b(A)$. Hence, we have

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(X,Y) \cong \operatorname{Hom}_{\mathcal{D}^b(A)}(GX,Y).$$

Since GX is a bounded complex of projective modules, Lemma 4.3.2 says that the vector space above is naturally isomorphic to

$$D\operatorname{Hom}_{\mathcal{D}^b(A)}(Y,\nu(GX)) = D\operatorname{Hom}_{\mathcal{D}^b(A)}(Y,(\mathbb{L}\nu)(X)),$$

as needed. □

Remark. When A is not of finite global dimension, the functor $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{D}^b(A)$ is not an equivalence but it is still fully faithful. Thus, if its essential image $\operatorname{per}(A)$, the so-called **perfect derived category** of A, coincides with the essential image of $\mathcal{K}^b(\operatorname{inj} A) \to \mathcal{D}^b(A)$, we can repeat the argument above to show that $\mathbb{L}\nu$ is a Serre functor on $\operatorname{per}(A)$.

This is the case when A is a **Gorenstein algebra**, that is, the injective dimensions of the regular modules A_A and ${}_AA$ are finite. Indeed, since every projective module is a direct summand of a power of A_A, they all have finite injective dimension, hence every bounded complex of projective modules admits a bounded injective resolution. Analogously, if ${}_AA$ has finite injective dimension, then $D({}_AA)$ has finite projective dimension and so every bounded complex of injective modules admits a bounded projective resolution. Therefore, the copies of $\mathcal{K}^b(\operatorname{proj} A)$ and $\mathcal{K}^b(\operatorname{inj} A)$ inside $\mathcal{D}^b(A)$ coincide.

We refer the reader to [26, Chapter 6] for more details.

Let us specialize to the case when A is hereditary. We will see that the derived Nakayama functor $\mathbb{L}\nu$ has a particular description.

Let M be an indecomposable A-module and view it as a complex concentrated in degree zero. On objects, a quasi-inverse G of the functor $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{D}^b(A)$ computes a projective resolution. Thus, in order to find $(\mathbb{L}\nu)(M)$, choose a minimal projective resolution

$$0 \longrightarrow P_1 \xrightarrow{p} P_0 \longrightarrow M \longrightarrow 0.$$

It has length at most one because we are assuming that A is hereditary. It follows that $(\mathbb{L}\nu)(M)$ is the complex

$$\cdots \longrightarrow 0 \longrightarrow \nu P_1 \xrightarrow{\nu p} \nu P_0 \longrightarrow 0 \longrightarrow \cdots .$$

[1] Notice that GX is $(iG)(X)$ since i is the inclusion map on objects.

By Proposition 4.1.1, this complex is isomorphic to its homology. Using the exact sequence in Lemma 3.2.4, we can then write $(\mathbb{L}\nu)(M)$ as

$$\cdots \longrightarrow 0 \longrightarrow \tau M \xrightarrow{0} \nu M \longrightarrow 0 \longrightarrow \cdots.$$

If M is projective, then $\tau M = 0$ and we are left with the stalk complex νM concentrated in degree zero. On the other hand, if M is not projective, then $M^t = \mathrm{Hom}_A(M, A)$ is zero, because the image of a homomorphism $M \to A$ has to be projective by the hypothesis on A and so it is a direct summand of the indecomposable module M. In this case, $\nu M = DM^t = 0$. To sum it up, we have

$$(\mathbb{L}\nu)(M) = \begin{cases} \nu M & \text{if } M \text{ is projective,} \\ (\tau M)[1] & \text{otherwise.} \end{cases}$$

Since $\mathbb{L}\nu$ is additive and commutes with the shift, this allows us to compute $(\mathbb{L}\nu)(X)$ for any $X \in \mathcal{D}^b(A)$ by Proposition 4.1.1.

Define $\underline{\tau}$ as the composition $[-1] \circ \mathbb{L}\nu$. For an indecomposable A-module M, we have

$$\underline{\tau} M = \begin{cases} (\nu M)[-1] & \text{if } M \text{ is projective,} \\ \tau M & \text{otherwise.} \end{cases}$$

A quasi-inverse for $\underline{\tau}$ is $\underline{\tau}^{-1} := \mathbb{R}\nu^{-1} \circ [1]$, which satisfies

$$\underline{\tau}^{-1} M = \begin{cases} (\nu^{-1} M)[1] & \text{if } M \text{ is injective,} \\ \tau^{-1} M & \text{otherwise,} \end{cases}$$

for M an indecomposable A-module. These two functors are a sort of extension of the Auslander-Reiten translations. In fact, they are the missing piece for solving the mystery in the introduction of the chapter!

Suppose Q is a Dynkin quiver and identify $\mathrm{mod}\, KQ$ as a subcategory of $\mathcal{D}^b(KQ)$. When applying the knitting algorithm to find the AR-quiver of KQ, we may use $\underline{\tau}^{-1}$ instead of τ^{-1} to compute the meshes. In the original algorithm, we only have to apply τ^{-1} to noninjective indecomposable modules, so $\underline{\tau}^{-1}$ gives the same result. However, if we try to carry on with the algorithm, then we have to apply $\underline{\tau}^{-1}$ to the injective indecomposables and, by the description above, we arrive at a permutation of the projective indecomposables, but they are now shifted. This is why we found in the introduction the opposite of the initial dimension vectors.

Remark. Suppose A is of finite global dimension. Let us sketch how to generalize Auslander-Reiten theory to $\mathcal{D}^b(A)$. Instead of looking at almost split *sequences*, we work with almost split *triangles*. They are distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

where X and Z are indecomposable objects and f and g are, respectively, left and right almost split (note that the definition we gave for modules is easily generalized to this context). One can show that for every indecomposable object $Z \in \mathcal{D}^b(A)$, there is a unique (up to isomorphism) almost split triangle whose third term is Z. In this case, the first term is isomorphic to $\underline{\tau} Z$, where $\underline{\tau} := [-1] \circ \mathbb{L}\nu$ as above. Hence, $\underline{\tau}$ has a similar role in $\mathcal{D}^b(A)$ to the one τ has in $\mathrm{mod}\, A$. Moreover, we can define the AR-quiver of $\mathcal{D}^b(A)$ and, if A is hereditary, it can be obtained from the AR-quiver of

A by gluing infinitely many copies of $\Gamma(\operatorname{mod} A)$ one after the other, as we saw in the introduction of the chapter.

For more information on this subject, see [18, Chapter I].

We finish with a lemma that will help us in the next section.

Lemma 4.3.4. If A is hereditary, then the functor $\underline{\tau}$ induces a map on $K_0(\mathcal{D}^b(A))$ which coincides with the Coxeter transformation of A defined in Section 1.4.

Proof. Since $\mathbb{L}\nu$ is a triangulated functor, it sends distinguished triangles to distinguished triangles. Hence, it induces a well-defined map $K_0(\mathcal{D}^b(A)) \to K_0(\mathcal{D}^b(A))$. In particular, by Lemma 4.2.2, $\underline{\tau} = [-1] \circ \mathbb{L}\nu$ also induces a map on $K_0(\mathcal{D}^b(A))$, which we denote again by $\underline{\tau}$ for simplicity.

Let us check that, after identifying $K_0(\mathcal{D}^b(A))$ with $K_0(A)$ by means of the isomorphism in Proposition 4.2.3, $\underline{\tau}$ coincides with the Coxeter transformation $c_A : K_0(A) \to K_0(A)$. Indeed, if P is an indecomposable projective A-module, then

$$\underline{\tau}([P]) = [\underline{\tau}P] = [(\nu P)[-1]] = -[\nu P] = c_A([P]),$$

where the last equality was discussed at the end of Section 1.4. But Lemma 1.4.5 says that the classes of the indecomposable projective modules generate $K_0(A)$, so $\underline{\tau}$ and c_A must agree on the whole Grothendieck group. □

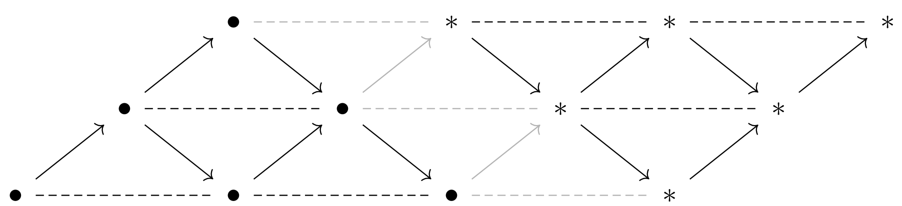
As a consequence, if A is the path algebra of a Dynkin quiver Q, then we know that $\underline{\tau}$ induces an automorphism of finite order on $K_0(\mathcal{D}^b(A))$. Its order is precisely the Coxeter number associated to Q (see Corollary A.3.4).

4.4 The main theorem

There is another noteworthy symmetry in the examples given in the introduction. The purpose of this final section is to formalize it and prove it.

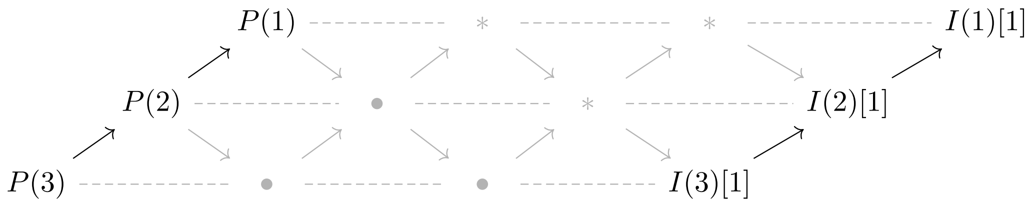
Let us illustrate what is happening in the case $Q = \mathbb{A}_3$, with the usual orientation. In Example 3.4.1, we saw that the AR-quiver of KQ has a triangular shape. In the way we pictured it, the τ-orbits in $\operatorname{mod} KQ$ correspond to the rows of $\Gamma(\operatorname{mod} KQ)$. Thus, there are three τ-orbits, but they are not of the same size.

Now, if we continue with the knitting algorithm using the functor $\underline{\tau}^{-1}$, we saw in the introduction that we get copies of $\Gamma(\operatorname{mod} A)$. Immediately after we draw the first copy, we have the following picture:



The triangle whose vertices are the symbol $\bullet$ is the original copy of $\Gamma(\operatorname{mod} KQ)$ and, in $\mathcal{D}^b(KQ)$, these vertices correspond to indecomposable stalk complexes concentrated in degree zero. On the other hand, each indecomposable object represented by $*$ is obtained by applying the shift [1] to a suitable vertex of $\Gamma(\operatorname{mod} KQ)$. This time, note that each row has the same number of vertices.

Explicitly, the vertical sides of the parallelogram above represent the following complexes:



Hence, if we apply $\underline{\tau}^{-1}$ to the right side, we get the left side shifted twice. In other words, we get that $\underline{\tau}^{-4}P \cong P[2]$ for every indecomposable projective module P. By Corollary 3.5.5, every indecomposable KQ-module is of the form $\tau^{-t}P$ for some $t \geq 0$ and some indecomposable projective P, so we also have $\underline{\tau}^{-4}M \cong M[2]$ for any indecomposable module M. Finally, since $\underline{\tau}$ is additive and commutes with shifts, it follows from Proposition 4.1.1 that $\underline{\tau}^{-4}$ and $[2]$ agree on all objects of $\mathcal{D}^b(KQ)$. We will prove that $\underline{\tau}^{-4}$ and $[2]$ are in fact isomorphic functors.

If we take $Q = \mathbb{D}_4$ with the orientation given in Example 3.4.3, one can check in the same way that $\underline{\tau}^{-6}X \cong X[2]$ for every $X \in \mathcal{D}^b(KQ)$. The number 6 appearing here is the Coxeter number associated to $\mathbb{D}_4$, just as the number 4 from the previous paragraph is the Coxeter number associated to $\mathbb{A}_3$ (see the table in Section A.2 and Proposition A.2.5). Actually, one can verify a stronger result for $\mathbb{D}_4$: we have $\underline{\tau}^{-3}X \cong X[1]$ for all $X \in \mathcal{D}^b(KQ)$. As we will see, this refinement comes from Proposition A.2.6.

These observations hold for any Dynkin quiver Q. They describe not only a special property of $\mathcal{D}^b(KQ)$ but also a symmetry of the AR-quiver of KQ, as we exemplified above. In order to prove them, let us formalize what we have discussed. We start with our main definition.

Definition 4.4.1. Let A be a finite-dimensional K-algebra of finite global dimension. We say that A is **fractionally Calabi-Yau** if there are integers m and $\ell \geq 1$ such that the functors $(\mathbb{L}\nu)^\ell$ and $[m]$ on $\mathcal{D}^b(A)$ are isomorphic. In this case, note that the rational number m/ℓ is uniquely determined by A. The **Calabi-Yau dimension** of A, which we denote by CY-dim A, is the pair (m, ℓ) satisfying the previous conditions with ℓ minimal.

Remark. We are using $\mathbb{L}\nu$ instead of $\underline{\tau}^{-1}$ because the former is the Serre functor of $\mathcal{D}^b(A)$. The definition above can be generalized to a general triangulated K-category admitting a Serre functor, and it was in this context that this notion was introduced. See [23] for more references on the subject and for different examples of Calabi-Yau triangulated categories appearing in representation theory.

With this nomenclature, we can state the main theorem of this thesis.

Theorem 4.4.2. Let Q be a Dynkin quiver and h the corresponding Coxeter number. Then KQ is a fractionally Calabi-Yau algebra and

$$\text{CY-dim}\, KQ = \begin{cases} (\frac{h}{2} - 1, \frac{h}{2}) & \text{if } Q \text{ is of type } \mathbb{A}_1,\ \mathbb{D}_n \text{ with } n \text{ even},\ \mathbb{E}_7 \text{ or } \mathbb{E}_8, \\ (h-2, h) & \text{otherwise.} \end{cases}$$

Most of the rest of the section is devoted to prove this result, so let us fix some conventions. We take $A = KQ$ and suppose Q is always a Dynkin quiver with n vertices (labelled as $1, \ldots, n$) and Coxeter number h. We also define (m, ℓ) to be either $(\frac{h}{2}-1, \frac{h}{2})$ or $(h-2, h)$ depending on the type of Q, as in the theorem. Note that $\ell - m$ is either 1 or 2.

Lemma 4.4.3. In $\mathcal{D}^b(A)$, the number of isomorphism classes of indecomposable stalk complexes concentrated in some fixed degree is $nh/2$.

Proof. This is the same as the number of indecomposable A-modules, so the lemma follows from Theorem 2.3.4 and Proposition A.2.5. □

Lemma 4.4.4. If $X \in \mathcal{D}^b(A)$ is an indecomposable stalk complex concentrated in degree $t \geq 0$, then there is $s \geq 0$ and an indecomposable projective module P such that $X \cong \underline{\tau}^{-s}P$.

Proof. By hypothesis, $X \cong M[t]$ for some indecomposable A-module M, which can be viewed as a complex concentrated in degree zero. In the proof of Corollary 3.5.5, we saw that $\Gamma(\operatorname{mod} A)$ is postprojective, so there exists an indecomposable projective module P and some $s \geq 0$ such that $M \cong \tau^{-s}P = \underline{\tau}^{-s}P$. If $t = 0$, we are done. Otherwise, we have

$$\underline{\tau}^{s+1}X \cong (\underline{\tau}P)[t] \cong (\nu P)[t-1].$$

This new indecomposable complex is concentrated in degree $t-1 \geq 0$, so the result follows by recursion. □

Proposition 4.4.5. For every $X \in \mathcal{D}^b(A)$, we have $\underline{\tau}^{-\ell}X \cong X[\ell - m]$.

Proof. Since $\underline{\tau}^{-1}$ is additive and commutes with the shift, we may assume that X is an indecomposable stalk complex concentrated in degree zero by Proposition 4.1.1. For simplicity, we identify X with the indecomposable module X_0.

By Lemma 4.3.4, $\underline{\tau}$ induces the Coxeter transformation of A on $K_0(\mathcal{D}^b(A))$. In particular, by the definition of Coxeter number and by Corollary A.3.4, $\underline{\tau}^{-h}$ is the identity on the Grothendieck group. By Lemma 4.2.4, we have $\underline{\tau}^{-h}X \cong X[2a]$ for some $a \in \mathbb{Z}$. Our goal is to prove that $a = 1$.

By the description of $\underline{\tau}^{-1}$ given in the previous section, $\underline{\tau}^{-h}X$ has to be concentrated in a nonnegative degree, so $a \geq 0$. If $a = 0$, then we have $\tau^{-h}X \cong X$ and the τ-orbit of X is periodic, but this contradicts Corollary 3.5.5. Hence, $a \geq 1$.

Now, let $P(1), \ldots, P(n)$ denote the indecomposable projective modules. For every $1 \leq i \leq n$, the argument above produces an integer $a_i \geq 1$ such that $\underline{\tau}^{-h}P(i) \cong P(i)[2a_i]$. Define S_i to be the set whose elements are the isomorphism classes of the complexes

$$P(i),\ \underline{\tau}^{-1}P(i),\ \underline{\tau}^{-2}P(i),\ \ldots\ ,\ \underline{\tau}^{-(h-1)}P(i),$$

for every $1 \leq i \leq n$. These sets have to be disjoint because, otherwise, there would be distinct i and j such that $P(i)$ and $P(j)$ are in the same τ-orbit, contradicting Corollary 3.5.5. In particular, the cardinality of the union of the sets $S_1, \ldots, S_n$ is exactly nh. On the other hand, by Lemma 4.4.4, every indecomposable stalk complex concentrated in degree $t \geq 0$ is of the form $\underline{\tau}^{-s}P(i)$ for some $s \geq 0$ and some i. Since $\underline{\tau}^{-h}P(i)$ is concentrated in degree $2a_i \geq 2$ and since applying $\underline{\tau}^{-1}$ can only increase this degree, we must have $0 \leq s < h$ if $t = 0$ or $t = 1$. This proves that the isomorphism class of any indecomposable stalk complex concentrated in degree zero or one is inside some S_i. By Lemma 4.4.3, the number of such isomorphism classes is also nh, so we conclude that the union of the sets $S_1, \ldots, S_n$ is precisely the set these isomorphism classes form. Consequently, for each $1 \leq i \leq n$, we deduce that $\underline{\tau}^{-(h-1)}P(i)$ is concentrated in degree zero or one, hence $\underline{\tau}^{-h}P(i) \cong P(i)[2a_i]$ is concentrated in degree one or two and we must have $a_i = 1$.

Lastly, coming back to X, Lemma 4.4.4 gives $s \geq 0$ and $1 \leq i \leq n$ such that $X \cong \underline{\tau}^{-s}P(i)$. Therefore, we have

$$\underline{\tau}^{-h}X \cong \underline{\tau}^{-(s+h)}P(i) \cong (\underline{\tau}^{-s}P(i))[2] \cong X[2],$$

as desired. This proves the proposition for $(m,\ell) = (h-2,h)$.

When $(m,\ell) = (\frac{h}{2}-1, \frac{h}{2})$, we are in the case where Proposition A.2.6 holds. Thus, $\underline{\tau}^{-h/2}$ acts as the opposite of the identity on $K_0(\mathcal{D}^b(A))$ and Lemma 4.2.4 yields $\underline{\tau}^{-h/2}X \cong X[2b+1]$ for some $b \in \mathbb{Z}$. Since applying $\underline{\tau}^{-1}$ can only increase the degree where X is concentrated, we must have $b = 0$ by what we showed before, finishing the proof. □

Corollary 4.4.6. For every $X \in \mathcal{D}^b(A)$, we have $(\mathbb{L}\nu)^\ell(X) \cong X[m]$. Moreover, ℓ is the smallest positive integer with this property.

Proof. Since $\underline{\tau} = [-1] \circ \mathbb{L}\nu$, Proposition 4.4.5 implies

$$(\mathbb{L}\nu)^\ell(X) = ([1] \circ \underline{\tau})^\ell(X) \cong (\underline{\tau}^\ell X)[\ell] \cong (X[m-\ell])[\ell] = X[m],$$

where we used that $\underline{\tau}$ commutes with the shift.

For the second part, suppose there are $a, b \in \mathbb{Z}$ with $b \geq 1$ such that $(\mathbb{L}\nu)^b(X) \cong X[a]$ for all $X \in \mathcal{D}^b(A)$. With a similar calculation as above, we get that $\underline{\tau}^b$ and $[a-b]$ coincide on objects. In particular, $\underline{\tau}^b$ acts on $K_0(\mathcal{D}^b(A))$ as $(-1)^{a-b}$ times the identity. Since $\underline{\tau}$ acts as the Coxeter transformation, which has order h, it follows[1] that ℓ divides b, proving the minimality condition. □

The only thing left to be proven is that the isomorphism from the corollary above can be taken to be natural on the variable X. Our proof does not suggest how to do this, so let us see how this could be achieved.

Recall from Proposition 1.4.3 that the Nakayama functor on $\operatorname{mod} A$ is isomorphic to the functor $-\otimes_A DA$, where we see DA as a bimodule over A. This implies that $\mathbb{L}\nu$ can be seen as the functor[2] $-\otimes^{\mathbb{L}}_A DA$. From the associativity of the derived tensor product, we have

$$(\mathbb{L}\nu)^\ell \cong - \otimes^{\mathbb{L}}_A (DA)^{\otimes^{\mathbb{L}}_A \ell}.$$

If the complex of bimodules $(DA)^{\otimes^{\mathbb{L}}_A \ell}$ is quasi-isomorphic to the stalk complex $A[m]$, where we see A as the regular bimodule, then $(\mathbb{L}\nu)^\ell$ is naturally isomorphic to

$$- \otimes^{\mathbb{L}}_A A[m] = - \otimes_A A[m] \cong [m],$$

where we used in the first equality that $A[m]$ is, in particular, a bounded complex of projective left A-modules. Indeed, by Corollary 4.4.6, we have

$$(DA)^{\otimes^{\mathbb{L}}_A \ell} \cong A \otimes^{\mathbb{L}}_A (DA)^{\otimes^{\mathbb{L}}_A \ell} \cong (\mathbb{L}\nu)^\ell(A) \cong A[m].$$

However, this is in principle an isomorphism in $\mathcal{D}^b(A)$ and not an isomorphism in the bounded derived category of A-bimodules! In order to finish the proof of Theorem 4.4.2, we will fix this subtle detail.

Let us introduce some new definitions. Denote by A^e the **enveloping algebra** of A, that is, $A^e := A^{\mathrm{op}} \otimes_K A$. Right modules over A^e are equivalent to bimodules over A, so we need to work with the category $\mathcal{D}^b(A^e)$. Denote by $\operatorname{End}(A)$ the space of

[1] We have to apply Proposition A.2.6 if $(-1)^{a-b} = -1$.

[2] See Section B.4 for more details on derived tensor products.

K-algebra endomorphisms of A. For $\varphi \in \operatorname{End}(A)$, define ${}_{\varphi}A_1$ to be the A-bimodule whose underlying K-vector space is A with the action

$$a \cdot x \cdot b \coloneqq \varphi(a)xb$$

for $a, x, b \in A$, where juxtaposition denotes the multiplication in A. Lastly, recall that A_A denotes the regular right A-module.

Lemma 4.4.7. Let M be an A-bimodule. If $M \cong A_A$ as right A-modules, then $M \cong {}_{\varphi}A_1$ as bimodules for some $\varphi \in \operatorname{End}(A)$.

Proof. Let $f : M \to A_A$ be an isomorphism of right A-modules. For $a \in A$, the map

$$\begin{aligned} g_a : M &\longrightarrow M \\ m &\longmapsto a \cdot m \end{aligned}$$

is a homomorphism of right A-modules. Thus, $f g_a f^{-1} : A_A \to A_A$ is also a homomorphism of right modules and is given by left multiplication by the element

$$\varphi(a) \coloneqq f g_a f^{-1}(1) = f(a \cdot f^{-1}(1)).$$

This defines a map $\varphi : A \to A$ which is easily checked to be a homomorphism of algebras. In this way, the map f satisfies

$$f(a \cdot m) = f(a \cdot f^{-1}(f(m))) = f(a \cdot f^{-1}(1) \cdot f(m)) = f(a \cdot f^{-1}(1))f(m) = \varphi(a)f(m)$$

for $a \in A$ and $m \in M$. Hence, f is a bimodule isomorphism from M to ${}_{\varphi}A_1$. □

Lemma 4.4.8. Let X be a bounded complex of A-bimodules. If X is isomorphic in $\mathcal{D}^b(A)$ to the complex A_A concentrated in degree zero, then $X \cong {}_{\varphi}A_1$ in $\mathcal{D}^b(A^e)$ for some $\varphi \in \operatorname{End}(A)$.

Proof. Remember that a complex whose homology is concentrated in degree zero is isomorphic in $\mathcal{D}^b(A)$ to its homology (see Proposition B.3.1). Therefore,

$$X \cong A_A \text{ in } \mathcal{D}^b(A) \iff H_0(X) \cong A_A \text{ and } H_n(X) = 0 \text{ for } n \neq 0$$

and, for $\varphi \in \operatorname{End}(A)$,

$$X \cong {}_{\varphi}A_1 \text{ in } \mathcal{D}^b(A^e) \iff H_0(X) \cong {}_{\varphi}A_1 \text{ and } H_n(X) = 0 \text{ for } n \neq 0.$$

The result then follows from Lemma 4.4.7. □

Returning to the context of Theorem 4.4.2, we saw that $(DA)^{\otimes^{\mathbb{L}}_A \ell} \cong A[m]$ in $\mathcal{D}^b(A)$. By the previous lemma, there is $\phi \in \operatorname{End}(A)$ such that $(DA)^{\otimes^{\mathbb{L}}_A \ell} \cong ({}_{\phi}A_1)[m]$ in $\mathcal{D}^b(A^e)$. We have to prove that ϕ can be taken to be the identity.

Lemma 4.4.9. The endomorphism ϕ is actually an automorphism.

Proof. Since $(DA)^{\otimes^{\mathbb{L}}_A \ell} \cong ({}_{\phi}A_1)[m]$ in $\mathcal{D}^b(A^e)$, the functor $(\mathbb{L}\nu)^{\ell}$ is the composition of the functor $F = - \otimes^{\mathbb{L}}_A ({}_{\phi}A_1)$ with the shift $[m]$. Consequently, we get that F is an equivalence on $\mathcal{D}^b(A)$. Let us see how this implies that ϕ is injective (and hence an automorphism).

Let $x \in \ker \phi$. Multiplication on the left by x induces a homomorphism of right modules $f : A_A \to A_A$. Viewing these modules as complexes concentrated in degree zero, we have a map in $\mathcal{D}^b(A)$ and we may apply the functor F. Because A_A is a complex of projective modules, we have

$$F(A_A) = A_A \otimes_A^{\mathbb{L}} ({}_\phi A_1) = A_A \otimes_A ({}_\phi A_1)$$

and we can work with the usual tensor product. In this case,

$$F(f)(a \otimes b) = f(a) \otimes b = xa \otimes b = 1 \otimes (xa) \cdot b = 1 \otimes \phi(xa)b = 0$$

for every pure tensor $a \otimes b \in A_A \otimes_A ({}_\phi A_1)$ since $x \in \ker \phi$. Thus, $F(f) = 0$. But F is an equivalence, so f must be zero and $x = f(1) = 0$, as desired. □

If $x \in A$ is invertible, denote by $\gamma_x : A \to A$ the **inner automorphism** determined by x, that is, $\gamma_x(a) = xax^{-1}$ for all $a \in A$.

Lemma 4.4.10. Let $\varphi, \psi \in \mathrm{End}(A)$. Then ${}_\varphi A_1 \cong {}_\psi A_1$ as bimodules if and only if $\psi = \gamma_x \varphi$ for some invertible element $x \in A$.

Proof. Suppose there is an isomorphism $f : {}_\varphi A_1 \to {}_\psi A_1$. In particular, it is an isomorphism of right modules and so it is given by left multiplication by the invertible element $x := f(1)$. Hence, for $a \in A$, we have

$$x\varphi(a) = f(\varphi(a)) = f(a \cdot 1) = a \cdot f(1) = \psi(a)x \implies \psi(a) = x\varphi(a)x^{-1}.$$

This proves that $\psi = \gamma_x \varphi$.

Conversely, if $\psi = \gamma_x \varphi$ for some invertible element $x \in A$, define $f : {}_\varphi A_1 \to {}_\psi A_1$ as left multiplication by x. This is an isomorphism of right A-modules and, by reversing the calculation above, we can verify that it also preserves the left-module structure. □

Lemma 4.4.11. We can assume that ϕ fixes every stationary path in A.

Proof. Since ϕ is an automorphism of the path algebra A, it sends the set of stationary paths $\{\varepsilon_1, \dots, \varepsilon_n\}$ to another complete set of primitive orthogonal idempotents. By [14, Theorem 3.4.1], these two sets are conjugated by some invertible element in A. Thus, in view of Lemma 4.4.10, we may assume there is a permutation σ of $\{1, \dots, n\}$ such that $\phi(\varepsilon_i) = \varepsilon_{\sigma(i)}$ for all $1 \le i \le n$.

Now, Corollary 4.4.6 and the previous discussion yield an isomorphism

$$\varepsilon_i A[m] \cong (\mathbb{L}\nu)^\ell(\varepsilon_i A) \cong \varepsilon_i A \otimes_A^{\mathbb{L}} ({}_\phi A_1)[m]$$

in $\mathcal{D}^b(A)$. But $\varepsilon_i A$ is a projective module and so we can work with the ordinary tensor product, obtaining

$$\varepsilon_i A[m] \cong (\varepsilon_i A \otimes_A ({}_\phi A_1))[m] \cong (\varepsilon_i \cdot ({}_\phi A_1))[m] = \phi(\varepsilon_i)A[m] = \varepsilon_{\sigma(i)}A[m].$$

This proves that the indecomposable projective A-modules $\varepsilon_i A$ and $\varepsilon_{\sigma(i)}A$ are isomorphic for every $1 \le i \le n$. Since A is basic, σ must be the identity permutation. □

Lemma 4.4.12. We can take ϕ to be the identity.

Proof. By Lemma 4.4.10, we just have to prove that ϕ is an inner automorphism. We can suppose $\phi(\varepsilon_i) = \varepsilon_i$ for $1 \le i \le n$ by Lemma 4.4.11. As a result, for an arrow $\alpha : i \to j$ in Q, we have

$$\phi(\alpha) = \phi(\varepsilon_i \alpha \varepsilon_j) = \phi(\varepsilon_i)\phi(\alpha)\phi(\varepsilon_j) = \varepsilon_i \phi(\alpha)\varepsilon_j.$$

This means that $\phi(\alpha)$ is a linear combination of paths starting at i and ending at j. But Q is a Dynkin quiver and, in particular, a tree, so there is only one such path: the arrow α itself. Hence, for every arrow $\alpha \in Q_1$, there is $\lambda_\alpha \in K$ such that $\phi(\alpha) = \lambda_\alpha \alpha$. Note that each λ_α is nonzero because ϕ is an automorphism.

For nonzero scalars $\mu_1, \dots, \mu_n \in K$, consider the element

$$x := \mu_1 \varepsilon_1 + \mu_2 \varepsilon_2 + \cdots + \mu_n \varepsilon_n \in A.$$

It is invertible with inverse

$$x^{-1} = \frac{1}{\mu_1}\varepsilon_1 + \frac{1}{\mu_2}\varepsilon_2 + \cdots + \frac{1}{\mu_n}\varepsilon_n.$$

We can then consider the inner automorphism γ_x. It is easy to see that $\gamma_x(\varepsilon_i) = \varepsilon_i$ for all $1 \leq i \leq n$ and that

$$\gamma_x(\alpha) = x\alpha x^{-1} = \frac{\mu_i}{\mu_j}\alpha$$

for every arrow $\alpha : i \to j$ in Q. Because the stationary paths and the arrows of Q generate the path algebra A, if we find the correct values of $\mu_1, \dots, \mu_n$ so that

$$\lambda_\alpha = \frac{\mu_i}{\mu_j}$$

for every arrow $\alpha : i \to j$, then $\phi = \gamma_x$. With these conditions, observe that choosing μ_i determines the value of μ_j, and vice versa. Since Q is connected, this implies that μ_1 should determine the value of $\mu_2, \dots, \mu_n$. Finally, since Q is a tree, there is only one path from any vertex to the vertex 1, thus, if we set $\mu_1 = 1$, there is indeed a unique and well-defined choice for the values of $\mu_2, \dots, \mu_n$ so that the conditions above are satisfied. Therefore, ϕ is an inner automorphism of A, as claimed. □

This concludes the proof of Theorem 4.4.2!

Remark. Returning to the general case, suppose A is an algebra of finite global dimension. As we saw above, in order to show that A is fractionally Calabi-Yau, it is not enough to find integers m and ℓ such that $(\mathbb{L}\nu)^\ell$ and $[m]$ agree on all objects of $\mathcal{D}^b(A)$. Nonetheless, this already guarantees the existence of an automorphism $\phi : A \to A$ and an isomorphism of functors

$$(\mathbb{L}\nu)^\ell \cong [m] \circ \phi^*,$$

where $\phi^* = - \otimes^{\mathbb{L}}_A ({}_\phi A_1)$. In this case, we say that A is *twisted* fractionally Calabi-Yau. We remark that the map ϕ is not any automorphism of A. For example, if A is the path algebra of a Dynkin quiver, we proved that ϕ has to be inner. More generally, it is an open question whether ϕ has to be an element of finite order in the outer automorphism group of A. This would imply that twisted fractionally Calabi-Yau algebras are actually fractionally Calabi-Yau. We refer the reader to [19], where this "twisted" definition was introduced, and to [12], where the authors expand on this problem.

We finish the chapter with a result relating some of the concepts we explored throughout the thesis.

Theorem 4.4.13. Let Q be a connected and acyclic quiver. The following assertions are equivalent:

(1) Q is a Dynkin quiver.

(2) KQ is of finite representation type.

(3) KQ is fractionally Calabi-Yau.

(4) The Coxeter transformation of KQ is an automorphism of finite order.

Proof. Let us start with the easy implications. The equivalence (1) $\iff$ (2) is Gabriel's theorem. The implication (1) $\implies$ (3) is Theorem 4.4.2. By a similar calculation as the one in the proof of Corollary 4.4.6, KQ is a fractionally Calabi-Yau if and only if some power of the functor $\underline{\tau}$ is isomorphic to a shift. But $\underline{\tau}$ acts on $K_0(\mathcal{D}^b(KQ))$ as the Coxeter transformation of KQ (Lemma 4.3.4), so (3) implies (4).

We just need to show that (4) implies (2). Suppose KQ is not of finite representation type. By Auslander's theorem, every connected component of $\Gamma(\operatorname{mod} KQ)$ has modules of arbitrarily high composition length, including its postprojective component (which exists by Proposition 3.5.3). This implies that the τ-orbit of some indecomposable projective module P is infinite and, moreover, that the set

$$\{[\tau^{-t}P] \in K_0(KQ) \mid t \geq 0\}$$

is also infinite. Hence, in $\mathcal{D}^b(KQ)$, $\underline{\tau}^{-t}P$ is always the complex $\tau^{-t}P$ concentrated in degree zero and its class in the Grothendieck group assumes infinitely many values as we vary t. By Lemma 4.3.4, the Coxeter transformation of KQ is not of finite order. □

Appendix A

Dynkin diagrams and root systems

This first appendix collects the results on Dynkin diagrams and root systems that are needed in Chapters 2 and 4. It is based on [3, Sections VII.2 to VII.4], [8, Sections 4.5 and 4.6], [13], [20, Chapter III] and [21].

A.1 A classification theorem

Let G be a finite connected graph whose set of vertices is $\{1, \ldots, n\}$. We allow loops and multiple edges. Denote by n_{ij} the number of edges between two vertices $i, j \in \{1, \ldots, n\}$. The **Tits form** $q_G : \mathbb{Z}^n \to \mathbb{Z}$ of G is the quadratic form given by

$$q_G(v) = \sum_{i=1}^{n} v_i^2 - \sum_{1 \leq i \leq j \leq n} n_{ij} v_i v_j$$

for $v \in \mathbb{Z}^n$. This agrees with the definition given in Section 2.1. There is also a symmetric bilinear form $(-,-)_G : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ defined by

$$(v, w)_G = q_G(v + w) - q_G(v) - q_G(w)$$

for $v, w \in \mathbb{Z}^n$. Notice that $q_G(v) = \frac{1}{2}(v, v)_G$ for $v \in \mathbb{Z}^n$, and, if $e_1, \ldots, e_n$ denote the vectors in the canonical basis of $\mathbb{Z}^n$, we have

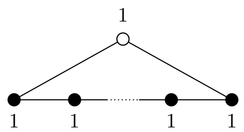
$$(e_i, e_j)_G = \begin{cases} -n_{ij} & \text{if } i \neq j, \\ 2 - 2n_{ii} & \text{if } i = j. \end{cases}$$

The goal of this section is to prove Theorem 2.1.4, which can be rephrased as follows:

Theorem A.1.1. With the notation above, q_G is positive definite if, and only if, G is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$.

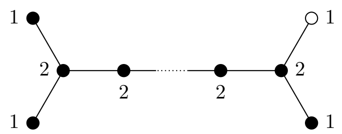
The idea of the proof is to study the more general case when q_G is **positive semi-definite**, that is, $q_G(v) \geq 0$ for all $v \in \mathbb{Z}^n$. Some *extended* Dynkin diagrams, also called **Euclidean diagrams**, appear. They are the following:

$\widetilde{\mathbb{A}}_m$:

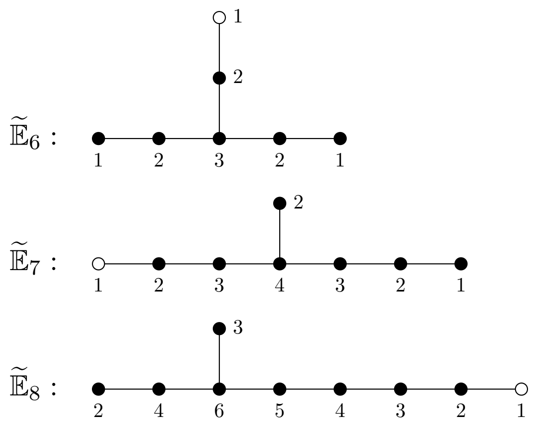


$(m \geq 0)$

$\widetilde{\mathbb{D}}_m$:



$(m \geq 4)$



Each diagram is obtained from the corresponding Dynkin diagram by adjoining the unfilled vertex in the picture. The only exception is $\widetilde{\mathbb{A}}_0$, which consists of a single vertex and a loop. Note that $\widetilde{\mathbb{A}}_1$ has two vertices and two edges joining them. Moreover, the number of vertices in each diagram is one more than the number in the subscript.

The labels in the vertices of the diagrams above are the coordinates of some special vectors in $\mathbb{Z}^n$. If G is one of these diagrams, enumerate its vertices and let $\delta \in \mathbb{Z}^n$ be the vector whose i-th coordinate is the number in the picture associated to the vertex i, for $1 \leq i \leq n$. One can check that δ is a **radical vector**, that is, it is an element of the **radical**:

$$\mathrm{rad}(G) := \{v \in \mathbb{Z}^n \mid (v, w)_G = 0, \forall w \in \mathbb{Z}^n\}.$$

To see this, it suffices to verify that $(\delta, e_i)_G = 0$ for all $1 \leq i \leq n$. One can readily check this for $G = \widetilde{\mathbb{A}}_0$ or $\widetilde{\mathbb{A}}_1$. In the other cases, there are no loops or multiple edges, so $(\delta, e_i)_G = 0$ becomes equivalent to

$$2\delta_i = \sum_{j \in V(i)} \delta_j,$$

where $V(i)$ denotes the set of neighbour vertices of i. This can be easily verified for each diagram.

Lemma A.1.2. Let G be a connected graph. If there is a nonzero $v \in \mathrm{rad}(G)$ with nonnegative coordinates, then q_G is positive semi-definite. Furthermore, all coordinates of v are nonzero and, for $w \in \mathbb{Z}^n$, we have

$$w \in \mathrm{rad}(G) \iff q_G(w) = 0 \iff w = \lambda v \text{ for some } \lambda \in \mathbb{Q}.$$

Proof. For all $1 \leq i \leq n$, we have the following equality:

$$0 = (v, e_i)_G = \sum_{j=1}^{n} v_j (e_j, e_i)_G = (2 - 2n_{ii}) v_i - \sum_{j \neq i} n_{ij} v_j. \qquad (*)$$

If $v_i = 0$ for some i, we would have $\sum_{j \neq i} n_{ij} v_j = 0$ and, since the coordinates of v are nonnegative, we would get $v_j = 0$ for every neighbour j of i. However, G is connected and so this would imply $v = 0$, a contradiction. Thus, the coordinates of v are nonzero.

For $w \in \mathbb{Z}^n$, observe that

$$\begin{aligned} q_G(w) &= \sum_{i=1}^{n}(1-n_{ii})w_i^2 - \sum_{i<j} n_{ij}w_iw_j \\ &= \sum_{i=1}^{n}(2-2n_{ii})v_i\frac{w_i^2}{2v_i} - \sum_{i<j} n_{ij}w_iw_j \\ &= \sum_{i\neq j} n_{ij}v_j\frac{w_i^2}{2v_i} - \sum_{i<j} n_{ij}w_iw_j \\ &= \sum_{i<j} n_{ij}\frac{v_iv_j}{2}\left(\frac{w_i^2}{v_i^2} - 2\cdot\frac{w_iw_j}{v_iv_j} + \frac{w_j^2}{v_j^2}\right) \\ &= \sum_{i<j} n_{ij}\frac{v_iv_j}{2}\left(\frac{w_i}{v_i} - \frac{w_j}{v_j}\right)^2, \end{aligned}$$

where we used ($*$) in the third equality. Since the coordinates of v are positive, this proves that q_G is positive semi-definite. Moreover, $q_G(w) = 0$ if and only if $w_i/v_i = w_j/v_j$ whenever i is a neighbour of j. Since G is connected, this equality holds for every vertices i and j, so $w = \lambda v$ where $\lambda \in \mathbb{Q}$ is this common fraction. Finally, any multiple of v is in $\mathrm{rad}(G)$ and, if $w \in \mathrm{rad}(G)$, then $q_G(w) = \frac{1}{2}(w,w)_G = 0$ by definition, concluding the proof. □

Corollary A.1.3. If G is an Euclidean diagram of type $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{E}}$, then q_G is positive semi-definite. If G is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, then q_G is positive definite.

Proof. In the first case, we can construct the radical vector δ by using the labels in the description of the Euclidean diagrams. Lemma A.1.2 implies that q_G is positive semi-definite and that every radical vector is an integer multiple of δ.

Now, suppose G is Dynkin of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, and let $\widetilde{G}$ be the associated Euclidean diagram. If $v \in \mathbb{Z}^n$, denote by $\widetilde{v} \in \mathbb{Z}^{n+1}$ the vector obtained by adjoining an extra coordinate of value zero. It is easy to see that $q_G(v) = q_{\widetilde{G}}(\widetilde{v})$. If $v \neq 0$, then $\widetilde{v}$ is not a multiple of δ and so $q_G(v) = q_{\widetilde{G}}(\widetilde{v}) > 0$. Therefore, q_G is positive definite. □

This proves one of the implications of Theorem A.1.1. We essentially studied Euclidean diagrams and used that the Dynkin diagrams are exactly their connected and proper subgraphs. We will now explore the “dual” fact that Euclidean diagrams appear inside every graph that is not Dynkin.

Lemma A.1.4. Let G be a connected graph. Either G is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, or there is an Euclidean diagram of type $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{E}}$ which is a subgraph of G. Both possibilities cannot occur simultaneously.

Proof. Suppose that G does not contain a copy of any of the Euclidean diagrams and let us show that it is one of the Dynkin diagrams.

Since G does not contain $\widetilde{\mathbb{A}}_m$ for all $m \geq 0$, G does not have loops, multiple edges or cycles. Therefore, G is a tree. Moreover, G has at most one branch point (i.e., a vertex with at least three edges), otherwise G would contain $\widetilde{\mathbb{D}}_m$ for some $m \geq 5$. If it does not have, then $G = \mathbb{A}_n$ for some $n \geq 1$. If it does, this branch point has degree three, otherwise G would contain $\widetilde{\mathbb{D}}_4$. Let $a, b, c \geq 1$ denote the length of each branch and suppose $a \leq b \leq c$. Since G does not contain $\widetilde{\mathbb{E}}_6$, we must have $a = 1$. If $b = 1$, then $G = \mathbb{D}_n$ for some $n \geq 4$, so we may suppose $b \geq 2$. If $b \geq 3$, then G would contain $\widetilde{\mathbb{E}}_7$, thus $b = 2$. We cannot have $c \geq 5$, otherwise G would contain $\widetilde{\mathbb{E}}_8$. Hence, $c = 2$, 3 or 4 and $G = \mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, respectively.

The last statement of the lemma is clear. □

Lemma A.1.5. Let G and G' be graphs with n and m vertices, respectively. Suppose that G is a subgraph of G'. If there exists $v \in \mathbb{Z}^n$ with positive coordinates such that $q_G(v) = 0$, then there is a nonzero $v' \in \mathbb{Z}^m$ with $q_{G'}(v') \leq 0$.

Proof. Denote by n_{ij} the number of edges in G between the vertices $i, j \in \{1, \dots, n\}$. Let m_{ij} denote the analogous number for G', where $i, j \in \{1, \dots, m\}$. Note that $n_{ij} \leq m_{ij}$ if $1 \leq i, j \leq n$. Let $v' \in \mathbb{Z}^m$ be the vector with $v'_i = v_i$ for $1 \leq i \leq n$, and $v'_i = 0$ for $n < i \leq m$. We have

$$q_{G'}(v') = \sum_{i=1}^{n} v_i^2 - \sum_{1 \leq i \leq j \leq n} m_{ij} v_i v_j \leq \sum_{i=1}^{n} v_i^2 - \sum_{1 \leq i \leq j \leq n} n_{ij} v_i v_j = q_G(v) = 0,$$

as desired. Observe that the first two sums go up to n because just the first n coordinates of v' are nonzero. □

Remark. If G is a proper subgraph of G', it is possible to find v'' with $q_{G'}(v'') < 0$. If one of the inequalities $n_{ij} \leq m_{ij}$ is strict, we can take $v'' = v'$ as above. If they are all equalities, we must have $n < m$ and we can find a vertex i of G' which is not in G but that connects to G. In this case, it is not hard to see that $v'' = 2v' + e_i$ works.

We conclude the proof of Theorem A.1.1.

Corollary A.1.6. If G is not a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, then q_G is not positive definite.

Proof. By Lemma A.1.4, G contains an Euclidean diagram as a subgraph. Since Euclidean diagrams have radical vectors with positive coordinates, Lemma A.1.5 gives us a nonzero $v \in \mathbb{Z}^n$ with $q_G(v) \leq 0$. □

Remark. Using the previous remark, one can show that G is an Euclidean diagram of type $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{E}}$ if, and only if, q_G is positive semi-definite but not positive definite.

A.2 Root systems

Following [20, Chapter III], let us recall the definition of a root system. Let V be a finite-dimensional real vector space endowed with a positive definite symmetric bilinear form $(-,-) : V \times V \to \mathbb{R}$. For any nonzero vector $v \in V$, we have the map $\sigma_v : V \to V$ defined by

$$\sigma_v(w) = w - \frac{2(w, v)}{(v, v)} v$$

for $w \in V$. This is a **reflection**, that is, an involution on V which fixes the hyperplane orthogonal to v and sends v to $-v$.

Definition A.2.1. With the notation above, a subset $\Phi \subseteq V$ is called a **root system** in V if it satisfies the following conditions:

(R1) Φ is finite, spans V and does not contain zero.

(R2) If $v \in \Phi$, the only multiples of v in Φ are $\pm v$.

(R3) If $v \in \Phi$, the reflection σ_v leaves Φ invariant.

(R4) If $v, w \in \Phi$, then
$$\frac{2(w,v)}{(v,v)} \in \mathbb{Z}.$$

Now, let G be a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, with n vertices. Following Definition 2.3.1, the set of roots of q_G is

$$\Phi_G \coloneqq \{v \in \mathbb{Z}^n \mid q_G(v) = 1\}.$$

The goal of this section is to study this set from the perspective of the theory of root systems.

The first thing to expect from the names is that Φ_G is a root system. For this purpose, we will consider Φ_G as a subset of $\mathbb{R}^n$. Note that q_G and $(-,-)_G$ can be naturally extended to a quadratic form and to a symmetric bilinear form on $\mathbb{R}^n$, respectively. Changing $\mathbb{Z}$ and $\mathbb{Q}$ to $\mathbb{R}$ in the previous chapter, the proofs of Lemma A.1.2 and Corollary A.1.3 still work, so q_G and thus $(-,-)_G$ are positive definite.

Proposition A.2.2. The set of roots Φ_G is a root system in $\mathbb{R}^n$ endowed with the bilinear form $(-,-)_G$.

Proof. For (R1), it is immediate that Φ_G does not contain zero. Note that the vectors $e_1, \ldots, e_n$ from the canonical basis of $\mathbb{R}^n$ are in Φ_G, so the roots span $\mathbb{R}^n$. Let us prove that Φ_G is finite. Since the unit sphere is a compact subset of $\mathbb{R}^n$ and q_G is continuous, q_G attains a minimum $\lambda \in \mathbb{R}$ over this set. We have $\lambda > 0$ because q_G is positive definite on $\mathbb{R}^n$. Thus, if $v \in \Phi_G$ has Euclidean norm $\|v\|$, then

$$\lambda \leq q_G\left(\frac{v}{\|v\|}\right) = \frac{q_G(v)}{\|v\|^2} = \frac{1}{\|v\|^2} \implies \|v\| \leq \frac{1}{\sqrt{\lambda}}.$$

This shows that Φ_G is a bounded set of $\mathbb{R}^n$ and, since its vectors have integer coordinates, it must be finite.

For (R2), we have
$$q_G(\lambda v) = \lambda^2 q_G(v) = \lambda^2$$
for $v \in \Phi_G$ and $\lambda \in \mathbb{R}$. Hence, $\lambda v \in \Phi_G$ if and only if $\lambda = \pm 1$. Moreover, if $v \in \Phi_G$, then
$$\frac{2(w,v)_G}{(v,v)_G} = \frac{2(w,v)_G}{2q_G(v)} = (w,v)_G,$$
which is an integer for every $w \in \mathbb{Z}^n$, so (R4) holds. This also shows that, for $v \in \Phi_G$, the reflection σ_v leaves $\mathbb{Z}^n$ invariant. Therefore, since reflections preserve the bilinear form $(-,-)_G$, (R3) follows as well. □

In fact, it turns out that Φ_G is an irreducible root system. To see this, we need a lemma.

Lemma A.2.3. If $v \in \Phi_G$, then the coordinates of v are either all nonnegative or all nonpositive.

Proof. Since $v \neq 0$ and $-v \in \Phi_G$, we may assume v has at least one positive coordinate. Let $v' \in \mathbb{Z}^n$ be the vector whose coordinates are the absolute values of the coordinates of v. Looking at the signs appearing in the definition of q_G, we can see that $q_G(v') \leq q_G(v)$, and the inequality is strict if v has some negative coordinate. But $q_G(v) = 1$ and $q_G(v')$ is a positive integer, so the inequality is in fact an equality and v has no negative coordinates. □

In other words, every element of Φ_G is a linear combination of the canonical vectors $e_1, \dots, e_n$ with integral coefficients which are all nonnegative or all nonpositive. This shows that $\{e_1, \dots, e_n\}$ is a base of the root system Φ_G. Thus, the Cartan matrix of Φ_G is the $n \times n$ matrix whose entry (i, j) is

$$\frac{2(e_i, e_j)_G}{(e_j, e_j)_G} = (e_i, e_j)_G = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \text{ is connected to } j \text{ in } G, \\ 0 & \text{otherwise,} \end{cases}$$

where we used that G has no loops or multiple edges. This is exactly the Cartan matrix of the irreducible root system with Dynkin diagram G! Since root systems are determined by their Cartan matrices, we have:

Theorem A.2.4. If G is a Dynkin diagram of type $\mathbb{A}$, $\mathbb{D}$ or $\mathbb{E}$, then Φ_G is the irreducible root system corresponding to G.

Proof. See [20, Sections 10 and 11] for the definitions and results used in the argumentation above. □

As a consequence we get the exact number of elements in Φ_G. It is given by the following table:

G	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert\Phi_G\rvert$	$n(n+1)$	$2n(n-1)$	72	126	240

There is also another way of obtaining these numbers. We recall that the **Weyl group** W of Φ_G is the subgroup of the group of linear automorphisms of $\mathbb{R}^n$ generated by the reflections σ_v for $v \in \Phi_G$. Since these reflections leave Φ_G invariant and this set spans $\mathbb{R}^n$, it follows that W is isomorphic to a subgroup of the symmetric group on Φ_G, hence it is finite.

Let Δ be a base for Φ_G (for example, the canonical basis of $\mathbb{R}^n$) and enumerate its elements as $s_1, \dots, s_n$. The reflections $\sigma_i := \sigma_{s_i}$ for $1 \leq i \leq n$ are the **simple reflections** of W (with respect to Δ). The product

$$\sigma_1 \sigma_2 \cdots \sigma_n \in W$$

is called a **Coxeter element** of W. This definition depends on the choice of Δ as well as on the way Δ is numbered. However, all Coxeter elements are conjugated (see [21, Section 3.16]) and so their order is always the same. It is usually denoted by h and called the **Coxeter number** of Φ_G.

Proposition A.2.5. We have $|\Phi_G| = nh$.

Proof. See [21, Section 3.18]. □

Another useful property about Coxeter elements is the following:

Proposition A.2.6. The Weyl group W contains $-\mathrm{id}_{\mathbb{R}^n}$ if, and only if, $G = \mathbb{A}_1$, $\mathbb{D}_n$ with n even, $\mathbb{E}_7$ or $\mathbb{E}_8$. In this case, h is even and, for any Coxeter element $c \in W$, we have $c^{h/2} = -\mathrm{id}_{\mathbb{R}^n}$.

Proof. See [21, Sections 3.7 and 3.19]. □

A.3 The Coxeter transformation

Let Q be a Dynkin quiver. We defined in Section 1.4 the Coxeter transformation $c_{KQ} : K_0(KQ) \to K_0(KQ)$. We will see in this section the reason for this name: c_{KQ} can be seen as a Coxeter element of the Weyl group associated to Q.

Let us fix some hypotheses and notations. We suppose that $Q_0 = \{1, \ldots, n\}$ and that this is an **admissible numbering**, that is, if there is an arrow $i \to j$, then $i > j$. Since Q is acyclic, it is possible to find such a numbering. Denote by n_{ij} the number of arrows from i to j or from j to i. Observe that this number is either 1 or 0 depending on whether i and j are connected or not.

In Section A.2, we saw that the set of roots of q_Q is the irreducible root system associated to Q. It has a base consisting of the vectors $e_1, \ldots, e_n$ of the canonical basis of $\mathbb{R}^n$. Let $\sigma_1, \ldots, \sigma_n$ be the corresponding simple reflections in the Weyl group, and consider the Coxeter element given by

$$c := \sigma_n \cdots \sigma_2 \sigma_1.$$

Our goal is to show that the matrix of c in the canonical basis is the Coxeter matrix of KQ, that is, it is $-C_{KQ}^t C_{KQ}^{-1}$ where C_{KQ} denotes the Cartan matrix of KQ (in the sense of Section 1.4).

Lemma A.3.1. Let $1 \leq i \leq n$. The matrix of σ_i in the canonical basis is the identity matrix, except that we replace the i-th row by the vector

$$v_i := (n_{i1}, n_{i2}, \ldots, n_{i,i-1}, -1, n_{i,i+1}, \ldots, n_{in}).$$

Proof. We have

$$\sigma_i(e_j) = e_j - (e_j, e_i)_Q e_i = \begin{cases} -e_i & \text{if } i = j, \\ e_j + n_{ij} e_i & \text{if } i \neq j. \end{cases}$$

The lemma follows. □

Lemma A.3.2. If we write $C_{KQ} = (\gamma_{ij})_{1 \leq i,j \leq n}$, then C_{KQ} is upper unitriangular and we have

$$\gamma_{ij} = \sum_{1 \leq k < j} \gamma_{ik} n_{kj} = \sum_{i < k \leq n} n_{ik} \gamma_{kj}$$

for $i < j$.

Proof. Recall that the columns of C_{KQ} are the dimension vectors of the indecomposable projective KQ-modules. Thus, γ_{ij} is the multiplicity of the simple $S(i)$ in the indecomposable projective $P(j)$. By Lemma 1.3.6, this is the number of paths in Q from j to i. If $i > j$, then there are no such paths because we fixed an admissible numbering in the beginning. If $i = j$, then there is only the stationary path and $\gamma_{ii} = 1$. This shows that C_{KQ} is upper unitriangular.

Now, suppose $i < j$. Every path from j to i is the concatenation of some arrow $j \to k$ with a path from k to i. Note that k is a successor of j if, and only if, $n_{kj} = 1$ and $j > k$. This shows that γ_{ij} equals the first sum above. A similar argument proves the second equality, but this time we consider a path from j to i as the concatenation of some path from j to a predecessor k of i and the arrow $k \to i$. □

Proposition A.3.3. The matrix of the Coxeter element c in the canonical basis of $\mathbb{R}^n$ is the Coxeter matrix of KQ.

Proof. Regard the simple reflections σ_i as matrices. We have to prove that

$$c = \sigma_n \cdots \sigma_1 = -C_{KQ}^t C_{KQ}^{-1},$$

or, equivalently,

$$-C_{KQ}^t = \sigma_n \cdots \sigma_1 C_{KQ}.$$

By Lemma A.3.1, for any matrix M, $\sigma_i M$ is exactly M except that we put the vector $v_i M$ in the i-th row. Therefore, $\sigma_1 C_{KQ}$ is obtained by replacing the first row of C_{KQ} with $v_1 C_{KQ}$. But $\sigma_2, \ldots, \sigma_n$ will not change the first row of $\sigma_1 C_{KQ}$, so this product should be the matrix C_1 whose first row is the first row of $-C_{KQ}^t$ and whose other rows coincide with those of C_{KQ}. In the same fashion, $\sigma_3, \ldots, \sigma_n$ will not change the first two rows of $\sigma_2\sigma_1 C_{KQ}$ and this product should be the matrix C_2 whose first two rows are the first two rows of $-C_{KQ}^t$ and whose other rows coincide with those of C_{KQ}. We can continue in this way. In the end, we have to prove the following. For $1 \leq i \leq n$, let C_{i-1} denote the matrix whose first $i-1$ rows are the first $i-1$ rows of $-C_{KQ}^t$ and whose last $n-i+1$ rows are the last $n-i+1$ rows of C_{KQ}. We just have to verify that $v_i C_{i-1}$ is the i-th row of $-C_{KQ}^t$.

Let $1 \leq j \leq n$ and write C_{KQ} as in Lemma A.3.2. The j-th entry of $v_i C_{i-1}$ is

$$\left(\sum_{1 \leq k < i} n_{ik} \cdot (-\gamma_{jk}) \right) + (-1) \cdot \gamma_{ij} + \sum_{i < k \leq n} n_{ik}\gamma_{kj}.$$

We have three cases to consider. First, suppose $i < j$. In the first sum, we always have $k < i < j$, so $\gamma_{jk} = 0$ as C_{KQ} is upper triangular. On the other hand, the second sum equals γ_{ij} by Lemma A.3.2, thus the expression above is zero. Note that this is indeed the entry (i,j) of $-C_{KQ}^t$ because this is a lower triangular matrix. Now, suppose $i > j$. This time, the triangularity implies that γ_{ij} and the second sum are zero. We are left with the first sum, which equals $-\gamma_{ji}$ by the first equality in Lemma A.3.2, and this is indeed the entry (i,j) of $-C_{KQ}^t$. Finally, if $i = j$, both sums are zero and we are left with $-\gamma_{ii}$, which is indeed the entry (i,i) of $-C_{KQ}^t$. This completes the proof. □

We get an important corollary.

Corollary A.3.4. The Coxeter transformation $c_{KQ} : K_0(KQ) \to K_0(KQ)$ is an automorphism of finite order. Its order is the Coxeter number of the root system associated to the Dynkin diagram Q.

Appendix B

Derived categories

This second appendix is a quick review of the theory of derived categories and contains the definitions and results used in Chapter 4. No proofs are given, but a good treatment of the subject can be found in [16, Chapter III], [22], [26, Chapter 4], [28, Chapter 10] and [29, Chapter 3].

Remark. We only deal with the derived category of $\operatorname{mod} A$, where A is a finite-dimensional K-algebra. Hence, from now on, A denotes such an algebra and all A-modules are of finite dimension. Although we present the subject in this context, most of the exposition can be adapted without effort for arbitrary abelian categories.

B.1 The definition

We start with some central concepts in homological algebra. A **chain complex** X of A-modules is a sequence $(X_n)_{n\in\mathbb{Z}}$ of A-modules together with maps $d_n : X_n \to X_{n-1}$, called the **differential maps**, such that $d_n \circ d_{n+1} = 0$ for every $n \in \mathbb{Z}$. We can depict X as a diagram:

$$X = \ \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots .$$

A **morphism of complexes** $f : X \to Y$ is a sequence of A-module homomorphisms $f_n : X_n \to Y_n$ which commute with the differential maps, that is, $f_{n-1}d_n^X = d_n^Y f_n$ for all $n \in \mathbb{Z}$. It is easy to see that chain complexes and morphisms between them form a category, which we denote by $\mathcal{C}(A)$. It is an abelian K-category (cf. [28, Section 1.2]).

A chain complex X is **bounded on the right** if $X_n = 0$ for $n \ll 0$. Similarly, X is **bounded on the left** if $X_n = 0$ for $n \gg 0$. We say that X is **bounded** if it is bounded on both sides. These restrictions define three full subcategories of $\mathcal{C}(A)$: $\mathcal{C}^-(A)$, $\mathcal{C}^+(A)$ and $\mathcal{C}^b(A)$, respectively. One should think for example that the symbol "$-$" corresponds to "left", because complexes that are bounded on the right are *concentrated* on the left. We are mostly interested in $\mathcal{C}^b(A)$, but it is worth presenting the other categories too.

By the condition imposed on the differential maps, note that $\operatorname{im} d_{n+1} \subseteq \ker d_n$ for all $n \in \mathbb{Z}$. Thus, it makes sense to study the **homology** of a complex X in degree n, which is the quotient

$$H_n(X) \coloneqq \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

It is not hard to verify that it defines a K-linear functor $H_n : \mathcal{C}(A) \to \operatorname{mod} A$. In this context, we say that a morphism of complexes $f : X \to Y$ is a **quasi-isomorphism** if $H_n(f) : H_n(X) \to H_n(Y)$ is an isomorphism of A-modules for all $n \in \mathbb{Z}$.

Many constructions in homological algebra, such as the Ext and Tor functors, are defined as the homology of certain complexes. However, instead of taking homology, it would be convenient to work with the complexes themselves. In this case, we would like to identify complexes having the same homology or, more precisely, we wish quasi-isomorphisms were true isomorphisms. This motivates[1] the following definition:

Definition B.1.1. The **derived category** $\mathcal{D}(A)$ of $\operatorname{mod} A$ is the localization of $\mathcal{C}(A)$ with respect to the collection of quasi-isomorphisms. In other words, $\mathcal{D}(A)$ is a category together with a functor $Q : \mathcal{C}(A) \to \mathcal{D}(A)$ such that:

- $Q(f)$ is an isomorphism for every quasi-isomorphism f in $\mathcal{C}(A)$;
- For every functor $F : \mathcal{C}(A) \to \mathcal{C}$ which sends quasi-isomorphisms to isomorphisms, there exists a unique functor $G : \mathcal{D}(A) \to \mathcal{C}$ such that the triangle

$$\begin{array}{ccc} \mathcal{C}(A) & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle Q}\downarrow & \nearrow_{G} & \\ \mathcal{D}(A) & & \end{array}$$

  commutes.

We can similarly define the bounded versions $\mathcal{D}^-(A)$, $\mathcal{D}^+(A)$ and $\mathcal{D}^b(A)$.

It is not clear that the derived category should exist. A naive construction is to take the objects of $\mathcal{D}(A)$ to be the same as the objects of $\mathcal{C}(A)$, formally add the inverses of the quasi-isomorphisms and consider morphisms in $\mathcal{D}(A)$ to be "zigzags" between these formal inverses and usual morphisms in $\mathcal{C}(A)$ (cf. [16, Section III.2] or [26, Section 1.1]). However, this raises some delicate set-theoretic issues (see the remarks on [28, Section 10.3]) and it is not convenient for computations.

Another way of constructing $\mathcal{D}(A)$ is by introducing an intermediate category. The **homotopy category** $\mathcal{K}(A)$ is the quotient of $\mathcal{C}(A)$ by the ideal formed by the **null-homotopic** maps, that is, morphisms of complexes $f : X \to Y$ for which there exists a sequence of maps $s_n : X_n \to Y_{n+1}$ such that

$$f_n = d^Y_{n+1} s_n + s_{n-1} d^X_n$$

for all $n \in \mathbb{Z}$. This means that the objects of $\mathcal{K}(A)$ are the same as the objects of $\mathcal{C}(A)$, but we quotient each Hom-space by the subspace generated by the null-homotopic maps. Thus, two maps in $\mathcal{C}(A)$ become the same in $\mathcal{K}(A)$ if they are **homotopic**, i.e., their difference is of the form given above. Note that an isomorphism in $\mathcal{K}(A)$ comes from a morphism $f : X \to Y$ in $\mathcal{C}(A)$ which is a **homotopy equivalence**: there exists a morphism $g : Y \to X$ such that gf and fg are homotopic to id_X and id_Y, respectively.

One can check that homotopic maps induce the same map on homology, so we get a well-defined functor $H_n : \mathcal{K}(A) \to \operatorname{mod} A$ for every $n \in \mathbb{Z}$. Hence, it still makes sense to talk about quasi-isomorphisms in the homotopy category and, moreover, homotopy equivalences in $\mathcal{C}(A)$ are quasi-isomorphisms. Now, by [28, Proposition 10.1.2], $\mathcal{K}(A)$ is the localization of $\mathcal{C}(A)$ with respect to the class of homotopy equivalences. Therefore, we can see $\mathcal{K}(A)$ as an intermediate between $\mathcal{C}(A)$ and $\mathcal{D}(A)$, and the localization of $\mathcal{K}(A)$ with respect to quasi-isomorphisms should be equivalent to $\mathcal{D}(A)$ (if it exists).

[1] For a more convincing and detailed motivation, see [16, Section III.1] and [22].

Quasi-isomorphisms in $\mathcal{K}(A)$ are better behaved and form what is called a **multiplicative system**. This allows us to finally construct $\mathcal{D}(A)$ using a **calculus of fractions**, which generalizes Ore localization for noncommutative rings to categories. More explicitly, the objects in $\mathcal{D}(A)$ are chain complexes of A-modules and a morphism $\varphi : X \to Y$ is the equivalence class (for a certain equivalence relation) of a right fraction

$$X \xleftarrow{\ s\ } Z \xrightarrow{\ f\ } Y$$

where f and s are morphisms in $\mathcal{K}(A)$ and, in addition, s is a quasi-isomorphism. Viewing f and s as morphisms in $\mathcal{D}(A)$ through the localization functor $\mathcal{K}(A) \to \mathcal{D}(A)$, we have $\varphi = fs^{-1}$, explaining why φ is called a "right fraction". A similar description with left fractions yields an equivalent category. We will not need to know how to work with the calculus of fractions, but more details can be found in [16, Section III.2], [28, Section 10.3] and [29, Section 3.5.3].

The derived category inherits some properties of $\mathcal{K}(A)$. For example, it is also a K-linear category. Moreover, finite direct sums in $\mathcal{D}(A)$ are computed in the same way as in $\mathcal{C}(A)$ and $\mathcal{K}(A)$, a property that we use in Proposition 4.1.1. This follows from the fact that the localization functor $\mathcal{K}(A) \to \mathcal{D}(A)$ is additive (see the end of [16, Section III.4] and [28, Corollary 10.3.11]). In the next section, we also comment about another structure of $\mathcal{K}(A)$ which descends to $\mathcal{D}(A)$.

Remark. We also have the categories $\mathcal{K}^-(A), \mathcal{K}^+(A)$ and $\mathcal{K}^b(A)$, and they give us an analogous description of $\mathcal{D}^-(A), \mathcal{D}^+(A)$ and $\mathcal{D}^b(A)$.

B.2 Triangulated categories

Suppose we have a short exact sequence

$$0 \longrightarrow X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z \longrightarrow 0 \tag{$*$}$$

in $\mathcal{C}(A)$. It is a well-known result in homological algebra that we get a long exact sequence in homology:

$$\cdots \longrightarrow H_{n+1}(Z) \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow \cdots .$$

In the philosophy of the previous section, we may wonder if this long exact sequence comes somehow from a sequence of complexes, because then we could work with it instead of having to take homology. For this purpose, we introduce some constructions.

For an object $X \in \mathcal{C}(A)$, we define the **shift** of X to be the chain complex $X[1]$ with

$$X[1]_n \coloneqq X_{n-1} \quad \text{and} \quad d_n^{X[1]} \coloneqq -d_{n-1}^X$$

for $n \in \mathbb{Z}$. This defines an isomorphism $[1] : \mathcal{C}(A) \to \mathcal{C}(A)$ called the **shift functor** (also known as the **translation** or **suspension** functor). For $n \in \mathbb{Z}$, $[n]$ denotes the n-th power of this functor.

The **cone** of a morphism $f : X \to Y$ in $\mathcal{C}(A)$ is the chain complex $\operatorname{cone}(f)$ defined by[1]

$$\operatorname{cone}(f)_n := X_{n-1} \oplus Y_n \quad \text{and} \quad d_n^{\operatorname{cone}(f)} := \begin{pmatrix} -d_{n-1}^X & 0 \\ f_{n-1} & d_n^Y \end{pmatrix}$$

for $n \in \mathbb{Z}$. Observe that $\operatorname{cone}(f)$ is almost the same as the direct sum $X[1] \oplus Y$, but the data of the morphism f appear in the differential maps. Even so, we still have a short exact sequence[2]

$$0 \longrightarrow Y \xrightarrow{\;i\;} \operatorname{cone}(f) \xrightarrow{\;\pi\;} X[1] \longrightarrow 0,$$

which induces the long exact sequence

$$\cdots \to H_{n+1}(X[1]) \xrightarrow{\delta_n} H_n(Y) \to H_n(\operatorname{cone}(f)) \to H_n(X[1]) \xrightarrow{\delta_{n-1}} H_{n-1}(Y) \to \cdots .$$

We have $H_{n+1}(X[1]) \cong H_n(X)$ and one can check that the connecting morphism $\delta_n : H_n(X) \to H_n(Y)$ is exactly $H_n(f)$. Therefore, this long exact sequence can be obtained from the sequence

$$\cdots \xrightarrow{\pi[-1]} X \xrightarrow{\;f\;} Y \xrightarrow{\;i\;} \operatorname{cone}(f) \xrightarrow{\;\pi\;} X[1] \xrightarrow{f[1]} \cdots \qquad (**)$$

by taking H_0.

Now, suppose the map $f : X \to Y$ fits into the short exact sequence $(*)$ from the first paragraph. In this case, there is a quasi-isomorphism $\varphi : \operatorname{cone}(f) \to Z$, so the long exact sequence above is isomorphic to the one we get from $(*)$. This implies that the sequence $(**)$ is the replacement we were searching for! In $\mathcal{D}(A)$, we can identify $\operatorname{cone}(f)$ and Z via φ, so we are working with a sequence of the form

$$\cdots \xrightarrow{h[-1]} X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z \xrightarrow{\;h\;} X[1] \xrightarrow{f[1]} \cdots ,$$

which extends $(*)$. Such sequences satisfy some special properties in the derived category that turn it into a *triangulated* category. Let us define what this means.

Let $\mathcal{C}$ be an additive category and $T : \mathcal{C} \to \mathcal{C}$ be an isomorphism[3] of categories. A **triangle** is a diagram in $\mathcal{C}$ of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow TX.$$

As before, we can apply successive powers of T to obtain an infinite sequence, but working with these three maps is enough. A **morphism of triangles** is a commutative diagram:

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle T\alpha} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

The triangles in the rows are **isomorphic** if α, β and γ are isomorphisms. Suppose we have also chosen a class of triangles in $\mathcal{C}$, which we call the class of **distinguished**

[1] Be careful when consulting the references because some authors have different conventions for the signs in the differential maps. We are following [29, Section 3.5.2].

[2] For this property of the cone and the subsequent ones, see [28, Section 1.5].

[3] Some authors consider the more general case where T is just an autoequivalence.

triangles. This data defines a **triangulated category** if the following four axioms are true:

(TR1) A triangle which is isomorphic to a distinguished triangle is itself distinguished. Furthermore, the triangle

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow TX$$

is always distinguished and any morphism $f : X \to Y$ can be completed to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX.$$

(TR2) The triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$$

is distinguished if and only if the triangle

$$Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$$

is distinguished.

(TR3) For any commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

whose rows are distinguished triangles, there is a morphism $\gamma : Z \to Z'$ such that

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & TX \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle T\alpha}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & TX' \end{array}$$

commutes.

(TR4) Suppose we have three distinguished triangles:

$$X_1 \xrightarrow{f} X_2 \longrightarrow Z_1 \longrightarrow TX_1,$$

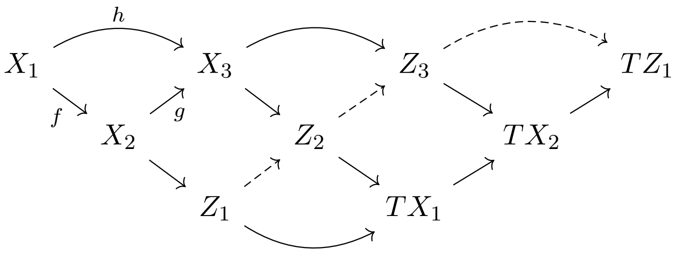
$$X_2 \xrightarrow{g} X_3 \longrightarrow Z_3 \longrightarrow TX_2,$$

$$X_1 \xrightarrow{h} X_3 \longrightarrow Z_2 \longrightarrow TX_1.$$

If $h = g \circ f$, then there is a distinguished triangle

$$Z_1 \longrightarrow Z_2 \longrightarrow Z_3 \longrightarrow TZ_1$$

such that the following diagram[1] commutes:



The homotopy category $\mathcal{K}(A)$ together with the shift becomes a triangulated category if we define the distinguished triangles to be those isomorphic (in $\mathcal{K}(A)$) to a triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{i} \operatorname{cone}(f) \xrightarrow{\pi} X[1],$$

where i and π are the maps from before ([29, Proposition 3.5.25]). This structure descends to the derived category $\mathcal{D}(A)$ through the localization functor, and the distinguished triangles become those isomorphic (but now in $\mathcal{D}(A)$) to a triangle of the previous form ([29, Proposition 3.5.40]).

As we discussed in the beginning of this section, any short exact sequence in $\mathcal{C}(A)$ defines a distinguished triangle in $\mathcal{D}(A)$. Conversely, every distinguished triangle comes from such a sequence: just take the short exact sequence relating the cone of $f : X \to Y$, Y and $X[1]$ that we saw before. Therefore, every distinguished triangle in $\mathcal{D}(A)$ should induce a long exact sequence after taking homology. This is, for example, [16, Section III.3, Theorem 6]. In the terminology of triangulated categories, this means that the functors H_n are **homological**.

Remark. The triangulated structure exists and it works in the same way for the bounded versions of the homotopy and the derived categories.

B.3 Identifying some categories

Recall that we can view $\operatorname{mod} A$ as a full subcategory of $\mathcal{C}(A)$. To do so, we consider the functor $\operatorname{mod} A \to \mathcal{C}(A)$ which sends an A-module M to the **stalk complex** M concentrated in degree zero:

$$\cdots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \cdots .$$

This is a fully faithful functor whose essential image consists of the complexes concentrated in degree zero. Composing it with the localization functor $\mathcal{C}(A) \to \mathcal{D}(A)$, we get a functor from $\operatorname{mod} A$ to $\mathcal{D}(A)$ which should also be an embedding.

Proposition B.3.1 ([16, Section III.5, Proposition 2])**.** The inclusion $\operatorname{mod} A \to \mathcal{D}(A)$ in degree zero is indeed a fully faithful functor. Its essential image consists of the complexes whose homology is concentrated in degree zero.

We can include $\operatorname{mod} A$ in any degree and the same result holds. Remarkably, these copies of $\operatorname{mod} A$ inside $\mathcal{D}(A)$ encode the Ext functors.

[1] This diagram can be drawn on an octahedron (cf. [29, Section 3.4]), so this is called the "octahedral axiom".

Proposition B.3.2 ([26, Proposition 4.2.11][1])**.** If M and N are A-modules viewed as complexes concentrated in degree zero, then

$$\mathrm{Hom}_{\mathcal{D}(A)}(M, N[n]) \cong \begin{cases} 0 & \text{if } n < 0, \\ \mathrm{Ext}^n_A(M, N) & \text{if } n \geq 0. \end{cases}$$

The copies of $\operatorname{mod} A$ in $\mathcal{C}(A)$ are made from stalk complexes concentrated in some degree. When we pass to the derived category, we relax this condition and only need the homology to be concentrated. In a similar way, the bounded versions $\mathcal{C}^-(A)$, $\mathcal{C}^+(A)$ and $\mathcal{C}^b(A)$ are obtained from $\mathcal{C}(A)$ by imposing some restrictions on the terms of its complexes, hence we can expect that $\mathcal{D}^-(A)$, $\mathcal{D}^+(A)$ and $\mathcal{D}^b(A)$ come from $\mathcal{D}(A)$ by the same restriction in homology. Note that we cannot immediately see these categories as subcategories of $\mathcal{D}(A)$ but, for $* = -, +, b$, the localization property gives a functor $\mathcal{D}^*(A) \to \mathcal{D}(A)$ induced by the inclusion $\mathcal{C}^*(A) \to \mathcal{C}(A)$.

Proposition B.3.3 ([26, Lemma 4.1.16 and Section 4.2])**.** For $* = -, +, b$, the canonical functor $\mathcal{D}^*(A) \to \mathcal{D}(A)$ is fully faithful. Its essential image $\mathcal{I}^*$ is given by:

$$\begin{aligned} \mathcal{I}^- &= \{X \in \mathcal{D}(A) \mid H_n(X) = 0 \text{ for } n \ll 0\}, \\ \mathcal{I}^+ &= \{X \in \mathcal{D}(A) \mid H_n(X) = 0 \text{ for } n \gg 0\}, \\ \mathcal{I}^b &= \{X \in \mathcal{D}(A) \mid H_n(X) = 0 \text{ for almost all } n\}. \end{aligned}$$

Remark. In some cases, we will use the identifications above without mentioning.

We conclude the section by discussing an alternative construction of the derived category in the bounded case. It is possible to avoid the calculus of fractions by describing it as a particular homotopy category. For this reason, we define $\mathcal{K}(\operatorname{proj} A)$ and $\mathcal{K}(\operatorname{inj} A)$ to be the full subcategories of $\mathcal{K}(A)$ whose objects are chain complexes of projective and injective A-modules, respectively. Since the definition of the homotopy category only depends on the additive structure of the initial category, we can indeed view these new subcategories as the homotopy categories of $\operatorname{proj} A$ and $\operatorname{inj} A$. Again, we can also define their bounded variants.

Lemma B.3.4 ([28, Corollary 10.4.7])**.** If $P \in \mathcal{K}^-(\operatorname{proj} A)$ and $I \in \mathcal{K}^+(\operatorname{inj} A)$, then the natural maps

$$\mathrm{Hom}_{\mathcal{K}(A)}(P, X) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(P, X) \quad \text{and} \quad \mathrm{Hom}_{\mathcal{K}(A)}(X, I) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(X, I)$$

induced by the localization functor are isomorphisms for all $X \in \mathcal{K}(A)$.

This lemma implies that the composition of the inclusion $\mathcal{K}^-(\operatorname{proj} A) \to \mathcal{K}^-(A)$ with the localization functor $\mathcal{K}^-(A) \to \mathcal{D}^-(A)$ is fully faithful. Similarly, we have a fully faithful functor $\mathcal{K}^+(\operatorname{inj} A) \to \mathcal{D}^+(A)$. Both of them are essentially surjective! This follows from the fact that any complex (bounded on the correct side) has a projective or injective resolution (see [16, Section III.5, Subsection 25] and [29, Proposition 3.5.43]). As a consequence, we have the following result:

Proposition B.3.5. The canonical functors $\mathcal{K}^-(\operatorname{proj} A) \to \mathcal{D}^-(A)$ and $\mathcal{K}^+(\operatorname{inj} A) \to \mathcal{D}^+(A)$ are equivalences.

Remark. Lemma B.3.4 also gives fully faithful functors $\mathcal{K}^b(\operatorname{proj} A) \to \mathcal{D}^b(A)$ and $\mathcal{K}^b(\operatorname{inj} A) \to \mathcal{D}^b(A)$. However, they are not necessarily essentially surjective because

[1] Only the second proof in this book addresses the case $n < 0$.

bounded complexes can have unbounded projective or injective resolutions. One way to guarantee that the resolutions are finite is to assume that A is of finite global dimension. If we do not want to assume hypotheses on A, we have to introduce the full subcategories $\mathcal{K}^{-,b}(\operatorname{proj} A)$ and $\mathcal{K}^{+,b}(\operatorname{inj} A)$ of $\mathcal{K}^{-}(\operatorname{proj} A)$ and $\mathcal{K}^{+}(\operatorname{inj} A)$, respectively, whose complexes have bounded homology. In this case, we do get equivalences $\mathcal{K}^{-,b}(\operatorname{proj} A) \to \mathcal{D}^b(A)$ and $\mathcal{K}^{+,b}(\operatorname{inj} A) \to \mathcal{D}^b(A)$.

B.4 Derived functors

Let B be another K-algebra of finite dimension. We are now interested in extending a K-linear functor $F : \operatorname{mod} A \to \operatorname{mod} B$ to the derived setting. The first step is to extend F to a functor $\mathcal{C}(A) \to \mathcal{C}(B)$ and then to a functor $\mathcal{K}(A) \to \mathcal{K}(B)$. This can be done easily by applying F in every term of a complex. Since the constructions of the category of chain complexes and the homotopy category only depend on the additive structure of $\operatorname{mod} A$ and $\operatorname{mod} B$, these extensions are well-defined. The new functor $F : \mathcal{K}(A) \to \mathcal{K}(B)$ is even a **triangulated functor**, that is, it commutes with the shift functor and sends distinguished triangles to distinguished triangles.

The problem arises when we want to pass from the homotopy category to the derived category. We would like to find a functor $F' : \mathcal{D}(A) \to \mathcal{D}(B)$ such that

$$\begin{array}{ccc} \mathcal{K}(A) & \xrightarrow{F} & \mathcal{K}(B) \\ {\scriptstyle Q_A}\downarrow & & \downarrow{\scriptstyle Q_B} \\ \mathcal{D}(A) & \xrightarrow[F']{} & \mathcal{D}(B) \end{array}$$

commutes, where the vertical arrows are the localization functors. Due to the universal property of the localization of categories, this can only happen if F sends quasi-isomorphisms to quasi-isomorphisms. In most cases, this condition is too restrictive. We can relax it if we just search for a functor F' such that the square above commutes *up to a natural transformation*:

$$\begin{array}{ccc} \mathcal{K}(A) & \xrightarrow{F} & \mathcal{K}(B) \\ {\scriptstyle Q_A}\downarrow & \nearrow\!\!\!\!\nearrow & \downarrow{\scriptstyle Q_B} \\ \mathcal{D}(A) & \xrightarrow[F']{} & \mathcal{D}(B) \end{array}$$

This diagram just symbolizes that there is a natural transformation from $F' \circ Q_A$ to $Q_B \circ F$. We can also consider natural transformations in the other direction. In order to get a unique extension, we can try to impose that F' satisfy some sort of universal property, so that it is the functor "closest" to making the square truly commute. This motivates the following definition:

Definition B.4.1. Let $* = \emptyset, -, +$ or b. Let $F : \mathcal{K}^*(A) \to \mathcal{K}(B)$ be a triangulated functor.

(1) A **left derived functor** of F is a functor $\mathbb{L}F : \mathcal{D}^*(A) \to \mathcal{D}(B)$ together with a natural transformation $\xi : \mathbb{L}F \circ Q_A \to Q_B \circ F$ which is universal in the following sense: if $F' : \mathcal{D}^*(A) \to \mathcal{D}(B)$ is another functor equipped with a natural transformation $\zeta : F' \circ Q_A \to Q_B \circ F$, then there is a unique natural transformation $\eta : F' \to \mathbb{L}F$ such that $\zeta_X = \xi_X \circ \eta_{Q_A(X)}$ for all $X \in \mathcal{K}^*(A)$.

(2) A **right derived functor** of F is a functor $\mathbb{R}F : \mathcal{D}^*(A) \to \mathcal{D}(B)$ together with a natural transformation $\xi : Q_B \circ F \to \mathbb{R}F \circ Q_A$ which is universal in the following sense: if $F' : \mathcal{D}^*(A) \to \mathcal{D}(B)$ is another functor equipped with a natural transformation $\zeta : Q_B \circ F \to F' \circ Q_A$, then there is a unique natural transformation $\eta : \mathbb{R}F \to F'$ such that $\zeta_X = \eta_{Q_A(X)} \circ \xi_X$ for all $X \in \mathcal{K}^*(A)$.

If they exist, these functors are unique up to a natural isomorphism.

In general, the derived functor of a functor defined over the whole category $\mathcal{K}(A)$ does not exist. This is the reason for restricting F to one of the subcategories $\mathcal{K}^-(A)$, $\mathcal{K}^+(A)$ or $\mathcal{K}^b(A)$ in the definition above. In this case, we have the following result.

Theorem B.4.2 ([16, Section III.6, Theorem 8], [28, Theorem 10.5.6])**.** Suppose $F : \mathcal{K}^-(A) \to \mathcal{K}(B)$ is a triangulated functor. The left derived functor $\mathbb{L}F : \mathcal{D}^-(A) \to \mathcal{D}(B)$ exists, it is triangulated and it is given by the composition

$$\mathcal{D}^-(A) \xrightarrow{G} \mathcal{K}^-(\operatorname{proj} A) \xrightarrow{F} \mathcal{K}(B) \xrightarrow{Q_B} \mathcal{D}(B),$$

where G is a quasi-inverse for the canonical equivalence $\mathcal{K}^-(\operatorname{proj} A) \to \mathcal{D}^-(A)$. Dually, if $F : \mathcal{K}^+(A) \to \mathcal{K}(B)$ is a triangulated functor, then the right derived functor $\mathbb{R}F : \mathcal{D}^+(A) \to \mathcal{D}(B)$ exists, it is triangulated and it is given by the composition

$$\mathcal{D}^+(A) \xrightarrow{G} \mathcal{K}^+(\operatorname{inj} A) \xrightarrow{F} \mathcal{K}(B) \xrightarrow{Q_B} \mathcal{D}(B),$$

where G is a quasi-inverse for the canonical equivalence $\mathcal{K}^+(\operatorname{inj} A) \to \mathcal{D}^+(A)$.

For a complex X, finding $G(X)$ with G as above amounts to choosing an object in $\mathcal{K}^-(\operatorname{proj} A)$ or in $\mathcal{K}^+(\operatorname{inj} A)$ which is isomorphic to X in the derived category. In other words, $G(X)$ is a projective or an injective resolution of X. This description allows us to view the classical derived functors of homological algebra, such as the Ext and Tor functors, as derived functors in the sense we defined here.

Remark. We have to pay attention again when considering $\mathcal{D}^b(A)$. If we start with a functor $F : \mathcal{K}(A) \to \mathcal{K}(B)$, then an analog of the previous theorem constructs $\mathbb{L}F$ and $\mathbb{R}F$ over $\mathcal{D}^b(A)$, where we use quasi-inverses for the equivalences $\mathcal{K}^{-,b}(\operatorname{proj} A) \to \mathcal{D}^b(A)$ and $\mathcal{K}^{+,b}(\operatorname{inj} A) \to \mathcal{D}^b(A)$. However, if F is defined only over $\mathcal{K}^b(A)$, we cannot do this in general. One case where it works is when A is of finite global dimension, because then these equivalences have $\mathcal{K}^b(A)$ as a domain.

To conclude, we discuss about a particular case of derived functor that we use in Section 4.4. Let X be a complex of right A-modules and Y a complex of left A-modules, both of them bounded on the right. We can define their tensor product $X \otimes_A Y$, which is a complex of K-vector spaces bounded on the right (see [8, Section 2.7]). Fixing X, we get a functor $X \otimes_A - : \mathcal{C}^-(A^{\mathrm{op}}) \to \mathcal{C}^-(K)$ which can be extended to the respective homotopy categories. By Theorem B.4.2, it has a left derived functor

$$X \otimes_A^{\mathbb{L}} - : \mathcal{D}^-(A^{\mathrm{op}}) \longrightarrow \mathcal{D}^-(K).$$

Analogously, we also have the left derived functor

$$- \otimes_A^{\mathbb{L}} Y : \mathcal{D}^-(A) \longrightarrow \mathcal{D}^-(K).$$

One can show ([28, Theorem 10.6.3]) that these two functors are compatible and can be assembled into a single bifunctor

$$- \otimes_A^{\mathbb{L}} - : \mathcal{D}^-(A) \times \mathcal{D}^-(A^{\mathrm{op}}) \longrightarrow \mathcal{D}^-(K),$$

which is called the **derived tensor product**. For complexes $X \in \mathcal{D}^-(A)$ and $Y \in \mathcal{D}^-(A^{\mathrm{op}})$, it can be computed in two ways:

$$X \otimes_A^{\mathbb{L}} Y \cong pX \otimes_A Y \cong X \otimes_A pY,$$

where pX and pY denote projective resolutions of X and Y, respectively.

We can upgrade the derived tensor product to complexes of bimodules in the following way. Let B_1 and B_2 be two K-algebras of finite dimension. Suppose X is a complex of B_1-A-bimodules and Y is a complex of A-B_2-bimodules, both of them bounded on the right. Using the A-module structure, we can form the tensor product $X \otimes_A Y$. This naturally becomes a complex of B_1-B_2-bimodules using the extra structure on X and Y. In a similar way, it is also possible to see the derived tensor product $X \otimes_A^{\mathbb{L}} Y$ as a complex of B_1-B_2-bimodules ([28, Exercise 10.6.2]). Identifying B_1-A-bimodules with modules over $B_1^{\mathrm{op}} \otimes_K A$ (and similarly for the other bimodule structures), we get a bifunctor

$$- \otimes_A^{\mathbb{L}} - : \mathcal{D}^-(B_1^{\mathrm{op}} \otimes_K A) \times \mathcal{D}^-(A^{\mathrm{op}} \otimes_K B_2) \longrightarrow \mathcal{D}^-(B_1^{\mathrm{op}} \otimes_K B_2).$$

Remark. If we want to compute the complex $X \otimes_A^{\mathbb{L}} Y$ using this new construction, we have to find, in principle, a projective resolution of X or of Y as complexes of A-modules, and not as complexes of bimodules. It is only after forming the usual derived tensor product that we add the bimodule structure. However, in our context, taking resolutions as complexes of bimodules also works and, in this way, it is easier to see how to define the extra structure (see Proposition 3.7.16 in [29] and the discussion that precedes it).

Finally, the derived tensor product is associative: if we have complexes $X \in \mathcal{D}^-(A)$, $Y \in \mathcal{D}^-(A^{\mathrm{op}} \otimes_K B)$ and $Z \in \mathcal{D}^-(B^{\mathrm{op}})$, then there is a natural isomorphism

$$X \otimes_A^{\mathbb{L}} (Y \otimes_A^{\mathbb{L}} Z) \cong (X \otimes_A^{\mathbb{L}} Y) \otimes_A^{\mathbb{L}} Z.$$

For a proof, see [28, Example 10.8.1 and Theorem 10.8.2]. This isomorphism also holds if X and Z are complexes of bimodules.

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