A categorification of combinatorial Auslander–Reiten quivers

Ricardo Canesin

Demi-journée de l'équipe GRG

June 6, 2024

Motivation

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and let $U_q(\widehat{\mathfrak{g}})$ be its quantum affine algebra.

If $\mathfrak{g}$ is of type ADE, then the representation theory of $U_q(\widehat{\mathfrak{g}})$ is intimately connected with the representation theory of a Dynkin quiver Q of the same type (cf. Hernandez–Leclerc, 2015).

For general type, the combinatorics coming from the AR theory of $\mathrm{Rep}(Q)$ and $\mathcal{D}^b(\mathrm{Rep}(Q))$ have been generalized to study $U_q(\widehat{\mathfrak{g}})$.

- Oh–Suh, 2019: Introduced combinatorial AR quivers, a generalization of the AR quiver of a Dynkin quiver.
- Fujita–Oh, 2021: Introduced Q-data, a generalization of a Dynkin quiver with height function.

Problem

Can we categorify these combinatorics?

Notation for root systems

- Δ: Dynkin diagram of type ADE.
- R: irreducible root system with Dynkin diagram Δ.
- $\alpha_i \in$ R: simple root for $i \in \Delta_0$.
- W: Weyl group.
- $s_i \in$ W: simple reflection for $i \in \Delta_0$.
- $w_0 \in$ W: longest element.

We have an involution $i \mapsto i^*$ on Δ_0 determined by $w_0(\alpha_i) = -\alpha_{i^*}$.

- $\underline{\boldsymbol{i}} = (i_1, \ldots, i_N) \in \Delta_0^N$: a fixed reduced word for w_0.
- $\widehat{\underline{\boldsymbol{i}}} = (i_k)_{k \in \mathbb{Z}}$: extension of $\underline{\boldsymbol{i}}$ by imposing $i_{k+N} = i_k^*$.

Combinatorial Auslander–Reiten quivers

Definition [Oh–Suh, 2019]

The combinatorial repetition quiver $\widehat{\Upsilon}_{\underline{\boldsymbol{i}}}$ is the quiver where:

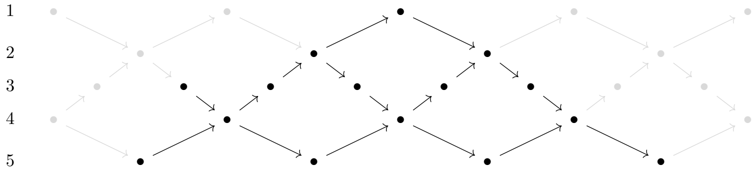
- the vertex set is $\mathbb{Z}$.
- there is an arrow from k to l if $k > l$, i_k is adjacent to i_l in Δ, and there is no index $l < j < k$ such that $i_j = i_l$ or $i_j = i_k$.

The combinatorial AR quiver $\Upsilon_{\underline{\boldsymbol{i}}}$ is the full subquiver of $\widehat{\Upsilon}_{\underline{\boldsymbol{i}}}$ with vertex set $\{1, 2, \ldots, N\}$, where N is the length of w_0.

These definitions depend only on the commutation class of $\underline{\boldsymbol{i}}$.

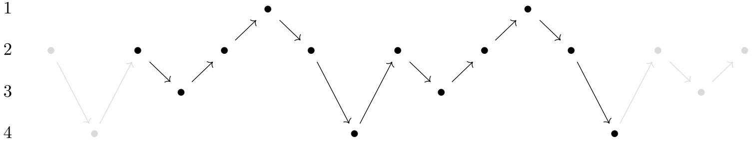
Examples

- $\Delta = \mathsf{A}_5$ and $\underline{\boldsymbol{i}} = (5, 4, 3, 2, 5, 3, 1, 4, 3, 2, 5, 3, 4, 3, 5)$:



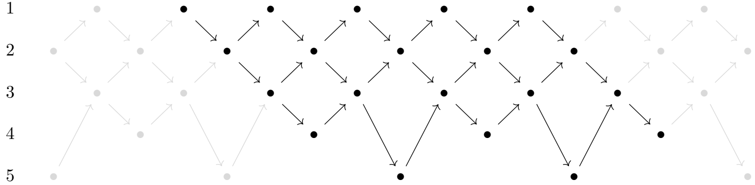
Examples

- $\Delta = \mathsf{D}_4$ and $\underline{\boldsymbol{i}} = (4, 2, 1, 2, 3, 2, 4, 2, 1, 2, 3, 2)$:



Examples

- $\Delta = \mathsf{D}_5$ and $\underline{\boldsymbol{i}} = (4,3,2,1,5,3,2,1,4,3,2,1,5,3,2,1,4,3,2,1)$:



Assigning roots

We define the coordinate map $\rho : (\widehat{\Upsilon}_{\underline{i}})_0 \to \mathsf{R}$ by

$$\rho(k) = \begin{cases} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k \geq 1, \\ -s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0. \end{cases}$$

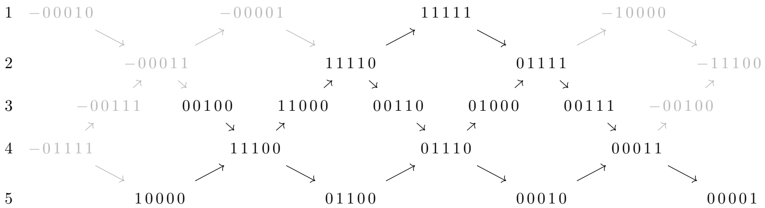
It restricts to a bijection between the vertex set of $\Upsilon_{\underline{i}}$ and the set of positive roots in R.

Theorem [Bédard, 1999; Oh–Suh, 2019]

If $\underline{i}$ is a reduced word for w_0 which is also a source sequence for an orientation Q of Δ, then $\Upsilon_{\underline{i}}$ is isomorphic to the AR quiver of the path algebra of Q. This isomorphism can be taken in such a way that the bijection above corresponds to the dimension vector map.

Example: the coordinate map

For the previous example in type A_5, the coordinate map is given as follows:



Mesh-additivity

Take a vertex $x \in (\widehat{\Upsilon}_{\underline{i}})_0 = \mathbb{Z}$.

- The translation of x is the smallest integer $\tau x \in (\widehat{\Upsilon}_{\underline{i}})_0$ such that $x < \tau x$ and $i_x = i_{\tau x}$.
- The set of abutters of x is the subset $V_{\underline{i}}(x) \subset (\widehat{\Upsilon}_{\underline{i}})_0$ formed by the vertices y such that $x < y < \tau x$ and i_y is adjacent to i_x in Δ.

Theorem [C.]

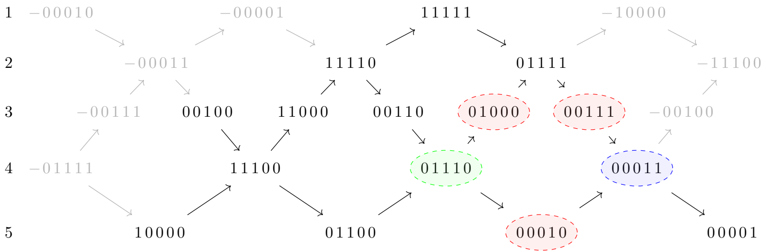
The coordinate map has the mesh-additivity property:

$$\rho(\tau x) + \rho(x) = \sum_{y \in V_{\underline{i}}(x)} \rho(y)$$

for all $x \in (\widehat{\Upsilon}_{\underline{i}})_0$.

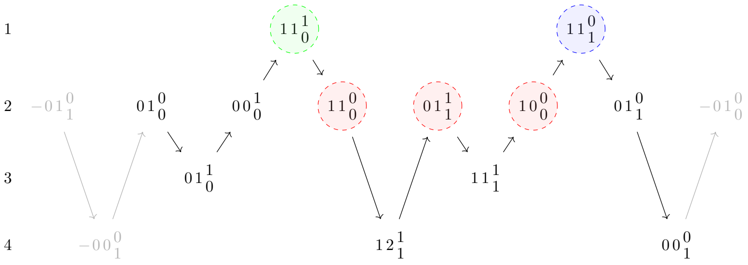
Examples

For the previous example in type A_5, we highlight a mesh:



Examples

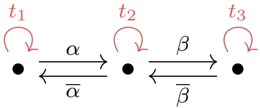
For the previous example in type D_4, we highlight a mesh:



The ambient category

Let Π be the 2-dimensional Ginzburg dg algebra associated with an orientation Q of Δ over some field K.

For example, if $\Delta = \mathsf{A}_3$, then Π is the dg path algebra given by



where black arrows have degree 0 and red arrows have degree -1. The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$.

It is connective and $H^0(\Pi)$ is the preprojective algebra of Q.

It is smooth and the perfectly valued derived category $\mathrm{pvd}(\Pi)$ is a 2-Calabi–Yau triangulated category.

A categorification of the root system

Each simple dg Π-module S_i corresponding to $i \in \Delta_0$ is 2-spherical and yields a spherical twist functor (see Seidel–Thomas, 2001)

$$T_i : \mathrm{pvd}(\Pi) \longrightarrow \mathrm{pvd}(\Pi)$$

The map $[S_i] \mapsto \alpha_i$ induces an isomorphism between $K_0(\mathrm{pvd}(\Pi))$ and the root lattice of Δ.

The action of T_i on $K_0(\mathrm{pvd}(\Pi))$ identifies with the action of the simple reflection $s_i \in$ W on the root lattice.

Categories associated with reduced words

For $k \in \mathbb{Z}$, we define the following object of $\mathrm{pvd}(\Pi)$:

$$M_k^{\boldsymbol{i}} = \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(S_{i_k}) & \text{if } k \geq 1, \\ \Sigma T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(S_{i_k}) & \text{if } k \leq 0, \end{cases}$$

where Σ denotes the suspension functor of $\mathrm{pvd}(\Pi)$.

Definition

- The repetition category $\mathcal{R}(\underline{\boldsymbol{i}})$ is the full additive subcategory of $\mathrm{pvd}(\Pi)$ generated by the indecomposable objects $M_k^{\boldsymbol{i}}$ for $k \in \mathbb{Z}$.
- The category of representations $\mathcal{C}(\underline{\boldsymbol{i}})$ is the full subcategory of $\mathcal{R}(\underline{\boldsymbol{i}})$ whose objects have cohomology concentrated in degree zero.

Categorification of combinatorial AR quivers

Proposition [Buan–Iyama–Reiten–Scott, 2009; Amiot–Iyama–Reiten–Todorov, 2012]

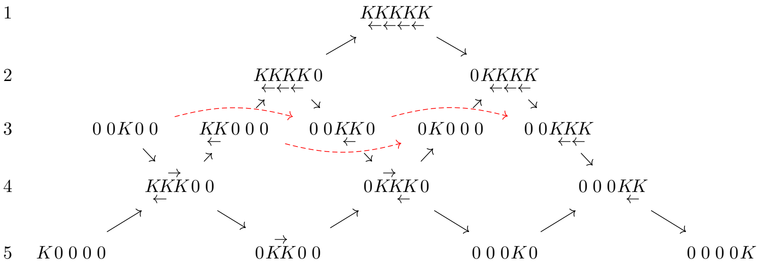
- The indecomposable objects of $\mathcal{C}(\underline{\boldsymbol{i}})$ are the $M_k^{\underline{\boldsymbol{i}}}$ for $1 \leq k \leq N$, where N is the length of w_0.
- As representations of the preprojective algebra, they coincide with certain modules called layers.
- If $\underline{\boldsymbol{i}}$ is a source sequence for an orientation Q of Δ, then $\mathcal{C}(\underline{\boldsymbol{i}})$ is equivalent to $\operatorname{mod} KQ$ (as a K-linear category).

Theorem [C.]

The combinatorial AR quiver $\Upsilon_{\underline{\boldsymbol{i}}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\underline{\boldsymbol{i}})$ by removing all arrows parallel to paths of length ≥ 2.

Example

For the previous example in type A_5, the Gabriel quiver of $\mathcal{C}(\underline{\boldsymbol{i}})$ is the following:



Categorification of the mesh-additivity property

Theorem [C.]

Given a vertex $x \in (\widehat{\Upsilon}_{\underline{\boldsymbol{i}}})_0$, choose an ordering $y_1, \ldots, y_t$ of the set of abutters $V_{\underline{\boldsymbol{i}}}(x)$ such that $k \leq l$ whenever there is a path from y_k to y_l in $\widehat{\Upsilon}_{\underline{\boldsymbol{i}}}$. Then there are indecomposable objects $X_1, \ldots, X_{t-1}$ in $\mathcal{R}(\underline{\boldsymbol{i}})$ and a diagram of the form

$$
\begin{array}{ccccccccccc}
\Sigma^{-1} M^{\boldsymbol{i}}_{x} = X_t & \longrightarrow & X_{t-1} & \to \cdots \to & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 = M^{\boldsymbol{i}}_{\tau x} \\
\nwarrow & & \swarrow & & \nwarrow & \swarrow & \nwarrow & \swarrow & \\
& M^{\boldsymbol{i}}_{y_t} & & & & M^{\boldsymbol{i}}_{y_2} & & M^{\boldsymbol{i}}_{y_1} &
\end{array}
$$

where the triangles above are distinguished triangles in $\mathrm{pvd}(\Pi)$.

Towards a derived category

We would like to define a category $\mathcal{D}(\underline{\boldsymbol{i}})$ analog to $\mathcal{D}^b(\operatorname{mod} KQ)$.

Problem: $\mathcal{C}(\underline{\boldsymbol{i}})$ is not an abelian category in general!

We have a faithful functor:

$$\mathcal{D}^b(\operatorname{mod} KQ) \longrightarrow \operatorname{pvd}(\Pi)$$

If $\underline{\boldsymbol{i}}$ is a source sequence for Q, then the objects in the image of this inclusion are those of $\mathcal{R}(\underline{\boldsymbol{i}})$.

Problem: The functor above is not full!

The combinatorially-derived category

We construct a certain ideal $\mathcal{I}$ of $\mathcal{R}(\underline{\boldsymbol{i}})$ and define the c-derived category $\mathcal{D}(\underline{\boldsymbol{i}})$ as the quotient $\mathcal{R}(\underline{\boldsymbol{i}})/\mathcal{I}$.

Proposition

If $\underline{\boldsymbol{i}}$ is a source sequence for an orientation Q of Δ, then $\mathcal{D}(\underline{\boldsymbol{i}})$ is equivalent to $\mathcal{D}^b(\operatorname{mod} KQ)$ (as a K-linear category).

Theorem [C.]

The combinatorial repetition quiver $\widehat{\Upsilon}_{\underline{\boldsymbol{i}}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{D}(\underline{\boldsymbol{i}})$ by removing all arrows parallel to paths of length ≥ 2.

The Euler form

The category $\mathcal{D}(\underline{\boldsymbol{i}})$ inherits the suspension functor Σ from $\mathrm{pvd}(\Pi)$.

We define the Euler form by

$$\langle M, N\rangle = \sum_{k\in\mathbb{Z}}(-1)^k \dim_K \mathrm{Hom}_{\mathcal{D}(\underline{\boldsymbol{i}})}(M, \Sigma^k N).$$

Theorem [C.]

For $x, y \in (\widehat{\Upsilon}_{\underline{\boldsymbol{i}}})_0$, we have

$$\langle M_x^{\boldsymbol{i}}, M_y^{\boldsymbol{i}}\rangle + \langle M_y^{\boldsymbol{i}}, M_x^{\boldsymbol{i}}\rangle = (\rho(x), \rho(y)),$$

where $(-,-)$ is the Cartan-Killing form on the root lattice of Δ.

Thank you for your attention!

References I

[AIRT12] Claire Amiot, Osamu Iyama, Idun Reiten, and Gordana Todorov. *Preprojective algebras and c-sortable words*. Proc. Lond. Math. Soc. (3) **104** (2012), no. 3, 513–539.

[Béd99] Robert Bédard. *On commutation classes of reduced words in Weyl groups*. European J. Combin. **20** (1999), no. 6, 483–505.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. *Cluster structures for 2-Calabi–Yau categories and unipotent groups*. Compos. Math. **145** (2009), no. 4, 1035–1079.

[Can25] Ricardo Canesin. *A categorification of combinatorial Auslander-Reiten quivers*. 2025. arXiv: 2505.06147 [math.RT].

References II

[FO21] Ryo Fujita and Se-jin Oh. *Q-data and representation theory of untwisted quantum affine algebras*. Comm. Math. Phys. **384** (2021), no. 2, 1351–1407.

[HL15] David Hernandez and Bernard Leclerc. *Quantum Grothendieck rings and derived Hall algebras*. J. Reine Angew. Math. **701** (2015), 77–126.

[OS19] Se-jin Oh and Uhi Rinn Suh. *Combinatorial Auslander–Reiten quivers and reduced expressions*. J. Korean Math. Soc. **56** (2019), no. 2, 353–385.

[ST01] Paul Seidel and Richard Thomas. *Braid group actions on derived categories of coherent sheaves*. Duke Math. J. **108** (2001), no. 1, 37–108.