Categorifying twisted Auslander-Reiten quivers

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Ricardo Canesin

Université Paris Cité

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Quantum affine algebras and quiver representations

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and let $U_q'(\widehat{\mathfrak{g}})$ be its quantum affine algebra.

If $\mathfrak{g}$ is of type ADE, then the representation theory of $U_q'(\widehat{\mathfrak{g}})$ is intimately connected with the representation theory of a Dynkin quiver Q of the same type (cf. Hernandez-Leclerc '15).

For general type, the combinatorics coming from the AR theory of $\mathrm{Rep}(Q)$ and $\mathcal{D}^b(\mathrm{Rep}(Q))$ have been generalized to study $U_q'(\widehat{\mathfrak{g}})$.

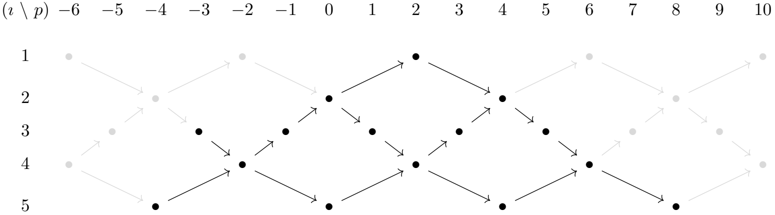
- Oh-Suh '19: Introduced twisted AR quivers.
- Fujita-Oh '21: Introduced Q-data to better study such quivers.

Problem

Can we categorify these combinatorics? Is there a "category of representations" for a Q-datum?

Twisted AR quiver: type B_3

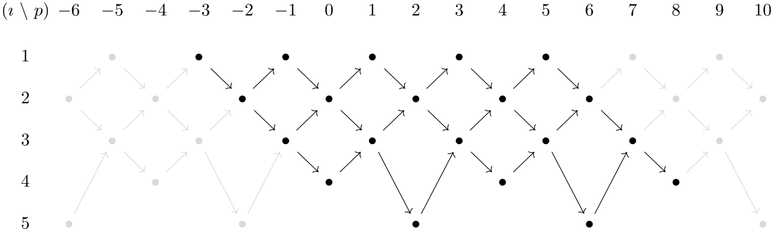
Highlighted subquiver: a twisted AR quiver of type B_3.



The vertices are in bijection with the positive roots of A_5.

Twisted AR quiver: type C_4

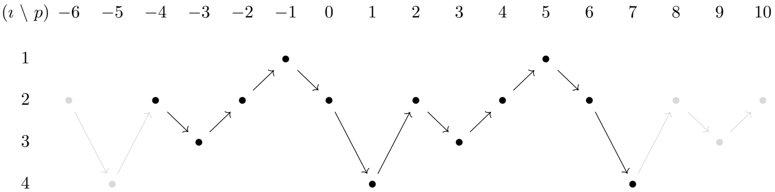
Highlighted subquiver: a twisted AR quiver of type C_4.



The vertices are in bijection with the positive roots of D_5.

Twisted AR quiver: type G_2

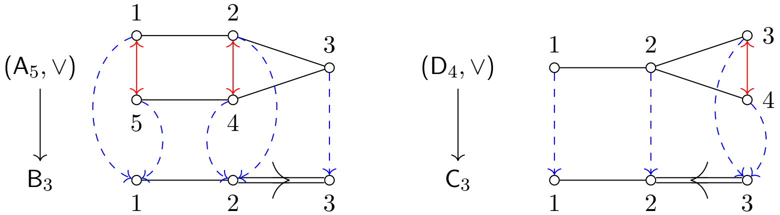
Highlighted subquiver: a twisted AR quiver of type G_2.



The vertices are in bijection with the positive roots of D_4.

Unfoldings

Let (Δ, σ) be the unfolding of $\mathfrak{g}$, where Δ is a simply-laced Dynkin diagram and σ is an automorphism of Δ.



Denote $I = \Delta_0 / \langle \sigma \rangle$. Quotient map: $\imath \in \Delta_0 \mapsto \bar{\imath} \in I$.

If $i \in I$, we define $d_i = |i| \in \{1, r\}$, where r is the order of σ.

Q-data

Definition [Fujita-Oh '21]

A Q-datum for $\mathfrak{g}$ is a triple $\mathcal{Q} = (\Delta, \sigma, \xi)$ where (Δ, σ) is the unfolding of $\mathfrak{g}$ and $\xi : \Delta_0 \to \mathbb{Z}$ satisfies:

- For adjacent $\imath, \jmath \in \Delta_0$ with $d_{\bar{\imath}} = d_{\bar{\jmath}}$, we have $|\xi_\imath - \xi_\jmath| = d_{\bar{\imath}} = d_{\bar{\jmath}}$.
- For adjacent $i, j \in I$ with $d_i = 1 < d_j = r$, there is a unique $\jmath \in j$ such that $|\xi_\imath - \xi_\jmath| = 1$ and $\xi_{\sigma^k(\jmath)} = \xi_\jmath - 2k$ for any $1 \leq k < r$, where $i = \{\imath\}$.

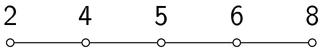
Q-data

Definition [Fujita-Oh '21]

A Q-datum for $\mathfrak{g}$ is a triple $\mathcal{Q} = (\Delta, \sigma, \xi)$ where (Δ, σ) is the unfolding of $\mathfrak{g}$ and $\xi : \Delta_0 \to \mathbb{Z}$ is a "generalized" height function.

If $\sigma = \mathrm{id}$, a Q-datum is the same as a Dynkin quiver of type Δ with a classical height function (i.e. $\xi_\imath = \xi_\jmath + 1$ for an arrow $\imath \to \jmath$).

For example, the following height function defines a Q-datum of type B_3:



Repetition quivers and twisted AR quivers

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum. The repetition quiver $\widehat{\Delta}^\sigma$ has vertices:

$$\widehat{\Delta}^\sigma_0 = \{(\imath, p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_\imath \in 2d_{\bar{\imath}}\mathbb{Z}\}.$$

There is an arrow $(\imath, p) \longrightarrow (\jmath, s)$ if $\imath$ and $\jmath$ are adjacent in Δ and $s - p = \min(d_{\bar{\imath}}, d_{\bar{\jmath}})$.

The twisted Auslander-Reiten quiver $\Gamma_{\mathcal{Q}}$ is the full subquiver of $\widehat{\Delta}^\sigma$ with vertex set

$$\{(\imath, p) \in \widehat{\Delta}^\sigma_0 \mid \xi_{\imath^*} - rh^\vee < p \leq \xi_\imath\},$$

where:

- $h^\vee$: dual Coxeter number of $\mathfrak{g}$.
- $\imath \mapsto \imath^*$: involution on Δ_0 induced by the longest element w_0 of the Weyl group associated with Δ.

Assigning positive roots

A compatible reading of $\Gamma_{\mathcal{Q}}$ is an ordering $(\imath_1, p_1), \ldots, (\imath_N, p_N)$ of $(\Gamma_{\mathcal{Q}})_0$ such that

$$\exists \text{ a path } (\imath_k, p_k) \rightsquigarrow (\imath_l, p_l) \text{ in } \Gamma_{\mathcal{Q}} \implies k > l.$$

Let R be the root system of Δ. To the vertex $(\imath_k, p_k)$, we assign

$$s_{\imath_1} s_{\imath_2} \cdots s_{\imath_{k-1}}(\alpha_{\imath_k}) \in \mathsf{R},$$

where $\alpha_\imath \in \mathsf{R}$ is the simple root associated with $\imath \in \Delta_0$ and $s_\imath$ is the corresponding simple reflection in the Weyl group W of R.

Theorem [Oh-Suh 19', Fujita-Oh 21']

The sequence $\underline{\boldsymbol{i}} = (\imath_1, \ldots, \imath_N)$ gives a reduced word for the longest element $w_0 \in \mathsf{W}$, hence the roots defined above are all the positive roots of R, without repetition. Moreover, the assignment independs on the choice of compatible reading of $\Gamma_{\mathcal{Q}}$.

Example: type B_3

If $\mathcal{Q}$ is the Q-datum of type B_3 from the previous example, then $\Gamma_{\mathcal{Q}}$ is given as follows:

$$
\begin{tikzcd}
(\imath \setminus p) & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & & & & & & 11111 \arrow[drr] & & & & & & \\
2 & & & & & 11110 \arrow[urr] \arrow[dr] & & & & 01111 \arrow[dr] & & & & \\
3 & & 00100 \arrow[dr] & & 11000 \arrow[ur] & & 00110 \arrow[dr] & & 01000 \arrow[ur] & & 00111 \arrow[dr] & & & \\
4 & & & 11100 \arrow[ur] \arrow[drr] & & & & 01110 \arrow[ur] \arrow[drr] & & & & 00011 \arrow[drr] & & \\
5 & 10000 \arrow[urr] & & & & 01100 \arrow[urr] & & & & 00010 \arrow[urr] & & & & 00001
\end{tikzcd}
$$

In each vertex, we indicate the corresponding positive root of A_5.

The ambient category

We fix a field K and an orientation Q° of Δ.

Let Π be the 2-Calabi-Yau completion of the path algebra KQ° (see Keller, 2011).

It is a smooth and connective dg algebra whose perfectly valued derived category $\mathrm{pvd}(\Pi)$ is a 2-Calabi-Yau triangulated category.

For example, if $\Delta = \mathsf{A}_3$, then Π is the dg path algebra given by

$$\begin{tikzcd} \bullet \arrow[loop above, "t_1"] \arrow[r, shift left, "\alpha"] & \bullet \arrow[loop above, "t_2"] \arrow[l, shift left, "\overline{\alpha}"] \arrow[r, shift left, "\beta"] & \bullet \arrow[loop above, "t_3"] \arrow[l, shift left, "\overline{\beta}"] \end{tikzcd}$$

where black arrows have degree 0 and red arrows have degree -1. The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$.

A categorification of the root system

Let $S_\imath$ denote the simple Π-module corresponding to $\imath \in \Delta_0$. It is 2-spherical and gives rise to the spherical twist functor (see Seidel-Thomas, 2001)

$$T_\imath : \mathrm{pvd}(\Pi) \longrightarrow \mathrm{pvd}(\Pi)$$

The map sending $[S_\imath]$ to $\alpha_\imath$ defines an isomorphism between $K_0(\mathrm{pvd}(\Pi))$ and the root lattice of Δ. The action of $T_\imath$ is identified with the action of the simple reflection $s_\imath$.

We can use $\mathrm{pvd}(\Pi)$ to categorify the previous constructions.

“Representations” of a Q-datum

Let $\mathcal{Q} = (\Delta, \sigma, \xi)$ be a Q-datum for $\mathfrak{g}$ and take a reduced word $\underline{\boldsymbol{i}} = (\imath_1, \ldots, \imath_N)$ for w_0 coming from a compatible reading of $\Gamma_{\mathcal{Q}}$.

For $1 \leq k \leq N$, define

$$M_k^{\boldsymbol{i}} = T_{\imath_1} T_{\imath_2} \cdots T_{\imath_{k-1}}(S_{\imath_k}) \in \operatorname{pvd}(\Pi).$$

We define the category of representations $\mathcal{C}(\mathcal{Q})$ to be the strictly full additive subcategory generated by these objects. It depends only on $\mathcal{Q}$.

Proposition

If $\mathcal{Q} = Q$ is a Dynkin quiver of type ADE, then $\mathcal{C}(\mathcal{Q})$ is equivalent to $\operatorname{mod} KQ$ (as a K-linear category).

"Representations" of a Q-datum

Proposition [C.]

The objects of $\mathcal{C}(\mathcal{Q})$ are dg modules whose cohomology is concentrated in degree zero. In particular, $\mathcal{C}(\mathcal{Q})$ can be viewed as a full subcategory of $\operatorname{mod} H^0(\Pi)$.

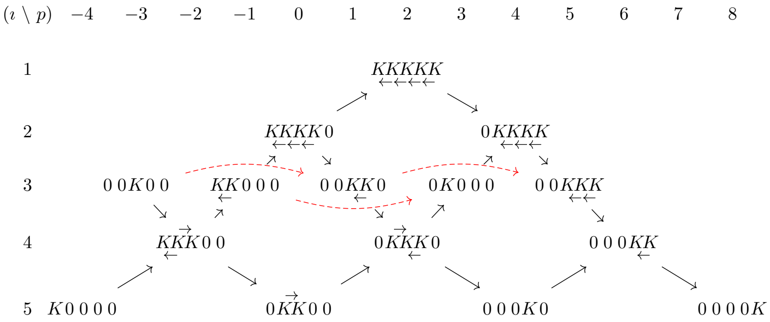
Notice that $H^0(\Pi)$ is isomorphic to the preprojective algebra of type Δ.

Theorem [C.]

The twisted AR quiver $\Gamma_{\mathcal{Q}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\mathcal{Q})$ by removing all arrows parallel to paths of length ≥ 2.

Example: the category $\mathcal{C}(\mathcal{Q})$

If $\mathcal{Q}$ is the Q-datum of type B_3 from previous examples, then $\mathcal{C}(\mathcal{Q})$ can be described by the following picture.



Objects are depicted as preprojective representations. Arrows between objects correspond to irreducible morphisms in $\mathcal{C}(\mathcal{Q})$.

The inverse quantum Cartan matrix

The quantum Cartan matrix $C(q)$ is a certain deformation of the Cartan matrix C of the simple Lie algebra $\mathfrak{g}$.

For example, if $\mathfrak{g} = \mathfrak{so}_5(\mathbb{C})$, we have

$$C = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad C(q) = \begin{bmatrix} q^2 + q^{-2} & -1 \\ -q - q^{-1} & q + q^{-1} \end{bmatrix}$$

Let $\widetilde{C}(q) = (\widetilde{C}_{ij}(q))$ be the inverse of $C(q)$. Each entry has a power series expansion: $\widetilde{C}_{ij}(q) = \sum_{u \geq 0} \widetilde{c}_{ij}(u) q^u$.

Example: $\widetilde{C}_{21}(q) = \frac{q^2+q^4}{1+q^6} = q^2 + q^4 - q^8 - q^{10} + q^{14} + \cdots$.

The simply-laced case

Suppose $\mathfrak{g}$ is of type ADE and let Q be an orientation of its Dynkin diagram. Let $\xi : Q_0 \to \mathbb{Z}$ be a height function on Q.

One can naturally identify $\operatorname{ind}(\mathcal{D}^b(\operatorname{mod} KQ))$ with the set

$$\widehat{\Delta}_0 = \{(i,p) \in I \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$

Each $(i,p) \in \widehat{\Delta}_0$ gives an indecomposable object $H_Q(i,p)$.

Theorem [Hernandez-Leclerc '15, Fujita '22]

For $(i,p),(j,s) \in \widehat{\Delta}_0$ with $s \geq p$, we have

$$\widetilde{c}_{ij}(s-p+1) = \langle H_Q(i,p), H_Q(j,s) \rangle$$

where $\langle -,- \rangle$ is the Euler form on $\mathcal{D}^b(\operatorname{mod} KQ)$. Every nonzero coefficient $\widetilde{c}_{ij}(u)$ can be written in this way.

The "derived category" of a Q-datum

For the general case, Fujita-Oh use twisted AR quivers to give a combinatorial formula for the coefficients $\widetilde{c}_{ij}(u)$, which we categorify as follows.

Let $\mathcal{R}(\mathcal{Q})$ be the full additive subcategory of $\mathrm{pvd}(\Pi)$ generated by objects of the form $\Sigma^k M$ for $M \in \mathcal{C}(\mathcal{Q})$ and $k \in \mathbb{Z}$.

We construct a certain ideal $\mathcal{I}$ of $\mathcal{R}(\mathcal{Q})$ and define the derived category $\mathcal{D}(\mathcal{Q})$ as the quotient $\mathcal{R}(\mathcal{Q})/\mathcal{I}$.

Proposition

If $\mathcal{Q} = Q$ is a Dynkin quiver of type ADE, then $\mathcal{D}(\mathcal{Q})$ is equivalent to $\mathcal{D}^b(\mathrm{mod}\, KQ)$ (as a K-linear category).

Final ingredients

- "Euler form": for $M, N \in \mathcal{D}(\mathcal{Q})$, we define

$$\langle M, N \rangle_{\mathcal{Q}} = \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \mathrm{Ext}^k_{\mathcal{Q}}(M, N),$$

  where $\mathrm{Ext}^k_{\mathcal{Q}}(M, N) = \mathrm{Hom}_{\mathcal{D}(\mathcal{Q})}(M, \Sigma^k N)$.
- Fujita-Oh define a twisted Coxeter element $\tau_{\mathcal{Q}} \in \mathsf{W}\sigma$. We can lift it to an equivalence $\tau_{\mathcal{Q}} : \mathcal{D}(\mathcal{Q}) \longrightarrow \mathcal{D}(\mathcal{Q})$.
- Folded repetition quiver: we define

$$\widehat{I} = \{(i, p) \in I \times \mathbb{Z} \mid \exists (\imath, p) \in \widehat{\Delta}^{\sigma}_0, \bar{\imath} = i\}.$$

  We can construct bijections $\widehat{I} \to \widehat{\Delta}^{\sigma}_0$ and $\widehat{\Delta}^{\sigma}_0 \to \mathrm{ind}(\mathcal{D}(\mathcal{Q}))$. Let $H_{\mathcal{Q}} : \widehat{I} \to \mathrm{ind}(\mathcal{D}(\mathcal{Q}))$ be their composition.

Reinterpreting Fujita-Oh's formula

Theorem [Fujita-Oh '21, C.]

For $(i,p),(j,s) \in \widehat{I}$ with $p \geq s$ and $\max\{d_i, d_j\} = r$, we have

$$\widetilde{c}_{ij}(p - s + d_i) = \left\langle H_{\mathcal{Q}}(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_{\mathcal{Q}}^{k}(H_{\mathcal{Q}}(i,p)) \right\rangle_{\mathcal{Q}}.$$

Every nonzero coefficient $\widetilde{c}_{ij}(u)$ (where $\max\{d_i, d_j\} = r$) can be written in this way.

Remarks:

- Fujita-Oh's formula works without any restriction on d_i and d_j.
- In type B, the formula above also works in general.

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