

Context

- In [HL15], Hernandez and Leclerc showed that a specialization of the quantum Grothendieck ring $\mathcal{K}_t(\mathcal{C}_\mathbb{Z})$ of a quantum loop algebra of type ADE is isomorphic to the derived Hall algebra of the derived category of a Dynkin quiver Q of the same type.
- Motivated by this result, Fujita and Oh [FO21] introduced the notion of a **Q-datum** and confirmed that the **twisted Auslander–Reiten quivers** should play the role of the Auslander–Reiten quiver of Q for non-simply laced types.
- We lift Fujita–Oh’s combinatorics to the categorical level in the slightly more general setting of **combinatorial Auslander–Reiten quivers**, as defined by Oh and Suh [OS19].

Notation for root systems

- Let R be an irreducible root system with Dynkin diagram Δ of type ADE. For each vertex $i \in \Delta_0$, we denote by $\alpha_i \in R$ the corresponding simple root.
- Denote by W the associated Weyl group and by s_i the simple reflection for $i \in \Delta_0$.
- Let $w_0 \in W$ be the longest element. It induces an involution $i \mapsto i^*$ on Δ_0 determined by $w_0(\alpha_i) = -\alpha_{i^*}$.
- We fix a reduced word $\mathbf{i} = (i_1, i_2, \dots, i_N) \in \Delta_0^N$ of w_0 . Its extension $\hat{\mathbf{i}} = (i_k)_{k \in \mathbb{Z}}$ is defined by imposing $i_{k+N} = i_k^*$ for all $k \in \mathbb{Z}$.

Combinatorial Auslander–Reiten quivers [OS19]

- The **combinatorial repetition quiver** $\hat{\Upsilon}_{\mathbf{i}}$ is the quiver with vertex set \mathbb{Z} where there is an arrow from k to l if $k > l$, i_k and i_l are adjacent in Δ and there is no index $l < j < k$ such that $i_j = i_l$ or $i_j = i_k$.
- The **combinatorial Auslander–Reiten quiver** $\Upsilon_{\mathbf{i}}$ is the full subquiver of $\hat{\Upsilon}_{\mathbf{i}}$ with vertex set $\{1, 2, \dots, N\}$, where N denotes the length of w_0 or, equivalently, the number of positive roots in R .
- We define the **coordinate map** $\rho : (\hat{\Upsilon}_{\mathbf{i}})_0 \rightarrow R$ by

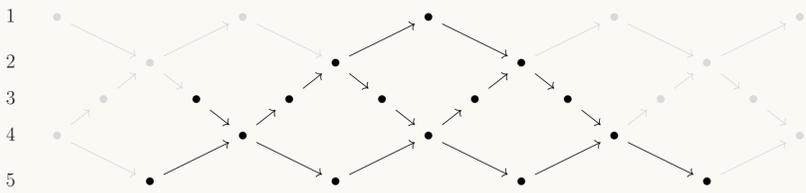
$$\rho(k) = \begin{cases} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k \geq 1, \\ -s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0. \end{cases}$$

Theorem 1: [Béd99; OS19]

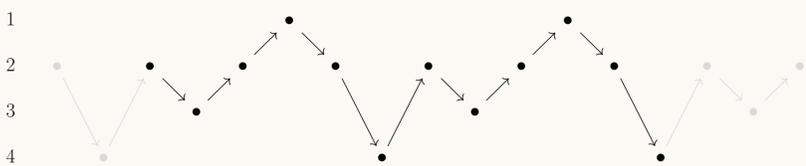
Let Q be a Dynkin quiver of type Δ . Let Γ be the Auslander–Reiten quiver of the path algebra of Q over a field. If $\mathbf{i} = (i_1, \dots, i_N)$ is a reduced word for w_0 which is also a source sequence for Q , then there is an isomorphism of quivers $\varphi : \Gamma \rightarrow \Upsilon_{\mathbf{i}}$ such that the composition $\rho \circ \varphi_0 : \Gamma_0 \rightarrow R$ restricts to the bijection between indecomposable representations of Q and positive roots given by Gabriel’s theorem.

Examples

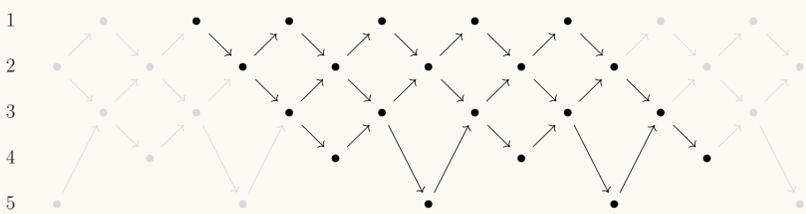
- $\Delta = A_5$ and $\mathbf{i} = (5, 4, 3, 2, 5, 3, 1, 4, 3, 2, 5, 3, 4, 3, 5)$ (corresponds to a Q-datum of type B_3):



- $\Delta = D_4$ and $\mathbf{i} = (4, 2, 1, 2, 3, 2, 4, 2, 1, 2, 3, 2)$ (corresponds to a Q-datum of type G_2):



- $\Delta = D_5$ and $\mathbf{i} = (4, 3, 2, 1, 5, 3, 2, 1, 4, 3, 2, 1, 5, 3, 2, 1, 4, 3, 2, 1)$ (corresponds to a Q-datum of type C_4):



Meshes in combinatorial AR quivers

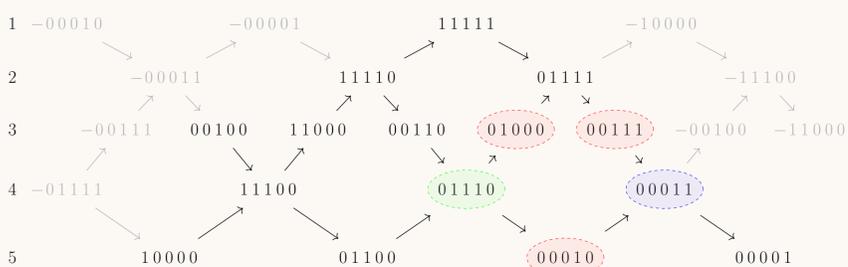
Take a vertex $x \in (\hat{\Upsilon}_{\mathbf{i}})_0 = \mathbb{Z}$ of the combinatorial repetition quiver.

- The **translation** of x is the smallest integer $\tau x \in (\hat{\Upsilon}_{\mathbf{i}})_0$ such that $x < \tau x$ and $i_x = i_{\tau x}$.
- The **set of abutters*** of x is the subset $V_i(x) \subset (\hat{\Upsilon}_{\mathbf{i}})_0$ formed by the vertices y such that $x < y < \tau x$ and i_y is adjacent to i_x in Δ .

*An abutter is the owner of an adjacent property.

Example: Coordinate map and meshes

For the previous example in type A_5 , over each vertex x of $\hat{\Upsilon}_{\mathbf{i}}$ we write the root $\rho(x)$ given by the coordinate map. The numbers indicate the coefficient of each simple root α_i in $\rho(x)$.



The translation of the blue vertex is the green one, and the red vertices form the set of abutters of the blue vertex.

Theorem 2: [FO21; Can25] Mesh-additivity property

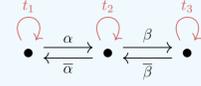
The coordinate map satisfies

$$\rho(\tau x) + \rho(x) = \sum_{y \in V_i(x)} \rho(y)$$

for all $x \in (\hat{\Upsilon}_{\mathbf{i}})_0$. This is a generalization of Fujita–Oh’s \mathfrak{g} -additive property.

Towards a categorification

- Let Q be an orientation of Δ . We define Π to be the **2-dimensional Ginzburg dg algebra** associated with Q over some field K .
- For example, if $\Delta = A_3$, then Π is the dg path algebra given by



where black arrows have degree 0 and red arrows have degree -1 . The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \bar{\alpha}] + [\beta, \bar{\beta}]$.

- Π is a connective dg algebra whose zeroth cohomology is the preprojective algebra of Q . It is smooth and its perfectly valued derived category $\text{pvd}(\Pi)$ is a 2-Calabi–Yau triangulated category.
- Each simple dg module S_i corresponding to $i \in \Delta_0$ is 2-spherical and yields a **spherical twist functor** $T_i : \text{pvd}(\Pi) \rightarrow \text{pvd}(\Pi)$, which is an autoequivalence of $\text{pvd}(\Pi)$.
- The map $[S_i] \mapsto \alpha_i$ induces an isomorphism between the Grothendieck group of $\text{pvd}(\Pi)$ and the root lattice of Δ . The action of T_i on the Grothendieck group identifies with the action of the simple reflection $s_i \in W$.

Categories associated with reduced words [Can25]

- For $k \in \mathbb{Z}$, we define the following object of $\text{pvd}(\Pi)$:

$$M_k^i = \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(S_{i_k}) & \text{if } k \geq 1, \\ \Sigma T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{k+1}}^{-1}(S_{i_k}) & \text{if } k \leq 0, \end{cases}$$

where Σ denotes the suspension functor of $\text{pvd}(\Pi)$.

- The **repetition category** $\mathcal{R}(i)$ is the full additive subcategory of $\text{pvd}(\Pi)$ generated by the indecomposable objects M_k^i for $k \in \mathbb{Z}$.
- The **category of representations** $\mathcal{C}(i)$ is the full subcategory of $\mathcal{R}(i)$ whose objects have cohomology concentrated in degree zero. In particular, we may view its objects as representations of the preprojective algebra.

Proposition 3: [AIRT12]

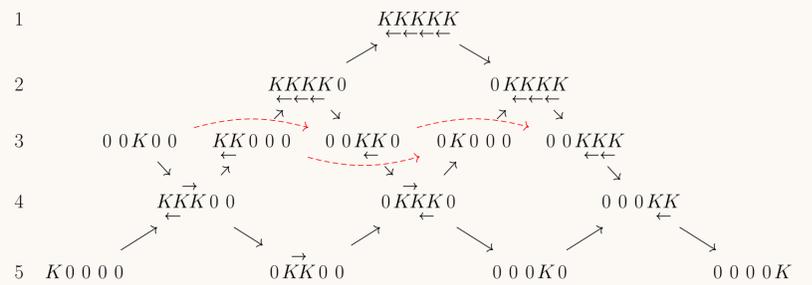
- The indecomposable objects of $\mathcal{C}(i)$ are the M_k^i for $1 \leq k \leq N$. As representations of the preprojective algebra, they coincide with certain modules called **layers** and defined in [AIRT12].
- If \mathbf{i} is a source sequence for an orientation Q of Δ , then $\mathcal{C}(i)$ is equivalent to the category of representations of the quiver Q as a K -linear category.

Theorem 4: [Can25] Categorification of combinatorial AR quivers

The combinatorial AR quiver $\Upsilon_{\mathbf{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(i)$ by removing all arrows parallel to paths of length at least two.

Example

For the previous example in type A_5 , the Gabriel quiver of $\mathcal{C}(i)$ is the following:



Each indecomposable object is viewed as a representation of the preprojective algebra of type A_5 .

Theorem 5: [Can25] Categorification of the mesh-additivity property

Given a vertex $x \in (\hat{\Upsilon}_{\mathbf{i}})_0$, choose an ordering y_1, \dots, y_t of the set of abutters $V_i(x)$ such that $k \leq l$ whenever there is a path from y_k to y_l in $\hat{\Upsilon}_{\mathbf{i}}$. Then there are indecomposable objects X_1, X_2, \dots, X_{t-1} in $\mathcal{R}(i)$ and a diagram of the form

$$\Sigma^{-1} M_x^i = X_t \longrightarrow X_{t-1} \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 = M_{\tau x}^i$$

$\swarrow \quad \searrow \quad \quad \quad \swarrow \quad \searrow \quad \quad \quad \swarrow \quad \searrow$
 $M_{y_t}^i \quad \quad \quad M_{y_2}^i \quad \quad \quad M_{y_1}^i$

where the triangles above are distinguished triangles in $\text{pvd}(\Pi)$.

The combinatorially derived category

In [Can25], we define the **combinatorially derived category** $\mathcal{D}(i)$ as a quotient of the repetition category $\mathcal{R}(i)$ by a certain ideal of morphisms.

- If \mathbf{i} is a source sequence for an orientation Q of Δ , then $\mathcal{D}(i)$ is equivalent to the bounded derived category of representations of the quiver Q as a K -linear category.
- The combinatorial repetition quiver $\hat{\Upsilon}_{\mathbf{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{D}(i)$ by removing all arrows parallel to paths of length at least two.
- When \mathbf{i} comes from a Q-datum, we are able to interpret certain results of Fujita and Oh in terms of $\mathcal{D}(i)$. For example, Fujita–Oh’s generalized Coxeter element yields a partially Serre functor on $\mathcal{D}(i)$.
- The distinguished triangles coming from the categorification of the mesh-additivity property could potentially allow us to define a type of derived Hall algebra for $\mathcal{D}(i)$.

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