A categorification of combinatorial Auslander–Reiten quivers

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Context

- In [Béd99], Bédard showed how to construct the Auslander–Reiten quiver of a Dynkin quiver from the Coxeter combinatorics of the corresponding Weyl group W.
- Oh and Suh used similar ideas in [OS19] to introduce a purely combinatorial analog of the AR quiver associated with any reduced word in W.
- Building on their work, Fujita and Oh [FO21] studied a particular class of such quivers called twisted AR quivers. Their framework has found important applications in the representation theory of quantum algebras.

Notation for root systems

- Let R be an irreducible root system with Dynkin diagram Δ of type ADE. For each vertex $i \in \Delta_0$, we denote by $\alpha_i \in \mathrm{R}$ the corresponding simple root.
- Denote by W the associated Weyl group and by s_i the simple reflection for $i \in \Delta_0$.
- Let $w_0 \in \mathrm{W}$ be the longest element. It induces an involution $i \mapsto i^*$ on Δ_0 determined by $w_0(\alpha_i) = -\alpha_{i^*}$.
- We fix a reduced word $\boldsymbol{i} = (i_1, i_2, \dots, i_N) \in \Delta_0^N$ of w_0. Its extension $\widehat{\boldsymbol{i}} = (i_k)_{k\in\mathbb{Z}}$ is defined by imposing $i_{k+N} = i_k^*$ for all $k \in \mathbb{Z}$.

Combinatorial Auslander–Reiten quivers [OS19]

- The **combinatorial repetition quiver** $\widehat{\Upsilon}_{\boldsymbol{i}}$ is the quiver with vertex set $\mathbb{Z}$ where there is an arrow from k to l if $k > l$, i_k and i_l are adjacent in Δ and there is no index $l < j < k$ such that $i_j = i_l$ or $i_j = i_k$.
- The **combinatorial Auslander–Reiten quiver** $\Upsilon_{\boldsymbol{i}}$ is the full subquiver of $\widehat{\Upsilon}_{\boldsymbol{i}}$ with vertex set $\{1, 2, \dots, N\}$.
- We define the **coordinate map** $\rho : (\widehat{\Upsilon}_{\boldsymbol{i}})_0 \to \mathrm{R}$ by

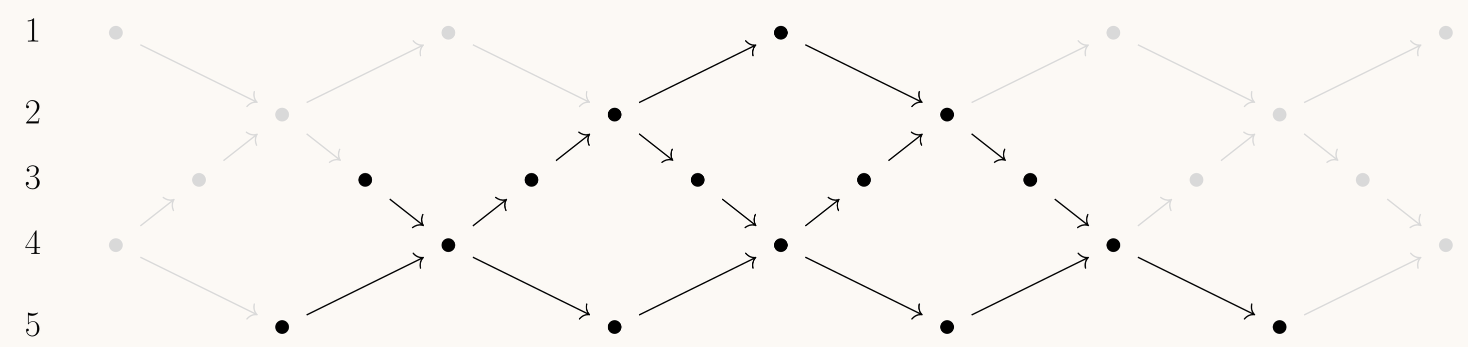
$$\rho(k) = \begin{cases} s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } k \geq 1, \\ -s_{i_0}s_{i_{-1}}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } k \leq 0. \end{cases}$$

Theorem 1: [Béd99; OS19]

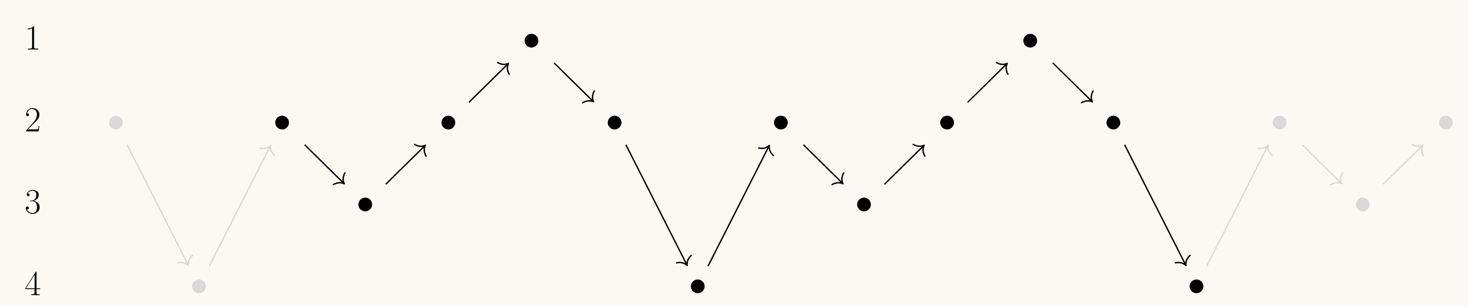
Let Q be a Dynkin quiver of type Δ. Let Γ be the Auslander–Reiten quiver of the path algebra of Q over a field. If $\boldsymbol{i} = (i_1, \dots, i_N)$ is a reduced word for w_0 which is also a source sequence for Q, then there is an isomorphism of quivers $\varphi : \Gamma \to \Upsilon_{\boldsymbol{i}}$ such that the composition $\rho \circ \varphi_0 : \Gamma_0 \to \mathrm{R}$ restricts to the bijection between indecomposable representations of Q and positive roots given by Gabriel's theorem.

Examples

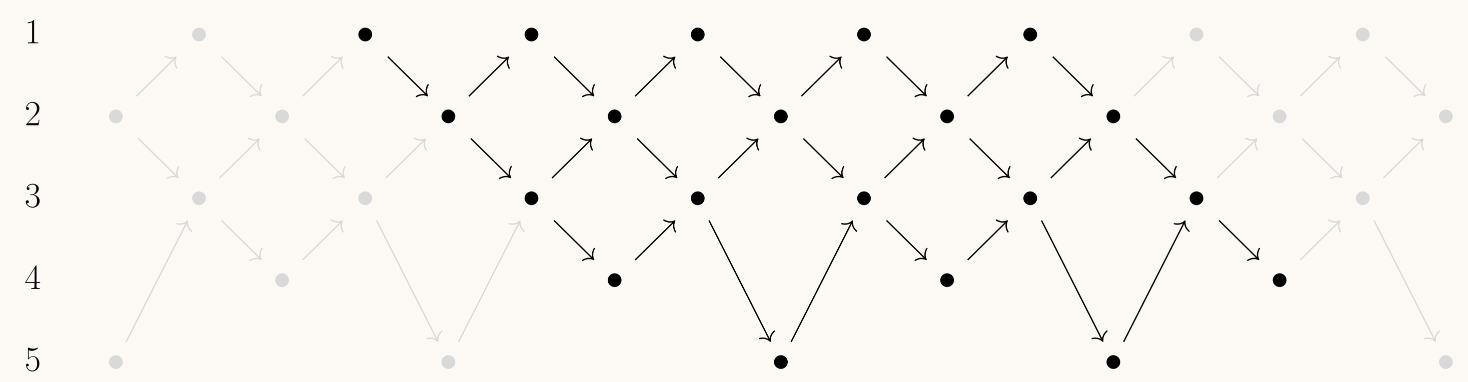
- $\Delta = \mathsf{A}_5$ and $\boldsymbol{i} = (5, 4, 3, 2, 5, 3, 1, 4, 3, 2, 5, 3, 4, 3, 5)$:



- $\Delta = \mathsf{D}_4$ and $\boldsymbol{i} = (4, 2, 1, 2, 3, 2, 4, 2, 1, 2, 3, 2)$:



- $\Delta = \mathsf{D}_5$ and $\boldsymbol{i} = (4, 3, 2, 1, 5, 3, 2, 1, 4, 3, 2, 1, 5, 3, 2, 1, 4, 3, 2, 1)$:



Meshes in combinatorial AR quivers

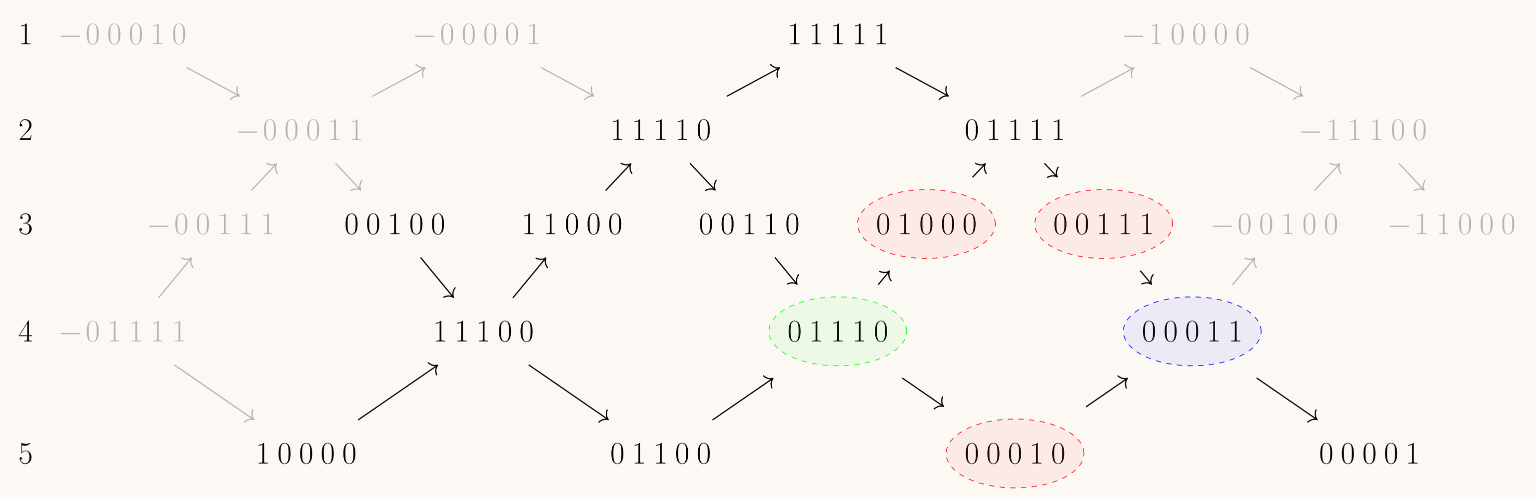
Take a vertex $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0 = \mathbb{Z}$ of the combinatorial repetition quiver.

- The **translation** of x is the smallest integer $\tau x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$ such that $x < \tau x$ and $i_x = i_{\tau x}$.
- The **set of abutters*** of x is the subset $V_{\boldsymbol{i}}(x) \subset (\widehat{\Upsilon}_{\boldsymbol{i}})_0$ formed by the vertices y such that $x < y < \tau x$ and i_y is adjacent to i_x in Δ.

*An abutter is the owner of an adjacent property.

Example: Coordinate map and meshes

For the previous example in type A_5, over each vertex x of $\widehat{\Upsilon}_{\boldsymbol{i}}$ we write the root $\rho(x)$ given by the coordinate map. The numbers indicate the coefficient of each simple root α_i in $\rho(x)$.



The translation of the blue vertex is the green one, and the red vertices form the set of abutters of the blue vertex.

Theorem 2: [Can25] Mesh-additivity property

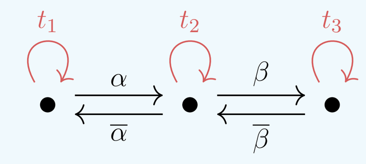
The coordinate map satisfies

$$\rho(\tau x) + \rho(x) = \sum_{y \in V_{\boldsymbol{i}}(x)} \rho(y)$$

for all $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$.

Towards a categorification

- Let Q be an orientation of Δ. We define Π to be the **2-dimensional Ginzburg dg algebra** associated with Q over some field K.
- For example, if $\Delta = \mathsf{A}_3$, then Π is the dg path algebra given by



where black arrows have degree 0 and red arrows have degree -1. The differential is determined by $d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$.
- Π is a connective dg algebra whose zeroth cohomology is the preprojective algebra of Q. It is smooth and its perfectly valued derived category $\mathrm{pvd}(\Pi)$ is a 2-Calabi–Yau triangulated category.
- Each simple dg module S_i corresponding to $i \in \Delta_0$ is 2-spherical and yields a **spherical twist functor** $T_i : \mathrm{pvd}(\Pi) \to \mathrm{pvd}(\Pi)$, which is an autoequivalence of $\mathrm{pvd}(\Pi)$.
- The map $[S_i] \mapsto \alpha_i$ induces an isomorphism between the Grothendieck group of $\mathrm{pvd}(\Pi)$ and the root lattice of Δ. The action of T_i on the Grothendieck group identifies with the action of the simple reflection $s_i \in \mathrm{W}$.

Categories associated with reduced words [Can25]

- For $k \in \mathbb{Z}$, we define the following object of $\mathrm{pvd}(\Pi)$:

$$M_k^{\boldsymbol{i}} = \begin{cases} T_{i_1}T_{i_2}\cdots T_{i_{k-1}}(S_{i_k}) & \text{if } k \geq 1, \\ \Sigma T_{i_0}^{-1}T_{i_{-1}}^{-1}\cdots T_{i_{k+1}}^{-1}(S_{i_k}) & \text{if } k \leq 0, \end{cases}$$

where Σ denotes the suspension functor of $\mathrm{pvd}(\Pi)$.
- The **repetition category** $\mathcal{R}(\boldsymbol{i})$ is the full additive subcategory of $\mathrm{pvd}(\Pi)$ generated by the indecomposable objects $M_k^{\boldsymbol{i}}$ for $k \in \mathbb{Z}$.
- The **category of representations** $\mathcal{C}(\boldsymbol{i})$ is the full subcategory of $\mathcal{R}(\boldsymbol{i})$ whose objects have cohomology concentrated in degree zero. In particular, we may view its objects as representations of the preprojective algebra.

Proposition 3: [AIRT12]

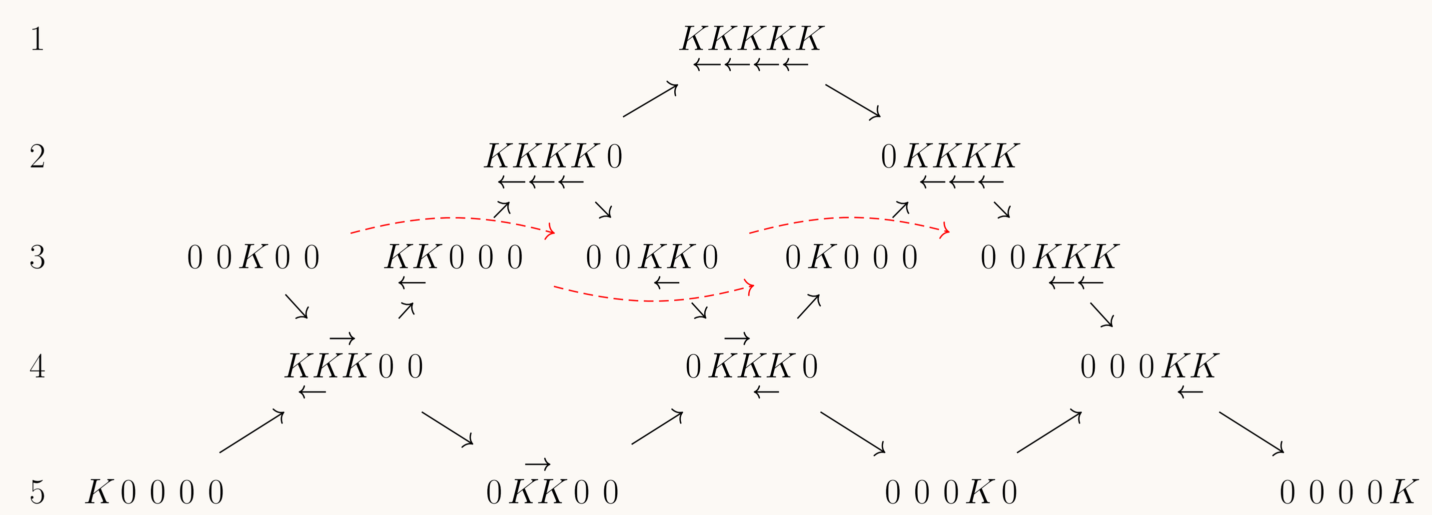
- *The indecomposable objects of $\mathcal{C}(\boldsymbol{i})$ are the $M_k^{\boldsymbol{i}}$ for $1 \leq k \leq N$. As representations of the preprojective algebra, they coincide with certain modules called* **layers** *and defined in [AIRT12].*
- *If $\boldsymbol{i}$ is a source sequence for an orientation Q of Δ, then $\mathcal{C}(\boldsymbol{i})$ is equivalent to the category of representations of the quiver Q as a K-linear category.*

Theorem 4: [Can25] Categorification of combinatorial AR quivers

The combinatorial AR quiver $\Upsilon_{\boldsymbol{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\boldsymbol{i})$ by removing all arrows parallel to paths of length at least two.

Example

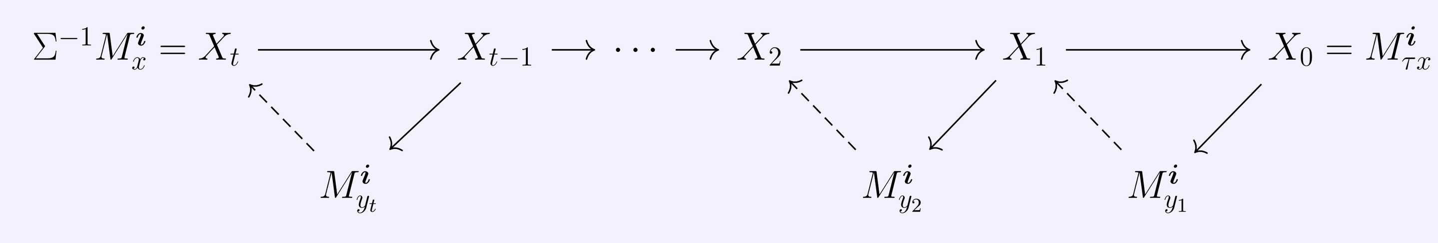
For the previous example in type A_5, the Gabriel quiver of $\mathcal{C}(\boldsymbol{i})$ is the following:



Each indecomposable object is viewed as a representation of the preprojective algebra of type A_5.

Theorem 5: [Can25] Categorification of the mesh-additivity property

Given a vertex $x \in (\widehat{\Upsilon}_{\boldsymbol{i}})_0$, choose an ordering $y_1, \dots, y_t$ of the set of abutters $V_{\boldsymbol{i}}(x)$ such that $k \leq l$ whenever there is a path from y_k to y_l in $\widehat{\Upsilon}_{\boldsymbol{i}}$. Then there are indecomposable objects $X_1, X_2, \dots, X_{t-1}$ in $\mathcal{R}(\boldsymbol{i})$ and a diagram of the form



where the triangles above are distinguished triangles in $\mathrm{pvd}(\Pi)$.

The combinatorially derived category

In [Can25], we define the **combinatorially derived category** $\mathcal{D}(\boldsymbol{i})$ as a quotient of the repetition category $\mathcal{R}(\boldsymbol{i})$ by a certain ideal of morphisms. It has the following main properties:

- If $\boldsymbol{i}$ is a source sequence for an orientation Q of Δ, then $\mathcal{D}(\boldsymbol{i})$ is equivalent to the bounded derived category of representations of the quiver Q as a K-linear category.
- The combinatorial repetition quiver $\widehat{\Upsilon}_{\boldsymbol{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{D}(\boldsymbol{i})$ by removing all arrows parallel to paths of length at least two.
- The suspension functor Σ of $\mathrm{pvd}(\Pi)$ descends to an autoequivalence of $\mathcal{D}(\boldsymbol{i})$. It allows us to define an **Euler form** on $\mathcal{D}(\boldsymbol{i})$ by

$$\langle M, N\rangle = \sum_{k\in\mathbb{Z}} (-1)^k \dim_K \mathrm{Hom}_{\mathcal{D}(\boldsymbol{i})}(M, \Sigma^k N).$$

Its symmetrization can be shown to agree with the Cartan–Killing form on the root lattice of Δ.

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