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Additive categorification of the monoidal Λ-invariant

joint work with Peigen Cao and Geoffrey Janssens

Plan for the talk

(1) Reminder on cluster algebras and Cao's tropical and F-invariants.

(2) Homological interpretation using additive categorification

(3) Link to monoidal categorification (reps. of quantum affine algebras)

1) Λ-cluster algebras

- $\tilde{B}_{m\times n} = \begin{bmatrix} B_{n\times n} \\ P \end{bmatrix}$: integer matrix $(m \geq n)$
- $\Lambda_{m\times m}$: skew-symmetric integer matrix

Def [Berenstein-Zelevinsky '05]: $(\tilde{B}, \Lambda)$ is a compatible pair if $\exists\, D = \mathrm{diag}(d_1, \dots, d_n)$ $(d_i \in \mathbb{Z}_{>0})$ s.t. $\tilde{B}^t \Lambda = (D \mid 0)$.

Ex:

$$\tilde{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \longleftrightarrow (Q, F) = \begin{tikzcd} \boxed{4} \arrow[dr] & & 2 \arrow[ll] \arrow[dr] & \\ & \boxed{3} \arrow[ur] & & 1 \arrow[ll] \end{tikzcd}$$

$$\Lambda = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} = |\det \hat{B}| \left(\hat{B}^{-T} - \hat{B}^{-1}\right)$$

$$(\hat{B}_{m\times m})_{ij} = \begin{cases} \#\{\text{unfrozen } i \to j\} - \#\{j \to i\}, & i \neq j \\ 1, & i = j \text{ frozen} \\ 0, & \text{else} \end{cases}$$

- $t = (X_t, \tilde{B}_t, \Lambda_t)$: seed
  - $\hookrightarrow (\tilde{B}_t, \Lambda_t)$: compatible pair
  - $\hookrightarrow X_t = (u_1, \dots, u_m)$: cluster, i.e., free generating set of $\mathbb{Q}(x_1, \dots, x_m)$.

  cluster variables

Rmk: Mutation produces new seeds.

Def: [Fomin-Zelevinsky '02, BZ '05]
The Λ-cluster algebra $\mathcal{A}$ assoc. with an initial seed $t_0 = ((x_1, \dots, x_m), \tilde{B}, \Lambda)$ is the subring of $\mathbb{Q}(x_1, \dots, x_m)$ gen. by all cluster variables obtained by iterated mutations of t_0.

Rmk: Λ yields a compatible Poisson structure on $\mathcal{A}$ $\Longrightarrow$ quantization

product of cluster variables in a cluster

[Cao '23] For cluster monomials $u, u' \in \mathcal{A}$:

- Tropical invariant : $\langle u, u' \rangle_{trop} \in \mathbb{Z}$
- F-invariant : $(u \| u')_F = \langle u, u' \rangle_{trop} + \langle u', u \rangle_{trop} \in \mathbb{Z}_{\geq 0}$

Thm [Cao '23]: (i) If u, u' are cluster variables in a seed t, then $\langle u, u' \rangle_{trop} =$ entry (u, u') in Λ_t.

(ii) Can be computed using F-polynomials and g-vectors.

(iii) F-invariant does not depend on Λ !

(iv) u, u' are cluster monomials associated with the same cluster $\iff (u \| u')_F = 0$.

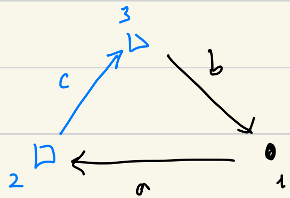
2) Additive categorification

- $k = \mathbb{C}$: base field
- (Q, F) : ice quiver ← possibly with frozen arrows
- W : non-degenerate potential on (Q, F)

completed

- $\hat{\Gamma} = \hat{\Gamma}(Q, F, W)$: relative Ginzburg dg algebra

Ex.:



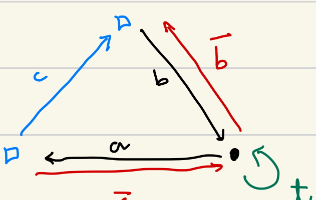
$W = abc$

$|a| = |b| = |c| = 0$

$|\bar{a}| = |\bar{b}| = -1$

$|t| = -2$

$\hat{\Gamma}$:



$d\bar{a} = \partial_a W = bc$

$d\bar{b} = \partial_b W = ca$

$dt = b\bar{b} - \bar{a}a$

Suppose $\dim H^0(\hat{\Gamma}) < \infty$.

Def [W, '23]: The relative cluster category is:

$$\mathcal{C} = \mathrm{per}\,\hat{\Gamma} \big/ \mathrm{thick}(S_i : i \notin F_0)$$

simple at i

The Higgs category $\mathcal{H} \subseteq \mathcal{C}$ is the full subcat. with obj. X s.t.

$$\mathrm{Ext}^p_{\mathcal{C}}(X, e_i\hat{\Gamma}) = \mathrm{Ext}^p_{\mathcal{C}}(e_i\hat{\Gamma}, X) = 0$$

for $p > 0$ and $i \in F_0$.

[Keller-Wu '23]: Construct a canonical cluster character:

$$CC : \mathcal{H} \longrightarrow \mathcal{A}_{\varphi, F}$$

Thm [KW '23]: The cluster character induces a bijection

$$\left\{ \begin{array}{c} \text{reachable rigid indecomposable} \\ \text{objects in } \mathcal{H} \end{array} \right\} \Big/ \cong \xrightarrow{\ CC\ } \left\{ \begin{array}{c} \text{cluster variables} \\ \text{in } \mathcal{A} \end{array} \right\}$$

Suppose $\hat{\Gamma}$ is proper: $\sum_{n \in \mathbb{Z}} \dim H^n(\hat{\Gamma}) < \infty$.

Define:

$$[M,N]_{\mathcal{H}} = \sum_{p \geq 0} (-1)^p \left(\dim \operatorname{Ext}^{-p}_{\mathcal{H}}(M,N) - \dim \operatorname{Ext}^{-p}_{\mathcal{H}}(N,M) \right)$$

Thm [CCJ]: If $\hat{\Gamma}$ is proper, then this formula induces a Λ-cluster alg. str. on $\mathcal{A}$. For reachable rigid objects $M, N \in \mathcal{H}$, we have:

$$\langle CC(M), CC(N) \rangle_{\text{top}} = \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}}$$

$$\big(CC(M) \,\|\, CC(N) \big)_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N)$$

3) Monoidal categorification

$\mathfrak{g}$: affine Kac-Moody Lie algebra

$U_q'(\mathfrak{g})$: quantum affine algebra

$\mathcal{C}_{\mathfrak{g}}$: cat. of f.d. (integrable) $U_q'(\mathfrak{g})$-modules

has duals

Rmk: $\mathcal{C}_{\mathfrak{g}}$ is a rigid monoidal abelian length category. Not braided!

[Kashiwara-Kim-Oh-Park '20]: For simple $V, W \in \mathcal{C}_{\mathfrak{g}}$,

- $\Lambda(V,W) \in \mathbb{Z}$
- $d(V,W) = \frac{1}{2}\left(\Lambda(V,W) + \Lambda(W,V)\right) \in \mathbb{Z}_{\geq 0}$

complete duality datum

[KKOP '24, '25]: Monoidal Serre subcategory $\mathcal{C}_{\mathfrak{g}}^{\mathbb{D}}(\mathbf{i}) \subseteq \mathcal{C}_{\mathfrak{g}}$

braid word

$\Longrightarrow$ Monoidal categorification :

Λ-cluster algebra

$$\varphi : K_0(\mathcal{C}_{\mathfrak{g}}^{\mathbb{D}}(\mathbf{i})) \xrightarrow{\sim} \mathcal{A}_{Q,F}$$

The Λ-matrix of a seed whose variables are $\varphi(V_1), \dots, \varphi(V_m)$ is given by:

$$\left(\Lambda(V_i, V_j)\right)_{1 \leq i,j \leq m}$$

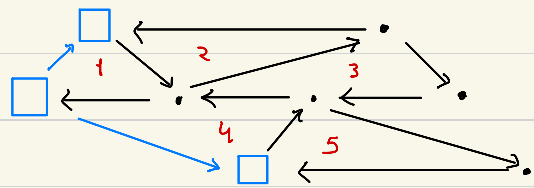
Thm [Cao '25]: For reachable simples V and W in $\mathcal{C}_{\mathfrak{g}}^{\mathbb{D}}(\mathbf{i})$, we have:

$$\langle \varphi(M), \varphi(N) \rangle_{trop} = \Lambda(M,N)$$

$$(\varphi(M) \| \varphi(N))_F = 2 \cdot d(M,N)$$

Ex: $\mathring{\mathbb{i}} = (2,1,2,3,2,1,2,3)$ in type A_3

$(Q,F) =$



$W = 1 + 2 + 3 + 4 + 5$

↳ unique non-degenerate potential

Rmk: In terms of weave combinatorics, (Q,F) is the ice quiver attached to the right inductive weave $\overrightarrow{\mathfrak{w}}(w_0 \mathring{\mathbb{i}})$, where w_0 is a lift of the longest element to the braid group.

See also: Huh-Jung-Kim-Park '26

Thm [CCJ]: Suppose $\hat{\Gamma}(Q,F,W)$ is proper. Let $V, W \in \mathcal{C}^{\mathcal{D}}_{\mathfrak{g}}(\mathring{\mathbb{i}})$ be reachable simples and let $M, N \in \mathcal{H}$ be the corresponding reachable rigid objects. We have:

$$\Lambda(V,W) = \dim \mathrm{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}}$$

Thm [CCJ]: The relative Ginzburg dg algebra $\hat{\Gamma}(Q,F,W)$ is proper if $\mathring{\mathbb{i}}$ is braid equivalent to a locally reduced word that is a source sequence for an orientation of the corresp. Dynkin diagram.

↳ every subword whose size is at most the length of the longest element is reduced

Conjecture: No restrictions required.

$\Longrightarrow$ See upcoming work by Casals-Christ