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Additive categorification of the monoidal Λ-invariant

joint work with Peigen Cao and Geoffrey Janssens

Plan for the talk

(1) Reminder on cluster algebras and Cao's tropical and F-invariants.

(2) Homological interpretation using additive categorification

(3) Link to monoidal categorification (reps. of quantum affine algebras)

1) Λ-cluster algebras

- $\tilde{B}_{m\times n} = \begin{bmatrix} B_{n\times n} \\ P \end{bmatrix}$: integer matrix $(m \geq n)$
- $\Lambda_{m\times m}$: skew-symmetric integer matrix

If B is skew-symmetric, $\tilde{B}$ comes from an ice quiver

Ex.:

$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \quad \leftrightsquigarrow \quad (Q,F) = \begin{tikzcd} & \boxed{3} \arrow[dr] & \\ \bullet_2 \arrow[ur] & & \bullet_1 \arrow[ll] \end{tikzcd}$$

frozen vertex

Def [Berenstein–Zelevinsky '05]: $(\tilde{B}, \Lambda)$ is a compatible pair if $\exists\, D = \mathrm{diag}(d_1, \dots, d_n)$ $(d_i \in \mathbb{Z}_{>0})$ s.t. $\tilde{B}^t \Lambda = (D \mid 0)$.

- $t = (X_t, \tilde{B}_t, \Lambda_t)$: seed
  - $\hookrightarrow$ $(\tilde{B}_t, \Lambda_t)$: compatible pair
  - $\hookrightarrow$ $X_t = (u_1, \ldots, u_m)$: cluster, i.e., free generating set of $\mathbb{Q}(x_1, \ldots, x_m)$.

cluster variables

There is a mutation procedure which produces new seeds from a given initial seed.

[Fomin-Zelevinsky '02, BZ '05]

Def: The $\underline{\Lambda\text{-cluster algebra}}$ $\mathcal{A}$ assoc. with an initial seed $t_0 = ((x_1, \ldots, x_m), \tilde{B}, \Lambda)$ is the subring of $\mathbb{Q}(x_1, \ldots, x_m)$ gen. by all cluster variables obtained by iterated mutations of t_0.

Rmk: $\mathcal{A}$ does not depend on Λ, but each choice of Λ yields a compatible Poisson structure on $\mathcal{A}$.

$\Longrightarrow$ quantization

product of cluster variables in a cluster

For cluster monomials $u, u' \in \mathcal{A}$, Cao defined:

- Tropical invariant : $\langle u, u' \rangle_{trop} \in \mathbb{Z}$
- F-invariant : $(u \| u')_F = \langle u, u' \rangle_{trop} + \langle u', u \rangle_{trop} \in \mathbb{Z}_{\geq 0}$

Thm [Cao '23]: (i) If u, u' are cluster variables in a seed t, then $\langle u, u' \rangle_{trop} = $ entry (u, u') in Λ_t.

(ii) They can be computed using F-polynomials and g-vectors.

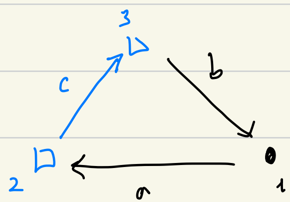
(iii) The F-invariant does not depend on Λ!

(iv) u, u' are cluster monomials associated with the same cluster $\iff (u \| u')_F = 0$.

2) Additive categorification

- $k = \mathbb{C}$: base field
- (Q, F) : ice quiver ← possibly with frozen arrows
- W : non-degenerate potential on (Q, F)
- $\hat{\Gamma} = \hat{\Gamma}(Q, F, W)$: relative Ginzburg dg algebra (← completed)

Ex :



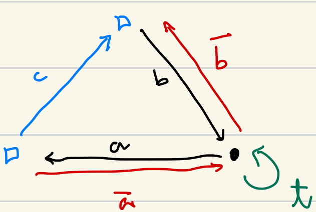
$W = abc$

$|a| = |b| = |c| = 0$

$|\bar{a}| = |\bar{b}| = -1$

$|t| = -2$

$\hat{\Gamma}$:



$d\bar{a} = \partial_a W = bc$

$d\bar{b} = \partial_b W = ca$

$dt = b\bar{b} - \bar{a}a$

Suppose $\dim H^0(\hat{\Gamma}) < \infty$.

Def [W, '23]: The <u>relative cluster category</u> is:

$$\mathcal{C} = \operatorname{per} \hat{\Gamma} / \operatorname{thick}(S_i : i \notin F_0)$$

(↑ simple at i)

The <u>Higgs category</u> $\mathcal{H} \subseteq \mathcal{C}$ is the full subcategory with objects X s.t.

$$\operatorname{Ext}^p_{\mathcal{C}}(X, e_i\hat{\Gamma}) = \operatorname{Ext}^p_{\mathcal{C}}(e_i\hat{\Gamma}, X) = 0$$

for $p \geq 0$ and $i \in F_0$.

[Keller-Wu '23]: Construct a canonical <u>cluster character</u>:

$$CC : \mathcal{H} \longrightarrow \mathcal{A}_{\varphi, F}$$

Thm [KW '23]: The cluster character induces a bijection

$$\left\{ \begin{array}{c} \text{reachable rigid indecomposable} \\ \text{objects in } \mathcal{H} \end{array} \right\} \Big/ \simeq \xrightarrow{CC} \left\{ \begin{array}{c} \text{cluster variables} \\ \text{in } \mathcal{A} \end{array} \right\}$$

—— " ——

Suppose $\hat{\Gamma}$ is <u>proper</u>: $\sum_{n \in \mathbb{Z}} \dim H^n(\hat{\Gamma}) < \infty$.

Define:

$$[M, N]_{\mathcal{H}} = \sum_{p \geq 0} (-1)^p \left(\dim \operatorname{Ext}^{-p}_{\mathcal{H}}(M, N) - \dim \operatorname{Ext}^{-p}_{\mathcal{H}}(N, M) \right)$$

Thm [CCJ]: If $\hat{\Gamma}$ is proper, then $\mathcal{A}$ admits a Λ-cluster algebra structure where the Λ-matrix $\Lambda_t = (\lambda^t_{ij})$ of a seed t is given by

$$\lambda^t_{ij} = [M^t_i, M^t_j]_{\mathcal{H}}$$

where M^t_i is reachable rigid indecomp. and $CC(M^t_i)$ is the i-th variable of t.

Thm [CCJ]: For reachable rigid objects $M, N \in \mathcal{H}$, we have:

$$\langle CC(M), CC(N) \rangle_{top} = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}}$$

$$\big(CC(M) \,\|\, CC(N) \big)_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N)$$

3) Monoidal categorification

$\mathfrak{g}$: affine Kac-Moody Lie algebra

$U_q'(\mathfrak{g})$: quantum affine algebra

$\mathcal{C}_{\mathfrak{g}}$: cat. of f.d. (integrable) $U_q'(\mathfrak{g})$-modules

has duals

Rmk: $\mathcal{C}_{\mathfrak{g}}$ is a rigid monoidal abelian length category. Not braided!

[Kashiwara-Kim-Oh-Park '20]: For simple $V, W \in \mathcal{C}_{\mathfrak{g}}$,

- $\Lambda(V,W) \in \mathbb{Z}$
- $d(V,W) = \frac{1}{2}\big(\Lambda(V,W) + \Lambda(W,V)\big) \in \mathbb{Z}_{\geq 0}$

[KKOP '24, '25]: Monoidal subcategory $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}} \subseteq \mathcal{C}_{\mathfrak{g}}$

$\Longrightarrow$ Monoidal categorification :

Λ-cluster algebra

$$\varphi : K_0\big(\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}}\big) \xrightarrow{\sim} \mathcal{A}$$

The Λ-matrix of a seed whose variables are $\varphi(V_1), \dots, \varphi(V_m)$ is given by:

$$\big(\Lambda(V_i, V_j)\big)_{1 \leq i,j \leq m}$$

Thm [Cao '25]: For reachable simples V and W in $\mathcal{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}}$, we have:

$$\langle \varphi(M), \varphi(N) \rangle_{trop} = \Lambda(M,N)$$

$$\big(\varphi(M) \,\|\, \varphi(N)\big)_F = 2 \cdot d(M,N)$$

- Δ : ADE Dynkin diagram related to $\mathfrak{g}$
- w_0 : longest element of the Weyl group of Δ
- $\underline{w_0} = (i_1, \dots, i_N)$: reduced expression for w_0.
- $\hat{\underline{w_0}} = (i_k)_{k\in\mathbb{Z}}$ extends $\underline{w_0}$ by $i_{k+N} = i_k^*$.

  involution on Δ_0 induced by w_0

For $s\in\mathbb{Z}$, $s^- := \max\{t\in\mathbb{Z} : t<s,\ i_t = i_s\}$.

Def: Let $a,b\in\mathbb{Z}$, $a\leq b$. We define an ice quiver with potential (Q,F,W):

- $Q_0 = [a,b] := \{s\in\mathbb{Z} : a\leq s\leq b\}$

  $F_0 = \{s\in[a,b] : s^- < a\}$

- $\exists\ s\to t$ if one of the following holds:

  (i) $t = s^-$

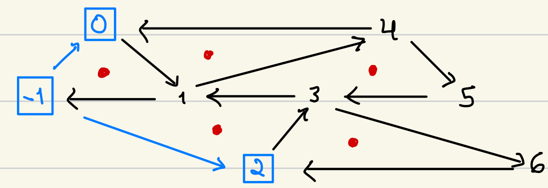
  (ii) $s^- < t^- < s < t$ and $i_s - i_t$ in Δ

  (iii) s,t frozen, $s<t$ and $i_s - i_t$ in Δ (frozen arrows)

- W = sum of all simple cordless cycles

Ex: $\Delta = A_3$, $\underline{w_0} = (2,3,2,1,2,3)$

$[a,b] = [-1,6]$:

$(Q,F) =$ 

W = sum of the five cycles •

Thm [CCJ]: The relative Ginzburg dg algebra $\hat{\Gamma}(Q, F, W)$ is proper if:

- $\underline{w}_0$ is a source sequence for an orientation of Δ
- or $\ell(w_0) \mid b-a+1$.

Thm [CCJ]: Suppose $\mathfrak{g}$ is of type $A^{(1)}\, D^{(1)}\, E^{(1)}$ (untwisted type) and $(\mathcal{D}, \underline{w}_0)$ is adapted to an orientation of Δ. Let $V, W \in \mathscr{C}_{\mathfrak{g}}^{[a,b], \mathcal{D}, \underline{w}_0}$ be reachable simples and let $M, N \in \mathcal{H}$ be the corresponding reachable rigid objects in the Higgs category. We have:

$$\Lambda(V,W) = \dim \operatorname{Ext}^1_{\mathcal{H}}(M, N) + [M, N]_{\mathcal{H}}$$

Conjecture: No restrictions required.