An additive interpretation of the monoidal Λ-invariant

ongoing joint work with Geoffrey Janssens and Peigen Cao

Plan for the talk

(1) Reminder on cluster algebras and Cao's tropical and F-invariants.

(2) Homological interpretation using additive categorification

(3) Link to monoidal categorification (reps. of quantum affine algebras)

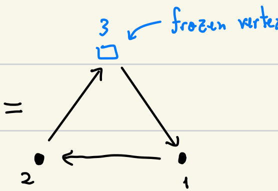
1) Λ-cluster algebras

- $\tilde{B}_{m\times n} = \begin{bmatrix} B_{n\times n} \\ P \end{bmatrix}$: integer matrix $(m \geq n)$
- $\Lambda_{m\times m}$: skew-symmetric integer matrix

If B is skew-symmetric, $\tilde{B}$ comes from an ice quiver

Ex.:

$$\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \leftrightsquigarrow (Q,F) =$$



Def [Berenstein–Zelevinsky '05]: $(\tilde{B}, \Lambda)$ is a compatible pair if $\exists\, D = \mathrm{diag}(d_1, \dots, d_n)$ $(d_i \in \mathbb{Z}_{>0})$ s.t. $\tilde{B}^t \Lambda = (D \mid 0)$.

- $t = (X_t, \tilde{B}_t, \Lambda_t)$: seed
  - $\hookrightarrow (\tilde{B}_t, \Lambda_t)$: compatible pair
  - $\hookrightarrow X_t = (u_1, \dots, u_m)$: cluster, i.e., free generating set of $\mathbb{Q}(x_1, \dots, x_m)$.

cluster variables

There is a mutation procedure which produces new seeds from a given initial seed.

Def: The <u>Λ-cluster algebra</u> $\mathcal{A}$ assoc. with an initial seed $t_0 = ((x_1, \dots, x_m), \tilde{B}, \Lambda)$ is the subring of $\mathbb{Q}(x_1, \dots, x_m)$ gen. by all cluster variables obtained by iterated mutations of t_0.

Rmk: $\mathcal{A}$ does not depend on Λ, but each choice of Λ yields a compatible Poisson structure on $\mathcal{A}$.

$\Longrightarrow$ quantization

product of cluster variables in a cluster

For cluster monomials $u, u' \in \mathcal{A}$, Cao defined:

- Tropical invariant : $\langle u, u' \rangle_{trop} \in \mathbb{Z}$
- F-invariant : $(u \| u')_F = \langle u, u' \rangle_{trop} + \langle u', u \rangle_{trop} \in \mathbb{Z}_{\geq 0}$

Thm [Cao '23]: (i) If u, u' are cluster variables in a seed t, then $\langle u, u' \rangle_{trop} =$ entry (u, u') in Λ_t.

(ii) They can be computed using F-polynomials and g-vectors.

(iii) The F-invariant does not depend on Λ!

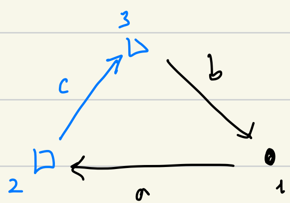
(iv) $u \cdot u'$ is a cluster monomial $\iff (u \| u')_F = 0$.

2) Additive categorification

- $k = \mathbb{C}$: base field
- (Q, F) : ice quiver ← possibly with frozen arrows
- W : non-degenerate potential on Q

- $\Gamma = \Gamma(Q, F, W)$: relative Ginzburg dg algebra

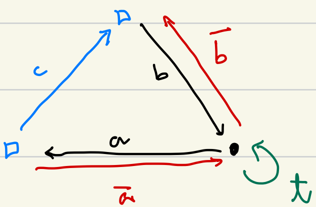
Ex :



$W = abc$

$|a| = |b| = |c| = 0$

Γ :



$|\bar{a}| = |\bar{b}| = -1$

$|t| = -2$

$d\bar{a} = \partial_a W = bc$

$d\bar{b} = \partial_b W = ca$

$dt = b\bar{b} - \bar{a}a$

Suppose $\dim H^0(\Gamma) < \infty$.

Def [W, '23] : The <u>relative cluster category</u> is :

$$\mathcal{C} = \operatorname{per}\Gamma \big/ \operatorname{thick}(S_i : i \notin F_0)$$

simple at i

The <u>Higgs category</u> $\mathcal{H} \subseteq \mathcal{C}$ is a certain extension closed full subcategory.

$\Longrightarrow$ $\mathcal{H}$ is an extriangulated category with a canonical choice of negative extensions.

Rmk : There is a cluster character $CC : \mathcal{H} \longrightarrow \mathcal{A}$

Suppose Γ is proper: $\sum_{n \in \mathbb{Z}} \dim H^n(\Gamma) < \infty$

We can then define:

$$[M,N]_{\mathcal{H}} = \sum_{n \geq 0} (-1)^n \left(\dim \operatorname{Ext}^{-n}_{\mathcal{H}}(M,N) - \dim \operatorname{Ext}^{-n}_{\mathcal{H}}(N,M) \right)$$

Thm [CJ+C]: The formula above defines a quantum structure on $\mathcal{H}$ in the sense of Grabowski–Pressland. If $M, N \in \mathcal{H}$ are reachable rigid objects, then:

$$\langle CC(M), CC(N) \rangle_{top} = \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}}$$

$$(CC(M) \| CC(N))_F = 2 \cdot \dim \operatorname{Ext}^1_{\mathcal{H}}(M,N)$$

[Cao '25]

Idea: $\tilde{B}$ is a submatrix of

$$C = \left(\langle S_i, S_j \rangle \right)_{i,j \in Q_0}$$

Euler form between simples.

Well-defined because Γ is smooth

When Γ is proper, C is invertible and

$$C^{-t} = \left(\langle e_i \Gamma, e_j \Gamma \rangle \right)_{i,j \in Q_0}$$

idempotents at vertices i,j

The formula above computes $\Lambda := C^{-t} - C^{-1}$, which yields a compatible pair $(\tilde{B}, \Lambda)$ with $D = 2 \cdot \mathrm{Id}$.

3) Monoidal categorification

$\mathfrak{g}$: affine Kac-Moody Lie algebra

$U_q(\mathfrak{g})$: quantum affine algebra

$\mathcal{C}_{\mathfrak{g}}$: cat. of f.d. (integrable) $U_q(\mathfrak{g})$-modules

Rmk: $\mathcal{C}_{\mathfrak{g}}$ is a rigid (has duals) monoidal abelian length category. It is "generically braided":

$$R_{V,W} : V \otimes W \longrightarrow W \otimes V$$

R-matrix

is an isomorphism for generic simples V, W.

[Kashiwara-Kim-Oh-Park '20] define from $R_{V,W}$:

- $\Lambda(V,W) \in \mathbb{Z}$
- $d(V,W) = \frac{1}{2}\big(\Lambda(V,W) + \Lambda(W,V)\big) \in \mathbb{Z}_{\geq 0}.$

[KKOP '24, '25] introduce a monoidal subcategory $\mathcal{C}_{\mathfrak{g}}^{[a,b], \mathcal{D}, \underline{w_0}} \subseteq \mathcal{C}_{\mathfrak{g}}$ and show that it provides a monoidal categorification

$$\varphi : K_0\big(\mathcal{C}_{\mathfrak{g}}^{[a,b], \mathcal{D}, \underline{w_0}}\big) \xrightarrow{\sim} \mathcal{A}$$

for a Λ-cluster algebra $\mathcal{A}$. The Λ-matrix of a seed whose variables are $\varphi(V_1), \dots, \varphi(V_m)$ is given by:

$$\big(\Lambda(V_i, V_j)\big)_{1 \leq i,j \leq m}$$

Thm [Cao'25]: For reachable simples V and W in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$, we have:

$$\langle \varphi(M), \varphi(N) \rangle_{trop} = \Lambda(M,N)$$

$$(\varphi(M) \| \varphi(N))_F = 2 \cdot d(M,N)$$

- Δ : ADE Dynkin diagram related to $\mathfrak{g}$
- w_0 : longest element of the Weyl group of Δ
- $\underline{w_0} = (i_1, \dots, i_N)$: reduced expression for w_0.
- $\hat{\underline{w_0}} = (i_k)_{k\in\mathbb{Z}}$ extends $\underline{w_0}$ by $i_{k+N} = i_k^{*}$.

involution on Δ_0 induced by w_0

For $s \in \mathbb{Z}$, $s^- := \max\{t \in \mathbb{Z} : t < s,\ i_t = i_s\}$.

Def: Let $a, b \in \mathbb{Z}$, $a \leq b$. We define an ice quiver with potential (Q, F, W):

- $Q_0 = [a,b] := \{s \in \mathbb{Z} : a \leq s \leq b\}$

  $F_0 = \{s \in [a,b] : s^- < a\}$

- $\exists\, s \longrightarrow t$ if one of the following holds:

  (i) $t = s^-$

  (ii) $s^- < t^- < s < t$ and $i_s - i_t$ in Δ

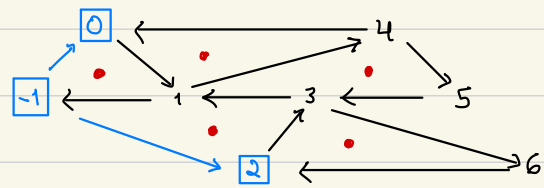
  (iii) s, t frozen, $s < t$ and $i_s - i_t$ in Δ (frozen arrows)

- W = sum of all simple cordless cycles

Thm [KKOP'24,'25]: (Q, F) is the ice quiver of an initial monoidal seed in $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$.

Ex: $\Delta = A_3$, $\underline{w_0} = (2,3,2,1,2,3)$

$[a,b] = [-1,6]$:

$(Q,F) =$ 

W = sum of the five cycles •

Thm [CJ]: The relative Ginzburg dg algebra $\Gamma(Q,F,W)$ is proper if:

- $\underline{w_0}$ is adapted to an orientation of Δ.
- or $\ell(w_0) \mid b-a+1$.

Conjecture: No restrictions required.

Thm [CJ]: Suppose $\mathfrak{g}$ is of type $A^{(1)} D^{(1)} E^{(1)}$ (untwisted type) and $(\mathcal{D}, \underline{w_0})$ is adapted to an orientation of Δ. Let $V, W \in \mathcal{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}}$ be reachable simples and let $M, N \in \mathcal{H}$ be the corresponding reachable rigid objects in the Higgs category. We have:

$$\Lambda(V,W) = \dim \mathrm{Ext}^1_{\mathcal{H}}(M,N) + [M,N]_{\mathcal{H}}$$

Idea: By Cao: $\mathfrak{d}(V,W) = \dim \mathrm{Ext}^1_{\mathcal{H}}(M,N)$. By KKOP:

$\mathcal{D}^{-1}$ = left dual in $\mathcal{C}_{\mathfrak{g}}$

$$\Lambda(V,W) = \mathfrak{d}(V,W) + \sum_{n \geq 1} (-1)^n \left(\mathfrak{d}(V, \mathcal{D}^{-n} W) - \mathfrak{d}(W, \mathcal{D}^{-n} V)\right)$$

relative cluster category

We have to show $\mathcal{D}^{-1}$ corresponds to the inverse shift Σ^{-1} in $\mathcal{C}$.

Both can be computed using maximal green sequences!