Graded quiver varieties and categories of split filtrations

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Context

- Graded Nakajima quiver varieties: geometric realization of modules over quantum affine algebra
- Hernandez-Leclerc, Leclerc-Plamondon, Keller-Scherotzke, ...: can be studied via the derived category of a Dynkin quiver.
- Goal: extend KS's setting to Nakajima's tensor product varieties.

1) Graded quiver varieties (after KS)

Q: ADE Dynkin quiver

Ex.: $Q = \bullet \longrightarrow \bullet \longrightarrow \bullet$ (type A_3)

$\mathbb{Z}Q$: repetition quiver

↳ vertices: (i,p), $i \in Q_0$, $p \in \mathbb{Z}$ (vertices of Q)

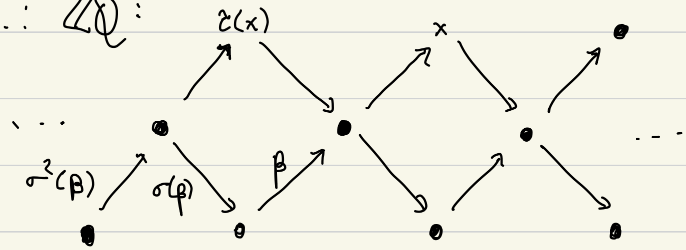
↳ arrows: for $\alpha: i \to j$ in Q and $p \in \mathbb{Z}$, we have

- $(\alpha, p): (i,p) \longrightarrow (j,p)$
- $\sigma(\alpha,p): (j,p-1) \longrightarrow (i,p)$

Denote $\tau(i,p) := (i, p-1)$.

$\sigma(\sigma(\alpha,p)) := (\alpha, p-1)$

Ex.: $\mathbb{Z}Q$:

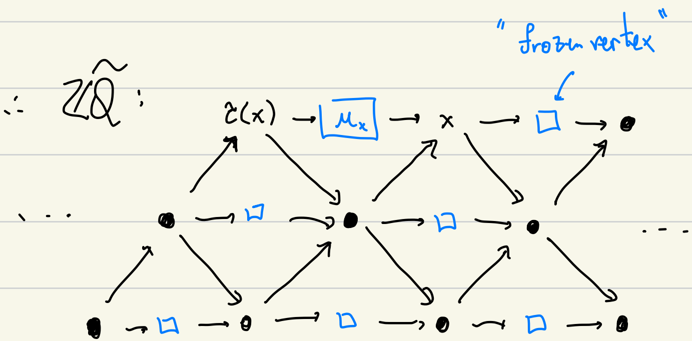


$\mathbb{Z}\tilde{Q}$: framed repetition quiver

↳ add a vertex μ_x for every $x \in Q_0$

and arrows $\tau(x) \xrightarrow{\sigma(\gamma)} \mu_x \xrightarrow{\gamma} x$

Ex.: $\mathbb{Z}\tilde{Q}$:



$k = \mathbb{C}$ base field

Def. [Keller-Scherotzke]:

Regular Nakajima category $\mathcal{R}$:

↳ objects: vertices of $\mathbb{Z}\tilde{Q}$

↳ morphisms: k-linear combinations of paths in $\mathbb{Z}\tilde{Q}$ modulo the mesh relations:

$$\sum_{\beta \text{ ends at } x} \sigma(\beta)\beta = 0 \quad , \quad \forall \text{ non-frozen vertices } x$$

Singular Nakajima category $\mathcal{S}$:

↳ full subcat. of $\mathcal{R}$ whose objects are the frozen vertices.

Def.: An S-module is a k-linear functor $S^{op} \longrightarrow \mathrm{Mod}\, k$.

Def.: A <u>dimension vector</u> for S is a function $w : \mathrm{ob}(S) \longrightarrow \mathbb{N}$ with finite support.

Def. [Nakajima, Plamondon-Leclerc, KS]
Let w be a dimension vector for S.
The <u>affine graded quiver variety</u> $\mathcal{M}_0(w)$ is the variety of S-modules $M : S^{op} \to \mathrm{Mod}\, k$ s.t. $M(u) = k^{w(u)}$ for $u \in S$.

$\mathcal{M}_0(w)$ is canonically an affine variety.

<u>Rmk:</u> Nakajima defines a stratification on $\mathcal{M}_0(w)$
↳ it can be defined using the category R.

<u>Thm</u> [KS]: There is a functor

$$\Phi : \mathrm{mod}\, S \longrightarrow \mathcal{D}^b(\mathrm{mod}\, kQ)$$

(↑ finite-dimensional)

s.t. $M_1, M_2 \in \mathcal{M}_0(w)$ lie in the same stratum iff $\Phi(M_1) \cong \Phi(M_2)$.

<u>Idea for construction:</u>

(1) S satisfies a weak version of the Iwanaga-Gorenstein property of dimension 1:

$$\mathrm{Ext}^i(M,P) = \mathrm{Ext}^i(I,M) = 0 \quad (i \geq 2)$$

where

- M is finite-dimensional
- P is finitely generated projective
- I is finitely cogenerated injective

(2) The syzygy of a f.d. module is a Gorenstein projective module.

(3) The stable category of Gorenstein projectives is triangle equivalent to $\mathcal{D}^b(\mathrm{mod}\, kQ)$

(4) $\Phi : \mathrm{mod}\, S \xrightarrow{\Omega} \underline{\mathrm{gpr}}\, S \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\, kQ)$

2) Tensor product varieties and split filtrations

Decompose $w = w_1 + w_2 + \dots + w_n$.

For $u \in S$, write $k^{w(u)} = k^{w_1(u)} \oplus k^{w_2(u)} \oplus \cdots \oplus k^{w_n(u)}$.

Def. [Nakajima] The n-fold affine graded tensor product variety $\mathcal{T}_0(w_1, \dots, w_n)$ is the closed subvariety of $\mathcal{M}_0(w)$ of all modules M such that, for all $1 \leq r \leq n$, the subspaces

$$k^{w_1(u)} \oplus \dots \oplus k^{w_r(u)} \subseteq k^{w(u)} = M(u)$$

are invariant under the S-module action on M.

for simplicity, you can think that A, B are algebras

Def. [C.]: Let $B \subseteq A$ be small k-categories.
Define the category $\mathrm{Filt}^n_B(A)$ by:

↳ objects: A-module M together with a filtration
$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$
and B-linear maps $M_i \to M_{i-1}$ which are retractions for the inclusions $M_{i-1} \hookrightarrow M_i$.

↳ morphisms: maps of A-modules respecting the extra structure

Ex.: A point in $\mathcal{J}_0(w_1, \ldots, w_n)$ is an object of $\mathrm{Filt}^n_{kS_0}(S)$ where

kS_0: discrete k-subcat. of S generated by the identities.

Thm. [C.]: If $\mathrm{ob}(B) = \mathrm{ob}(A)$, then $\mathrm{Filt}^n_B(A)$ is equivalent to the module category of a category $T^n_B(A)$.
Moreover, $T^n_B(A)$ is a triangular matrix category.

Ex.: $n = 2$, $X := \ker(A \otimes_B A \to A)$

induced by composition in A

$$T^2_B(A) = \begin{pmatrix} A & 0 \\ X & A \end{pmatrix}$$

objects: two copies x^1, x^2 for each $x \in A$.

morphisms: $T^2_B(A)(x^i, y^j) = \begin{cases} A(x,y) & \text{if } i = j \\ X(x,y) & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}$

Def.: $S^{n\text{-filt}} := T^n_{kS_0}(S)$

Rmk: We recover $\mathcal{J}_0(w_1, \dots, w_n)$ as a variety of modules over $S^{n\text{-filt}}$.

Prop. [C.]: The following hold:

(1) $S^{n\text{-filt}}$ is weakly Gorenstein of dimension 1.

(2) The syzygy of a f.d. $S^{n\text{-filt}}$-module is Gorenstein projective.

Thm. [C.]: We have a triangle equivalence

$$F: \underline{\mathrm{Gpr}}(S^{n\text{-filt}}) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}\, kQ \otimes k\vec{A}_n)$$

↑ lower triangular matrices in kQ.

where $\vec{A}_n = 1 \leftarrow 2 \leftarrow \cdots \leftarrow n$.

Define $\Phi^n: \mathrm{mod}\, S^{n\text{-filt}} \longrightarrow \mathcal{D}^b(\mathrm{mod}\, kQ \otimes k\vec{A}_n)$ as the composition $F \circ \Omega$. ← if $n=1$, this gives the functor of KS

Thm. [C.]: Let M, N be f.d. $S^{n\text{-filt}}$-modules. If $\Phi^n(M) \cong \Phi^n(N)$, then

$$\Phi\left({}^{M_j}/_{M_i}\right) \cong \Phi\left({}^{N_j}/_{N_i}\right) \quad \forall\, i < j.$$