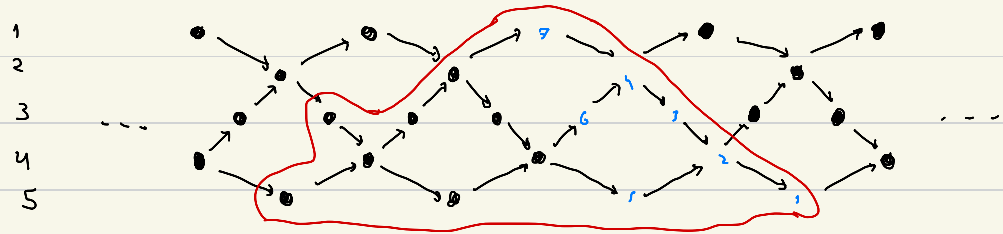
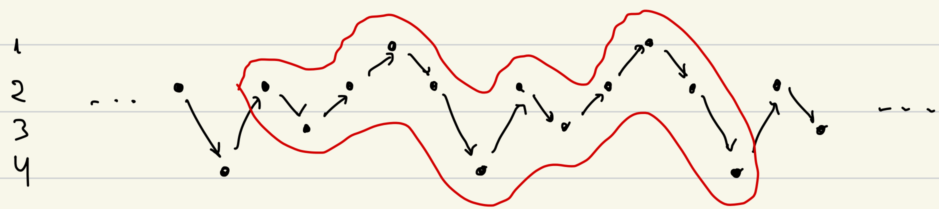
Draw before the talk :

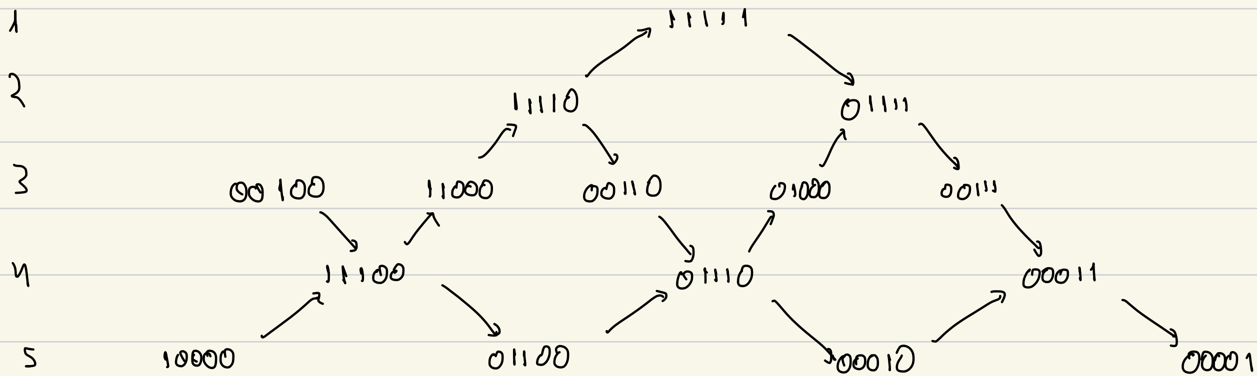
- $\Delta = A_5$, $\underline{i} = (5,4,3,2,5,3,1,4,3,2,5,3,4,3,5)$:

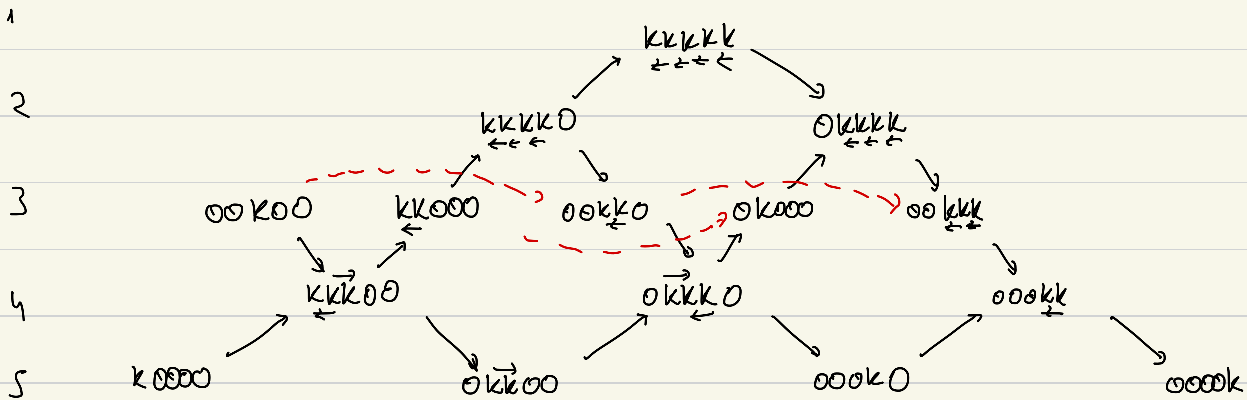


- $\Delta = D_4$, $\underline{i} = (4,2,1,2,3,2,4,2,1,2,3,2)$:



Draw before the talk:





A categorification of combinatorial AR quivers

(arXiv:2505.06147)

1) Context

- Bédard '99: constructed the AR quiver of a Dynkin quiver using Coxeter combinatorics.
- Oh-Suh '19: generalized the construction to arbitrary reduced words in the Weyl group.
- Fujita-Oh '21: Q-data and applications to quantum affine algebras.

↳ just say: motivate the need for a categorification

2) The combinatorics

Δ: ADE Dynkin diagram

R: root system

$\alpha_i \in R$: simple root $(i \in \Delta_0)$

W: Weyl group

$s_i \in W$: simple reflection $(i \in \Delta_0)$

$w_0 \in W$: longest element

$\underline{i} = (i_1, \dots, i_N) \in \Delta_0^N$: reduced word for w_0

↳ just say: $N = \# R^+$

Extend $\underline{i}$ to $(i_k)_{k \in \mathbb{Z}}$ by $i_{k+N} = i_k^*$

just say: involution induced by w_0 on Δ_0

Def. [Oh-Suh]:

Combinatorial repetition quiver $\hat{\Gamma}_{\underline{i}}$:

↳ Vertices: $\mathbb{Z}$

↳ $k \to l$ iff $k > l$, i_k adjacent to i_l in Δ and no index $l < j < k$ s.t. $i_j = i_l$ or $i_j = i_k$

Combinatorial AR quiver $\Gamma_{\underline{i}}$:

↳ Full subquiver with vertex set $\{1, 2, \dots, N\}$, where $N = \ell(w_0)$.

[show examples drawn on the board]

just say: interpretation as a Hasse quiver

Coordinate map $\rho : (\hat{\Gamma}_{\underline{i}})_0 \longrightarrow R$

$$\rho(k) = \begin{cases} s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k}), & k \geq 1 \\ -s_{i_0} s_{i_{-1}} \cdots s_{i_{k+1}} (\alpha_{i_k}), & k \leq 0 \end{cases}$$

↳ just say: positive roots on $\Gamma_{\underline{i}}$.

[continue example]

Thm. [Happel; Bédard; OS]: Let Q be an orientation of Δ. If $\underline{i}$ is a source sequence for Q, then $\Gamma_{\underline{i}}$ is isomorphic to the AR quiver of the path algebra of Q and $\hat{\Gamma}_{\underline{i}}$ to the AR quiver of the derived cat.

↳ just say: agrees with Gabriel's bijection.

For $x \in (\widehat{\Gamma}_{\underline{i}})_0$, define:

- the <u>translate</u> of x is the smallest integer τx s.t. $x < \tau x$ and $i_x = i_{\tau x}$.

"Anlieger"

- the <u>set of abutters</u> of x is the subset $V_{\underline{i}}(x)$ of vertices y s.t. $x < y < \tau x$ and i_y is adjacent to i_x in Δ.

<u>Thm</u> [C.] We have:

$$\rho(\tau x) + \rho(x) = \sum_{y \in V_{\underline{i}}(x)} \rho(y)$$

[demonstrate in the example]

3) The categorification

just say: we want first to categorify the root system

$K = \overline{k}$: field

just say: 'dg enhancement'

Π : derived preprojective algebra of type Δ over k.

$\mathrm{pvd}(\Pi)$: perfectly valued derived cat. of Π

just say: 2-CY triangulated cat.

<u>Ex.</u>: $\Delta = A_3$, Π is the dg path algebra of

$$\begin{tikzcd} \bullet \arrow[loop above, "t_1"] \arrow[r, bend left, "\alpha"] & \bullet \arrow[loop above, "t_2"] \arrow[l, bend left, "\overline{\alpha}"] \arrow[r, bend left, "\beta"] & \bullet \arrow[loop above, "t_3"] \arrow[l, bend left, "\overline{\beta}"] \end{tikzcd}$$

$|t_i| = -1$

$|\text{other arrows}| = 0$

$$d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$$

just say: H^0 is the preproj. alg.

S_i : simple dg module $(i \in \Delta_0)$

Lemma: S_i is a 2-spherical object of $\mathrm{prd}(\Pi)$.

Seidel–Thomas

$\Longrightarrow$ $T_i : \mathrm{prd}(\Pi) \longrightarrow \mathrm{prd}(\Pi)$

spherical twist functor

↳ just say: braid group action

Lemma: $K_0(\mathrm{prd}(\Pi)) \xrightarrow{\sim}$ root lattice of Δ

$[S_i] \longmapsto \alpha_i$

and the action of T_i corresponds to the action of s_i.

For $k \in \mathbb{Z}$, define

$$M^{\underline{i}}_k = \begin{cases} T_{i_1} T_{i_2} \cdots T_{i_{k-1}} (S_{i_k}) , & k \geq 1 \\ \Sigma\, T_{i_0}^{-1} T_{i_{-1}}^{-1} \cdots T_{i_{k+1}}^{-1} (S_{i_k}) , & k \leq 0 \end{cases}$$

just say: Σ is the suspension functor

T_i is not involutive

Def:

- **Repetition category** $\mathcal{R}(\underline{i})$: full additive subcategory of $\mathrm{prd}(\Pi)$ generated by the $M^{\underline{i}}_k$.
- **Category of representations** $\mathcal{C}(\underline{i})$: full subcategory of $\mathcal{R}(\underline{i})$ of objects concentrated in degree 0.

Prop. [Baur-Iyama-Reiten-Scott, Amiot-I-R-Todorov]

- The indecomposable objects of $\mathcal{C}(\underline{i})$ are the $M^{\underline{i}}_k$ for $1 \le k \le N = \ell(w_0)$.

  just say: they are called "layers"

- If $\underline{i}$ is a source sequence for an orientation Q of Δ, then $\mathcal{C}(\underline{i}) \simeq \operatorname{mod} KQ$.

Thm. [C.]: $\Gamma_{\underline{i}}$ is isomorphic to the quiver obtained from the Gabriel quiver of $\mathcal{C}(\underline{i})$ by removing all arrows parallel to paths of length at least two. Similarly for $\hat{\Gamma}_{\underline{i}}$ and $R(\underline{i})$.

[show example drawn on the board]

Thm. [C.]: Given $x \in (\hat{\Gamma}_{\underline{i}})_0$, choose an ordering $y_1, \dots, y_t$ of the set of abutters $V_{\underline{i}}(x)$ s.t. $k \le \ell$ whenever there is a path from y_k to y_ℓ in $\hat{\Gamma}_{\underline{i}}$.

Then there are indec. obj. $X_1, X_2, \dots, X_{t-1} \in R(\underline{i})$ and a diagram of the form:

$$\begin{tikzcd}[column sep=small]
\Sigma^{-1} M^{\underline{i}}_x = X_t \arrow[rr] & & X_{t-1} \arrow[dl] \arrow[rr] & & \cdots \arrow[rr] & & X_2 \arrow[rr] & & X_1 \arrow[rr] \arrow[dl] & & X_0 = M^{\underline{i}}_{2x} \arrow[dl] \\
& M^{\underline{i}}_{y_t} \arrow[ul, dotted] & & & & & & M^{\underline{i}}_{y_2} \arrow[ul, dotted] & & M^{\underline{i}}_{y_1} \arrow[ul, dotted] &
\end{tikzcd}$$

distinguished triangles in pvd(Π)

Just say (if time allows): derived cat., Q-data, ...