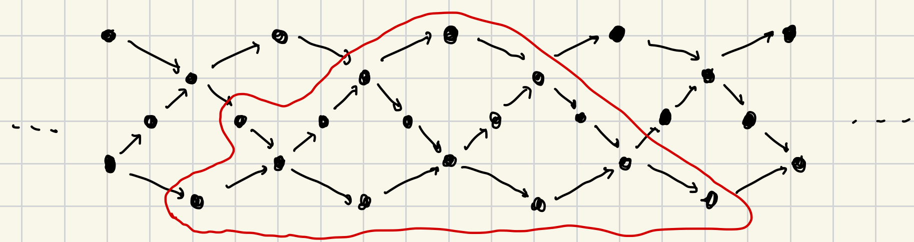
Draw before the talk :

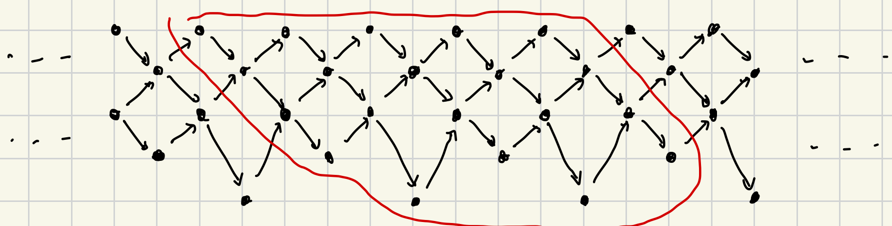
Type B_3 :

twisted AR quiver



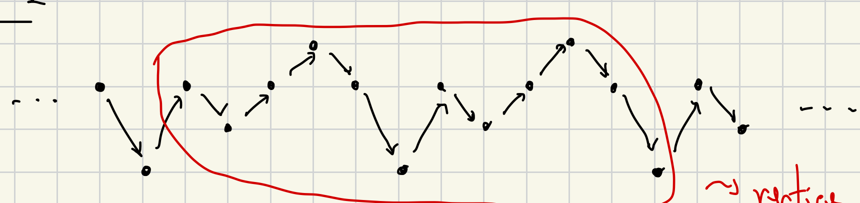
↳ vertices $\longleftrightarrow$ positive roots of A_5

Type C_4 :



↳ vertices $\longleftrightarrow$ positive roots of D_5

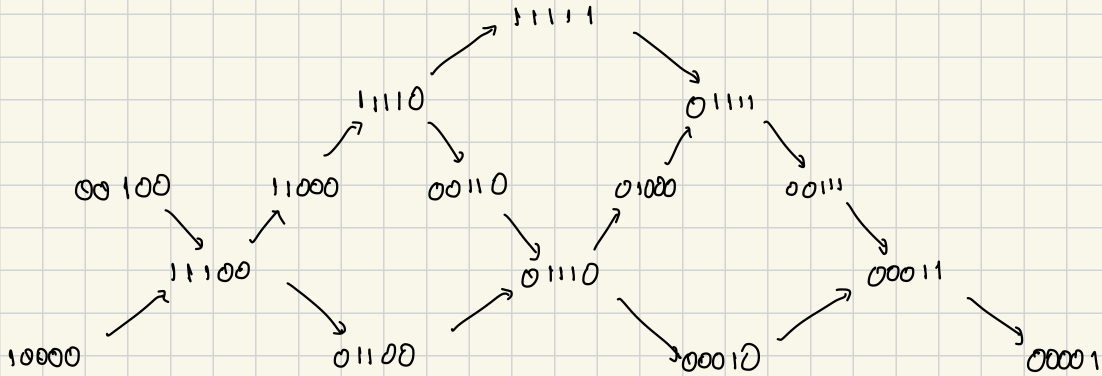
Type G_2 :

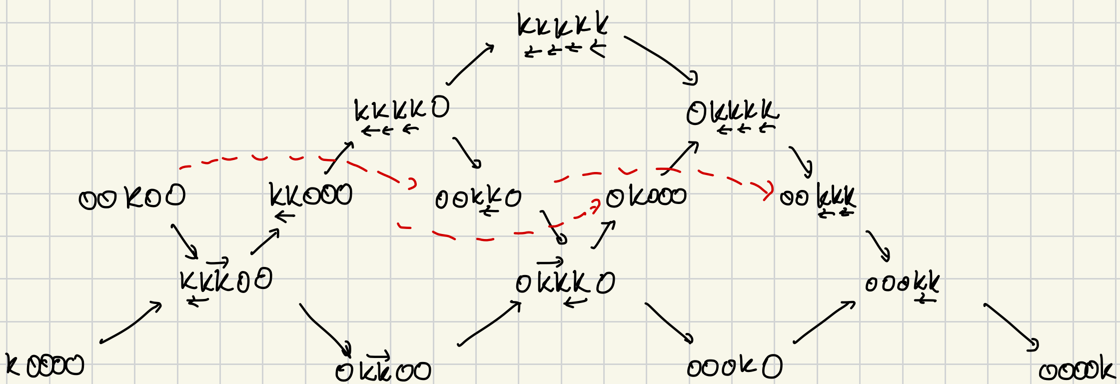


$\rightsquigarrow$ vertices $\longleftrightarrow$ positive roots of D_4

Draw before the talk:

1
2
3
4
5





Categorifying twisted Auslander-Reiten quivers

1) Motivation

- $\mathfrak{g}$: f.d. complex simple Lie alg. e.g. $\mathfrak{so}_5(\mathbb{C})$
- C = Cartan matrix of $\mathfrak{g}$ e.g. $\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$

- $C(q)$: quantum Cartan matrix of $\mathfrak{g}$

e.g. $C(q) = \begin{bmatrix} q^2+q^{-2} & -1 \\ -(q+q^{-1}) & q+q^{-1} \end{bmatrix}$

- $\tilde{C}(q) := C(q)^{-1} \rightsquigarrow \tilde{C}_{ij}(q) = \sum_{u \geq 0} \tilde{c}_{ij}(u) q^u$

ADE case

- Q : orientation of Dynkin diagram of $\mathfrak{g}$
- $\xi : Q_0 \to \mathbb{Z}$ height function
  (i.e. $\xi_i = \xi_j + 1$ for $i \to j$ in Q)
- K : field

Happel: $H_Q : \hat{\Delta}_0 \xrightarrow{\sim} \mathrm{ind}(\mathcal{D}^b(\mathrm{mod}\, KQ))$

$(i,p) \longmapsto \tau^{(\xi_i - p)/2}(I_i)$

$$\hat{\Delta}_0 := \{ (i,p) \in Q_0 \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z} \}$$

Thm [Hernandez-Leclerc '15, Fujita '22]: For $(i,p), (j,s) \in \hat{\Delta}_0$ s.t. $s \geq p$, we have:

$$\tilde{c}_{ij}(s-p+1) = \langle H_Q(i,p), H_Q(j,s) \rangle$$

Euler form on $\mathcal{D}^b(\mathrm{mod}\, KQ)$

General case:

Fujita-Oh '21:

- Q-datum
- Twisted AR quivers (Oh-Suh 19')
- Combinatorial formula for $\tilde{c}_{ij}(u)$

Goal: · Categorify these combinatorics.
· Reinterpret Fujita-Oh's formula

We'll construct: $\mathcal{C}(\mathfrak{q})$ and $\mathcal{D}(\mathfrak{q})$

"cat. of reps. of $\mathfrak{q}$"

"derived cat."

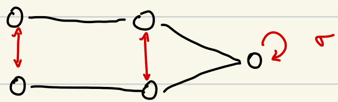
[Show example of twisted AR quivers and comment a little about their properties, e.g. "mesh relations"]

[Use slides or draw the examples before]

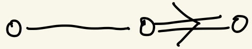
2) Q-data combinatorics

- (Δ, σ) : unfolding of $\mathfrak{g}$
  - ↳ Δ : Dynkin diag. of type ADE
  - ↳ σ : automorphism of Δ

Ex.: A_5



B_3 :



Denote $I = \Delta_0 / \langle \sigma \rangle$

$i \in \Delta_0 \Rightarrow d_i := \#$ orbit containing i

$i \in I \Rightarrow d_i := \# i$

Rmk: $d_i \in \{1, r\}$, where r is the order of σ

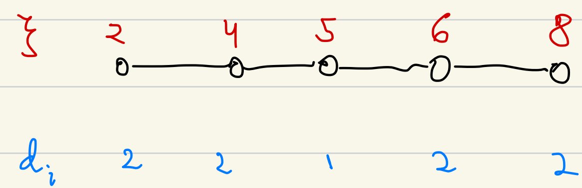
Def [FO]: A Q-datum for $\mathfrak{g}$ is $Q = (\Delta, \sigma, \xi)$ where:

- (Δ, σ) is the unfolding of $\mathfrak{g}$
- $\xi : \Delta_0 \longrightarrow \mathbb{Z}$ is a "generalized" height function

(intuition : $|\xi_i - \xi_j| = \min(d_i, d_j)$, $i - j$ in Δ)

Rmk: $\sigma = id \Rightarrow$ Q-datum = Dynkin quiver of type Δ + height function

Ex.: Type B_3 :



<u>Repetition quiver</u> $\hat{\Delta}^\sigma$:

$$\hat{\Delta}^\sigma_0 = \{ (i,p) \in \Delta_0 \times \mathbb{Z} \mid p - \xi_i \in 2d_i \mathbb{Z} \}$$

$$\hat{\Delta}^\sigma_1 = \left\{ (i,p) \to (j,s) \;\middle|\; \begin{array}{l} i - j \text{ in } \Delta \\ s-p = \min(d_i, d_j) \end{array} \right\}$$

<u>Twisted AR quiver</u> Γ_Q : full subquiver of $\hat{\Delta}^\sigma$ with :

$$(\Gamma_Q)_0 = \{ (i,p) \in \hat{\Delta}^\sigma_0 \mid \xi_{i^*} - rh^\vee < p \leq \xi_i \}$$

$h^\vee$: dual Coxeter number of $\mathfrak{g}$

$i \mapsto i^*$: involution induced by longest element of Weyl group of Δ

<u>Def.</u>: A <u>compatible reading</u> of Γ_Q is an enumeration $(i_1, p_1), \dots, (i_N, p_N)$ of Γ_Q s.t.

$\exists$ path $(i_k, p_k) \rightsquigarrow (i_\ell, p_\ell)$ in $\Gamma_Q \Rightarrow k > \ell$.

To the vertex (i_k, p_k), we assign a root of Δ :

$$s_{i_1} s_{i_2} \cdots s_{i_{k-1}} (\alpha_{i_k})$$

↳ α_i : simple root for $i \in \Delta_0$

↳ s_i : simple reflection for $i \in \Delta_0$

<u>Thm</u> [OS, FO] : $\underline{i} = (i_1, \dots, i_N)$ is a reduced word for w_0.

$\Longrightarrow$ We get all positive roots, without repetition.

The assignment independs on the compatible reading.

3) Categorification

- K : field
- Q° : orientation of Δ

- Π : 2-CY completion of KQ°
- $\mathrm{prd}(\Pi)$: perfectly valued derived cat. of Π
  ↳ 2-CY triang. cat.

Ex.: If $\Delta = A_3$, Π is the dg path algebra given by:

$$\begin{tikzcd}
\circ \arrow[loop above, "t_1"] \arrow[r, bend left, "\alpha"] & \circ \arrow[loop above, "t_2"] \arrow[l, bend left, "\overline{\alpha}"] \arrow[r, bend left, "\beta"] & \circ \arrow[loop above, "t_3"] \arrow[l, bend left, "\overline{\beta}"]
\end{tikzcd}$$

$|t_i| = -1$

$|\text{other arrows}| = 0$

$$d(t_1 + t_2 + t_3) = [\alpha, \overline{\alpha}] + [\beta, \overline{\beta}]$$

$i \in \Delta_0 \implies$ simple module $S_i \in \mathrm{prd}(\Pi)$

Lemma: S_i is a 2-spherical obj. in $\mathrm{prd}(\Pi)$

$\implies T_i : \mathrm{prd}(\Pi) \longrightarrow \mathrm{prd}(\Pi)$
spherical twist functor

Lemma: $K_0(\mathrm{prd}(\Pi)) \xrightarrow{\sim}$ root lattice of Δ

$$[S_i] \longmapsto \alpha_i$$

Under this identification, T_i corresponds to s_i

q : Q-datum for $\mathfrak{g}$

$\underline{i}$: reduced word for w_0 from compatible reading of Γ_q

$= (i_1, \ldots, i_N)$

For $1 \leq k \leq N$, define :

$$M_k = T_{i_1} T_{i_2} \cdots T_{i_{k-1}} (S_{i_k}) \in \mathrm{prd}(\Pi)$$

$$\boxed{\begin{array}{c}\mathscr{C}(q) := \text{strictly full additive subcat. of } \mathrm{prd}(\Pi) \\ \text{generated by } M_k, \ 1 \leq k \leq N\end{array}}$$

<u>Prop:</u> If q is a Dynkin quiver of type ADE, then

$$\mathscr{C}(q) \simeq \mathrm{mod}\, Kq .$$

<u>Prop</u> [C.]: The objects of $\mathscr{C}(q)$ are dg modules whose cohomology is concentrated in deg. 0.

$$\Longrightarrow \quad \mathscr{C}(q) \subseteq \mathrm{mod}\, \underbrace{H^0(\Pi)}$$

preprojective alg. of type Δ

<u>Thm</u> [C.]: Γ_q is isom. to the quiver obtained from the Gabriel quiver of $\mathscr{C}(q)$ by removing all arrows parallel to paths of length ≥ 2.

[show picture already drawn on the board]

$R(Q)$: full additive subcat. of $\mathrm{prd}(\Pi)$ generated by $\Sigma^k M$, $M \in \mathcal{C}(Q)$, $k \in \mathbb{Z}$.

We construct a certain ideal $\mathcal{J}$ of $R(Q)$.

<u>Def.</u>: The "derived cat." of Q is:

$$\boxed{\mathcal{D}(Q) := R(Q) / \mathcal{J}}$$

<u>Prop</u>: If Q is a Dynkin quiver of type ADE, then $\mathcal{D}(Q) \simeq \mathcal{D}^b(\mathrm{mod}\, kQ)$.

For $M, N \in \mathcal{D}(Q)$, define:

$$\langle M, N \rangle_Q := \sum_{k \in \mathbb{Z}} (-1)^k \dim \operatorname{Hom}_{\mathcal{D}(Q)}(M, \Sigma^k N)$$

Fujita-Oh: twisted Coxeter element $\tau_Q \in W\sigma$

$\Longrightarrow$ we lift it to an equivalence $\tau_Q : \mathcal{D}(Q) \longrightarrow \mathcal{D}(Q)$.

We can identify $\operatorname{ind}(\mathcal{D}(Q))$ with $\hat{\Delta}_0^\sigma$ and

$$\hat{I} := \{ (i,p) \in I \times \mathbb{Z} \mid p - \xi_j \in 2\mathbb{Z},\ \forall j \in i \}$$

$\rightsquigarrow H_Q : \hat{I} \xrightarrow{\sim} \operatorname{ind}(\mathcal{D}(Q))$

$$(i,p) \in \hat{\Delta}_0^\sigma \longmapsto \tau_Q^{\frac{\xi_i - p}{2}}(\gamma_i)$$

Thm: [FO, C.] For $(i,p), (j,s) \in \hat{I}$ s.t. $p \geq s$ and $\max\{d_i, d_j\} = r$, we have

$$\boxed{\tilde{c}_{ij}(p-s+d_i) = \left\langle H_q(j,s), \bigoplus_{k=0}^{\lceil d_j/d_i \rceil - 1} \tau_q^k (H_q(i,p)) \right\rangle_q}$$

Rmk: Type B: we can remove the restriction on d_i and d_j.

(TS)